Enumerating the Fake Projective Planes

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A fake projective plane is a smooth compact complex surface P which is not biholomorphic to the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, but has the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$, namely 1, 0, 1, 0, 1.

The first fake projective plane was constructed by Mumford in 1979. He also showed that there could only be a finite number of these surfaces.

Two more examples were found by Ishida and Kato in 1998, and another by Keum in 2006.

In their 2007 Inventiones paper [PY], Gopal Prasad and Sai-Kee Yeung almost completely classified fake projective planes. They showed that these fall into 33 “classes”.

For 28 of the classes, they give at least one fake projective plane. These 28 classes are all defined using unitary groups associated with certain **cubic division algebras**.

For each of the remaining 5 classes, they left open the question of existence of fake projective planes in that class, but conjectured that there are none. These 5 classes are all defined using certain unitary **matrix groups**.

All the classes are associated with pairs (k, ℓ) of fields, and extra data. It turns out that k is either $\mathbb{Q}$ or a real quadratic extension of $\mathbb{Q}$, and ℓ is always a totally complex quadratic extension of k.

What Tim and I have done is

(a) found **all** the fake projective planes, up to isomorphism, in each of the 28 division algebra classes, and

(b) shown that there are indeed no fake projective planes in the remaining 5 matrix algebra classes.

In (a), we find that there are, altogether, **50** fake projective planes.

This count depends on what "isomorphism" means. If it means "biholomorphism", then we should multiply this number by 2. We are in fact classifying the fpp's according to their **fundamental groups**. It follows from a result of Siu, that if two fpp's have isomorphic fundamental groups, then they are either biholomorphic or conjugate-biholomorphic.

Recall that $U(2,1)$ is the group of 3×3 complex matrices g such that $g^* F_0 g = F_0$, where

$$F_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and that $PU(2,1)$ is $U(2,1)$, modulo scalars. This is the automorphism group of $\mathbb{B}(\mathbb{C}^2) = \{(z_1, z_2) \ : \ |z_1|^2 + |z_2|^2 < 1\}$, with the hyperbolic metric.

Theorem (Klingler, Yeung). The fundamental group Π of a fake projective plane is a torsion-free cocompact arithmetic subgroup of $PU(2,1)$.

So a fake projective plane is a ball quotient $\mathbb{B}(\mathbb{C}^2)/\Pi$ for such a Π. By Hirzebruch proportionality,

$$3\mu(PU(2,1)/\Pi) = \chi(\mathbb{B}(\mathbb{C}^2)/\Pi) = 3$$

So Π has **covolume 1** in $PU(2,1)$.

If $\Pi \subset \bar{\Gamma}$, where $\bar{\Gamma}$ is maximal arithmetic, then $\mu(PU(2,1)/\bar{\Gamma})$ must equal $1/[\bar{\Gamma} : \Pi]$.

Prasad and Yeung work out the possibilities for maximal arithmetic subgroups $\bar{\Gamma}$ of $PU(2,1)$ such that $\mu(PU(2,1)/\bar{\Gamma}) = 1/N$ for an integer N.

Theorem [PY]. The fundamental groups of the fake projective planes in this class are the **torsion-free** subgroups Π of $\bar{\Gamma}$ such that

- $[\bar{\Gamma} : \Pi] = N$, and

- $\Pi/[\Pi, \Pi]$ is finite.

For the 28 cubic division algebra cases, the only possibilities for N are 1, 3, 9 and 21. In the 5 matrix algebra cases, N is one of 48, 288, 600, and 864.

Our main result: For each of the $\bar{\Gamma}$'s we have found **generators** (explicit elements of the corresponding cubic division algebra or matrix algebra). We have found a **presentation** of $\bar{\Gamma}$ in each case.

This allows us to find all subgroups Π in $\bar{\Gamma}$ (up to conjugacy) satisfying (a), (b) and (c). The computer algebra package Magma, for example, has a command `LowIndexSubgroups(G,n)` which lists all conjugacy classes of subgroups of a given index n in a finitely presented group G.

Magma's command `Rewrite(G,H)` finds a presentation of a given subgroup H of finite index in a given finitely presented group G.

So the fundamental groups Π of the fpp's can be listed, and presentations given for each of them.

Example. For the class we call $(a = 7, p = 2, \{7\})$, $\bar{\Gamma}$ has the presentation

$$
\begin{aligned}
\bar{\Gamma} = \langle\ b, z\ |\ \\
& b^3 = 1, \\
& z^7 = 1, \\
& (bz^{-2}bz^{-1})^3 = 1, \\
& b^{-1}zbz^2bz^2b^{-1}z^{-1}bz^2b^{-1}z = 1, \\
& bz^2b^{-1}z^{-1}bz^{-1}bz^2b^{-1}z^{-1}bz^{-1}bz^{-3} = 1, \\
& bz^2bzbz^{-2}b^{-1}zbz^{-1}bz^{-2}b^{-1}z^2 = 1\rangle.
\end{aligned}
$$

In this case $\mu(PU(2,1)/\bar{\Gamma}) = 1/21$. Magma's command

`LowIndexSubgroups`$(\bar{\Gamma}, 21)$

lists all (conjugacy classes of) subgroups of $\bar{\Gamma}$ of index at most 21.

Example (continued). It finds that there are four subgroups of index 21:

$$\Pi_a = \langle bzb^{-1}z^{-2},\ bz^{-1}b^{-1}z^2,\ zbz^3b^{-1}\rangle$$

$$\Pi_b = \langle zbz^{-1}b^{-1},\ z^3bzb^{-1},\ z^2b^{-1}z^{-1}b,\ zb^{-1}zbz\rangle$$

$$\Pi_c = \langle zbz^{-1}b^{-1},\ z^2bz^2b^{-1},\ zb^{-1}zbz\rangle$$

$$\Pi_d = \langle zb^{-1},\ z^{-3}b,\ b^{-1}zbzb\rangle$$

All are torsion-free with finite abelianization.

The group Π_d is the fundamental group of Mumford's original fake projective plane. We know this because

- $N_{\bar{\Gamma}}(\Pi_d) = \Pi_d$, which means that $\mathrm{Aut}(B(\mathbb{C}^2)/\Pi_d)$ is trivial,
- There is a surjective homomorphism $\bar{\Gamma} \to PSL(2, \mathbb{F}_7)$.

Getting back to the Prasad-Yeung classes: The 28 division algebra classes are each associated with pairs (k,ℓ) of fields. Only 9 pairs (k,ℓ) arise:

- For $k=\mathbb{Q}$: $\ell=\mathbb{Q}(\sqrt{-a}\,)$, $a=$ 1, 2, 7, 15 or 23.
- $k=\mathbb{Q}\Big(\sqrt{5}\Big)$, $\ell=k\Big(\sqrt{-3}\Big)$. "$\mathcal{C}_2$ case".
- $k=\mathbb{Q}\Big(\sqrt{2}\Big)$, $\ell=k\Big(\sqrt{-5+2\sqrt{2}}\,\Big)$. "$\mathcal{C}_{10}$ case".
- $k=\mathbb{Q}\Big(\sqrt{6}\Big)$, $\ell=k\Big(\sqrt{-3}\Big)$. "$\mathcal{C}_{18}$ case".
- $k=\mathbb{Q}\Big(\sqrt{7}\Big)$, $\ell=k\Big(\sqrt{-1}\Big)$. "$\mathcal{C}_{20}$ case".

For each of these 9 pairs (k,ℓ), there are at least 2 classes.

Associated to each pair (k, ℓ) there is a division algebra $\mathcal{D}$.

For each pair (k, ℓ), a particular prime p is specified. We require that

- there are exactly two p-adic places v on ℓ,
- $\mathcal{D} \otimes \ell_v$ is a division algebra for these two v's, and
- $\mathcal{D} \otimes \ell_{v'} \cong \mathrm{Mat}_{3\times 3}(\ell_{v'})$ for any other non-archimedean place v' of ℓ.

There is an involution ι of the second kind on $\mathcal{D}$. Form the group

$$G(k) = \{\xi \in \mathcal{D} \ : \ \iota(\xi)\xi = 1 \text{ and } \mathrm{Nrd}(\xi) = 1\}.$$

The involution ι is chosen so that $G(k_{v_0}) \cong SU(2,1)$ for one archimedean place v_0 on k, and so that $G(k_v)$ is compact for the other archimedean place (when $k \neq \mathbb{Q}$).

We realize the groups $\bar{\Gamma}$ inside the adjoint group

$$\bar{G}(k) = \{\xi \in \mathcal{D} \ : \ \iota(\xi)\xi = 1\}/\{t \in \ell \mid \bar{t}t = 1\}.$$

For the purpose of calculations, we realize each $\mathcal{D}$ as a “cycle simple algebra”:

Suppose that L is a field, and that M is a degree 3 Galois extension of L, with Galois group generated by φ. Let $D \in L$. Form

$$\mathcal{D} = \{a + b\sigma + c\sigma^2 : a, b, c \in M\}.$$

We define multiplication so that $\sigma^3 = D$ and $\sigma x \sigma^{-1} = \varphi(x)$ for all $x \in M$. Then $\mathcal{D}$ is a 9-dimensional simple algebra over L with centre L.

We can embed $\mathcal{D}$ in $\mathrm{Mat}_{3\times 3}(M)$:

$$\Psi : \quad x \mapsto \begin{pmatrix} x & 0 & 0 \\ 0 & \varphi(x) & 0 \\ 0 & 0 & \varphi^2(x) \end{pmatrix} \quad \text{(for } x \in M\text{), and} \quad \sigma \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ D & 0 & 0 \end{pmatrix}$$

If D is the norm $\mathsf{N}_{M/L}(\eta)$ of an element of M, then $\mathcal{D} \cong \mathsf{Mat}_{3\times 3}(L)$. We choose a basis $\theta_0, \theta_1, \theta_2$ of M over L, let $\zeta_0, \zeta_1, \zeta_2$ be the basis of M dual to this with respect to trace: $\mathsf{Trace}_{M/L}(\theta_i \zeta_j) = \delta_{ij}$. We find that for any $\xi \in \mathcal{D}$, the following has entries in L:

$$C_2 C_1 \Psi(\xi) C_1^{-1} C_2^{-1},$$

where

$$C_1 = \begin{pmatrix} \eta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/\varphi(\eta) \end{pmatrix} \quad \text{and} \quad C_2 = \begin{pmatrix} \theta_0 & \varphi(\theta_0) & \varphi^2(\theta_0) \\ \theta_1 & \varphi(\theta_1) & \varphi^2(\theta_1) \\ \theta_2 & \varphi(\theta_2) & \varphi^2(\theta_2) \end{pmatrix}.$$

This is because C_2^{-1} has (i,j)th entry $\varphi^i(\zeta_j)$, and so for each $x \in M$,

$$\left(C_2 C_1 \Psi(x) C_1^{-1} C_2^{-1}\right)_{i,j} = \mathsf{Trace}_{M/L}(\theta_i x \zeta_j) \in L,$$
$$\left(C_2 C_1 \Psi(\sigma) C_1^{-1} C_2^{-1}\right)_{i,j} = \mathsf{Trace}_{M/L}(\theta_i \eta \varphi(\zeta_j)) \in L.$$

Example: the case $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-7})$. There are 6 classes in this case. We use the field $m = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_7$, which is a degree 3 extension of ℓ with Galois group $\mathrm{Gal}(m/\ell) = \langle \varphi \rangle$, where $\varphi(\zeta) = \zeta^2$, and let $D = \frac{3+\sqrt{-7}}{4}$. Note that $\sqrt{-7} = 1 + 2\zeta + 2\zeta^2 + 2\zeta^4$, so $\ell \subset m$. Thus

$$\mathcal{D} = \Big\{ a + b\sigma + c\sigma^2 \ : \begin{array}{l} \bullet\ a, b, c \in m, \\ \bullet\ \sigma x \sigma^{-1} = \varphi(x) \text{ for all } x \in m, \\ \bullet\ \sigma^3 = \dfrac{3+\sqrt{-7}}{4}. \Big\} \end{array}$$

The special prime here is **2**.

- $\mathcal{D} \otimes \ell_v$ is a division algebra for the two 2-adic valuations on ℓ.

- $\mathcal{D} \otimes \ell_{v'} \cong \mathrm{Mat}_{3\times 3}(\ell_{v'})$ for all the other v''s.

The reason is: $v'(D) = 0$, and so $D = \mathrm{N}_{m_{v''}/\ell_{v'}}(\eta)$ for some $\eta \in m_{v''}$.

There is an involution ι_0 on $\mathcal{D}$ which maps σ to σ^{-1} and ζ to ζ^{-1}, but this needs to be modified. In terms of the embedding $\Psi : \mathcal{D} \to \mathrm{Mat}_{3\times 3}(m)$, we have

$$\Psi(\iota_0(\sigma)) = \Psi(\sigma^{-1}) = \Psi(\sigma)^{-1} = \begin{pmatrix} 0 & 0 & \bar{D} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \Psi(\sigma)^*,$$

and in general $\iota_0(\xi)\xi = 1$ means that $\Psi(\xi)^*\Psi(\xi) = I$. So $G(\mathbb{R})$ would be $SU(3)$, not the desired $SU(2,1)$. We replace ι_0 by $\iota : \xi \mapsto w^{-1}\iota_0(\xi)w$, where $w = \zeta + \zeta^{-1}$. Now

$$\Psi(\iota(\sigma)) = \Psi(w^{-1}\sigma^{-1}w) = \Psi(w)^{-1}\Psi(\sigma)^*\Psi(w) = F^{-1}\Psi(\sigma)F$$

for

$$F = \begin{pmatrix} w & 0 & 0 \\ 0 & \varphi(w) & 0 \\ 0 & 0 & \varphi^2(w) \end{pmatrix},$$

and in general, $\iota(\xi)\xi = 1$ means that $F^{-1}\Psi(\xi)^*F\Psi(\xi) = I$, that is, $\Psi(\xi)^*F\Psi(\xi) = F$. So $G(\mathbb{R}) \cong SU(2,1)$ because the real numbers w, $\varphi(w)$ and $\varphi^2(w)$ do not all have the same sign.

Returning to the general situation, we know that for a particular prime p, there is a unique p-adic valuation v on k, and that $\mathcal{D} \otimes \ell_{\tilde{v}}$ is a division algebra for the two valuations $\tilde{v}$ on ℓ extending v.

For the other archimedean valuations v on k, either

$$\bar{G}(k_v) \cong \begin{cases} PGL(3, k_v) & \text{if } v \text{ splits in } \ell, \\ PU(h_v) & \text{if } v \text{ does not split in } \ell. \end{cases}$$

Here h_v is a nondegenerate hermitian form on ℓ_v^3.

We need to describe the maximal parahoric subgroups $\bar{P}_v \leq \bar{G}(k_v)$.

When $\bar{G}(k_v) \cong PGL(3, k_v)$, the maximal parahorics are the conjugates of $PGL(3, \mathcal{O}_v)$, where $\mathcal{O}_v$ is the valuation ring of k_v. $PGL(k_v)$ acts on the homothety classes of $\mathcal{O}_v$-lattices in k_v^3 and $PGL(3, \mathcal{O}_v)$ is the stabilizer of $[\mathcal{O}_v^3]$. The building involved is of type $\tilde{A}_2$.

When $\bar{G}(k_v) \cong PU(h_v)$, we write v for the unique place of ℓ over v. Let $\mathcal{O}_v$ denote the valuation ring of ℓ_v, and π_v a uniformizer of ℓ_v. If $\mathcal{L} \subset \ell_v^3$ is an $\mathcal{O}_v$-lattice, define its dual to be the lattice

$$\mathcal{L}' = \{y \in \ell_v^3 : h_v(x,y) \in \mathcal{O}_v \text{ for all } x \in \mathcal{L}\}.$$

A **type 1** maximal parahoric is the stabilizer of a self-dual lattice $\mathcal{L}_1$.

A **type 2** maximal parahoric is the stabilizer of a lattice $\mathcal{L}_2$ such that $\pi_v \mathcal{L}_2 \subsetneqq \mathcal{L}_2' \subsetneqq \mathcal{L}_2$.

The building involved here is a **tree**, with $\mathcal{L}_1$ adjacent to $\mathcal{L}_2$ if $\pi_v \mathcal{L}_2 \subset \mathcal{L}_1 \subset \mathcal{L}_2$.

When ℓ_v is a ramified extension of k_v, this tree is homogeneous, with each vertex having $q_v + 1$ neighbours. When ℓ_v is a unramified extension of k_v, this tree is semihomogeneous, with each type 1 vertex having $q_v^3 + 1$ neighbours, and each type 2 vertex having $q_v + 1$ neighbours.

Return to our example $k = \mathbb{Q}$ and $\ell = \mathbb{Q}(\sqrt{-7})$. If a prime $p \neq 2$ splits in ℓ, let $\bar{P}_p$ be the stabilizer of $[\mathbb{Z}_p^3]$. For the primes p which do not split in ℓ, arrange the isomorphisms $\bar{G}(\mathbb{Q}_p) \cong PU(h_p)$ so that $\mathcal{L}_p = \mathbb{Z}_p(\sqrt{-7})^3$ is self-dual, and let $\bar{P}_p$ be its stabilizer. Prasad and Yeung tell us that

$$\bar{\Gamma} = \bar{G}(\mathbb{Q}) \cap \prod_p \bar{P}_p$$

has covolume $1/21$ in $PU(2,1)$.

If we replace the parahoric $\bar{P}_7$ by the stabilizer of a type 2 neighbour of $\mathcal{L}_7 = \mathbb{Z}_7(\sqrt{-7})^3$ we get the **same** covolume, because the 7-adic building is a homogeneous tree. If for some prime $q \neq 7$ which does not split in ℓ we replace $\mathcal{L}_q = \mathbb{Z}_q(\sqrt{-7})^3$ by a type 2 neighbour, then the covolume of the corresponding $\bar{\Gamma}$ is

$$\frac{1}{21} \times \frac{q^3+1}{q+1}.$$

So if $q = 3$, the covolume is $1/3$; if $q = 5$, the covolume is 1; otherwise we do not get a covolume of the form $1/N$.

In general, if

$$\bar{\Gamma} = \bar{G}(k) \cap \prod_v \bar{P}_v$$

has covolume $1/N$, we denote by $\mathcal{T}_1$ the set of non-archimedean places of k which do not split in ℓ, and for which $\bar{P}_v$ is of type 2.

In our example, the possibilities for $\mathcal{T}_1$ are

$$\emptyset,\ \{3\},\ \{5\},\ \{7\},\ \{3,7\} \text{ and } \{5,7\}.$$

There are 6 classes of fake projective places corresponding to the pair $(k,\ell) = (\mathbb{Q}, \mathbb{Q}(\sqrt{-7}\,))$. we denote them $(a = 7, p = 2, \mathcal{T}_1)$ for the above $\mathcal{T}_1$'s. The covolumes of the corresponding $\bar{\Gamma}$'s are, respectively,

$$\frac{1}{21},\ \frac{1}{3},\ \frac{1}{1},\ \frac{1}{21},\ \frac{1}{3},\ \text{and } \frac{1}{1}.$$

Finding generators and relations for the groups $\bar{\Gamma}$. Let's consider the example $\bar{\Gamma}_{(a=7,p=2,\{7\})}$. We are looking for elements $\xi \in \mathcal{D}$ which fix the "standard" vertex in the p-adic building for all primes $p \neq 2, 7$, and which fix a type 2 vertex of the 7-adic building. We need to look for elements

$$\xi = \sum_{i=0}^{5} \sum_{j=-1}^{1} c_{ij} \zeta^i \sigma^j,$$

of $\mathcal{D}$ such that (a) $\iota(\xi)\xi = 1$, as well as

(b) the coefficients c_{ij} are in $\mathbb{Z}[1/2, 1/7]$,

(c) $\mathrm{Nrd}(\xi)$ is a power of $(3 + \sqrt{-7})/4$, and

(d) the vector $\mathbf{c}$ of coefficients c_{ij} satisfies a condition of the form

$M\mathbf{c}$ has entries in $\mathbb{Z}_7$ (for a certain 18×18 matrix M.)

Our main method of finding elements ξ with these properties is the **Cayley transform**: for $S \in \mathcal{D}$,

$$\iota(S) = -S \quad \Rightarrow \quad \xi = (1+S)(1-S)^{-1} \text{ satisfies } \iota(\xi)\xi = 1.$$

Writing

$$S = \sum_{i=0}^{5} \sum_{j=-1}^{1} s_{ij}\zeta^i\sigma^j,$$

the condition $\iota(S) = -S$ is linear in the s_{ij}'s, and reduces the number of free variables from 18 to 9 rational numbers.

C programs were written which ran through 10 integer variables looking for S's such that $\xi = (1+S)(1-S)^{-1}$ satisfies the integrality conditions (b) and (d). Two separate searches were done imposing the conditions $\mathrm{Nrd}(\xi) = 1$ and $\mathrm{Nrd}(\xi) = (3+\sqrt{-7})/4$.

We ordered the list of ξ's produced by these C programs according to how far they moved the origin for the hyperbolic distance on the ball $\mathbb{B}(\mathbb{C}^2)$.

Fact. The Hilbert-Schmidt norm $\|g\|_{HS} = \left(\sum_{i,j} |g_{ij}|^2\right)^{1/2}$ of $g \in U(2,1)$ satisfies

$$\|g\|_{HS}^2 = 3 + 4\sinh^2(d_H(g(0), 0)).$$

This allows us to simply work with the Hilbert-Schmidt norm of the images of the ξ's under an explicit isomorphism $\bar{G}(\mathbb{R}) \cong PU(2,1)$.

We list our output ξ's in order of increasing Hilbert-Schmidt norm, then form products of these, and add a product to the list if its Hilbert-Schmidt norm is small. Reorder the list and repeat.

At a certain point, two (small) elements b and z appear on the list, and we checked that all output is in the group generated by these two.

The element z is just the image of ζ_7. The element b is the image of

$$\frac{1}{7}\sum_{i=0}^{5}\sum_{j=-1}^{1} b_{ij}\zeta^i\sigma^j,$$

where the coefficients b_{ij} are the 18 numbers

$$-9, -3, 6, -4, 1, -2, 1, -2, -3, -1, -5, 3, -3, -8, 2, 2, -4, -6$$

in the order

$$b_{0,-1},\ b_{0,0},\ b_{0,1},\ b_{1,-1},\quad \cdots \quad,\ b_{5,-1},\ b_{5,0},\ b_{5,1}.$$

How do we show that b and z really generate $\bar{\Gamma}$?

To show that these two elements really do generate $\bar{\Gamma}$, we embed $\bar{\Gamma}$ into $PU(2,1)$:

As mentioned, [PY] tell us that the covolume of the embedded lattice is 1/21. So the normalized hyperbolic volume of the fundamental domain

$$\mathcal{F} = \{z \in \mathbb{B}^2_{\mathbb{C}} \ : \ d(0,z) \leq d(g(0),z) \text{ for all } g \in \bar{\Gamma}\}$$

is 1/21. Let Γ' be the subgroup of $\bar{\Gamma}$ generated by b and z. Tim has written a program to estimate the hyperbolic volume of

$$\mathcal{F}' = \{z \in \mathbb{B}^2_{\mathbb{C}} \ : \ d(0,z) \leq d(g(0),z) \text{ for all } g \in \Gamma'\}$$

The idea of the program is to consider, for a large set of $z = (z_1, z_2)$ such that $|z_1|^2 + |z_2|^2 = 1$,

$$t_z = \sup\{t \in [0,1) \ : \ d(0,tz) \leq d(g(0),tz) \text{ for all } g \in \Gamma' \text{ s.t. } d(g(0),0) \leq M\}.$$

The program also estimates (correct to several decimal places) the radius

$$r_0 = \max\{d(z, 0) : z \in \mathcal{F}\}$$

of the fundamental domain. We then use a standard theorem to obtain a presentation for $\bar{\Gamma}$.

Theorem. The elements g satisfying $d(g(0), 0) \leq 2r_0$ generate $\bar{\Gamma}$, and the relations of the form $g_1 g_2 g_3 = I$, where $d(g_i(0), 0) \leq 2r_0$ for $i = 1, 2, 3$, are sufficient to give a presentation of $\bar{\Gamma}$.

So the presentation initially involves hundreds of relations. This can be given to Magma, which has a command `Simplify`, which resulted in the presentation given above.

Probably human intervention can further simplify this presentation.

In terms of matrices, z and b are the images of

$$\begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^4 \end{pmatrix}$$

and $1/14$ times

$$\left[\begin{matrix} -8\zeta^5-16\zeta^4-10\zeta^3-4\zeta^2+2\zeta-6 & -12\zeta^5+4\zeta^4+6\zeta^3-6\zeta^2-4\zeta+12 & 10\zeta^5+6\zeta^4+2\zeta^3+12\zeta^2+8\zeta-1 \\ 2\zeta^5+4\zeta^4+6\zeta^3-6\zeta^2-4\zeta-16 & 10\zeta^5+6\zeta^4+2\zeta^3+12\zeta^2-6\zeta+4 & -6\zeta^5-12\zeta^4-18\zeta^3-10\zeta^2-2\zeta+ \\ 10\zeta^5+13\zeta^4+2\zeta^3+19\zeta^2+\zeta+18 & -6\zeta^5-12\zeta^4-4\zeta^3-10\zeta^2-2\zeta-22 & -2\zeta^5+10\zeta^4+8\zeta^3-8\zeta^2+4\zeta+2 \end{matrix}\right.$$

respectively.

Checking that a subgroup Π of $\bar{\Gamma}$ is torsion-free.

We have formed a list $g_1, \ldots, g_m$ of all $g \in \bar{\Gamma}$ such that $d(g(0), 0) \leq 2r_0$.

Let $\gamma_1, \ldots, \gamma_N$ be a set of coset representatives of Π in $\bar{\Gamma}$.

Fact: If $g \in PU(2,1)$ has finite order, then it fixes a point of $B(\mathbb{C}^2)$.

Suppose that there is a non-trivial $g \in \Pi$ of finite order.

If g fixes z, choose a $\gamma \in \bar{\Gamma}$ such that $z' = \gamma(z)$ lies in our fundamental domain. Then $g' = \gamma g \gamma^{-1}$ fixes z', and $d(z', 0) \leq r_0$. Hence $d(g'(0), 0) \leq 2r_0$. So $\gamma g \gamma^{-1}$ is one of the g_i, and it fixes the coset $\gamma\Pi$, which equals $\gamma_j \Pi$ for some j.

Thus $\gamma_j^{-1} g_i \gamma_j \in \Pi$ for some i, j.

So to check that Π is torsion-free, we check that it contains no such element.

Our calculations give us a list of conjugacy classes of elements of finite order in each case.

This helps us give some interesting singular surfaces X_G corresponding to groups G such that $\Pi < G \leq \bar{\Gamma}$, because then

- $\pi_1(X_G) \cong G/\langle$ torsion elements in $G\,\rangle$.

For the above example, $\bar{\Gamma}$ is generated by the elements b and z, which have finite order. So $\pi_1(X_{\bar{\Gamma}})$ is trivial.

Also, the subgroup

$$G = \langle \bar{\Gamma} \mid z,\ bzb^{-1} \rangle$$

has index 3 in $\bar{\Gamma}$. For three of the above Π's, $\Pi < G \leq \bar{\Gamma}$. Also, G is clearly generated by elements of finite order, and so $\pi_1(X_G)$ is trivial.

There are several examples of this sort in our list. Most arise from examples with $[G : \Pi] = 3$, and then the $\pi_1(X_G)$'s (coming from various classes) are:

$$\{1\},\ C_2,\ C_3,\ C_4,\ C_7,\ C_{13},\ C_2 \times C_3,\ C_2 \times C_7,$$
$$C_2 \times C_2,\ C_2 \times C_4,\ S_3,\ D_8,\ Q_8.$$

We also have examples of G's with $\pi_1(X_G)$ trivial and examples with $\pi_1(X_G) = C_2$ for $[G : \Pi] = 7$, 9 and 21.

By knowing a presentation for each Π, we can easily determine whether Π can be lifted to $SU(2,1)$.

In geometric terms, this is equivalent to asking whether the canonical line bundle of $\mathbb{B}^2_{\mathbb{C}}/\Pi$ is divisible by 3.

This was proved in [PY] to be true for most cases, but in the $\mathcal{C}_2$ and $\mathcal{C}_{18}$ cases this issue was left open.

In the $\mathcal{C}_{18}$ classes, it turns out that there are Π's which **do not** lift to $SU(2,1)$.

In the above example, the generators b and z are initially realized as elements of reduced norm $(3+\sqrt{-7})/4$ and 1 respectively. Replacing b by tb, where $t^3 = (3-\sqrt{-7})/4$, one checks the relations in the presentation hold in $SU(2,1)$, and gets a lift of all of $\bar{\Gamma}$ to $SU(2,1)$.

Most of the Π are congruence subgroups, but some are not. For simplicity, let us consider the cases $k = \mathbb{Q}$. Let

$$V = \{\xi \in \mathcal{D} : \iota(\xi) = \xi \text{ and } \mathrm{Trace}(\xi) = 0\}.$$

This is an 8-dimensional vector space over $\mathbb{Q}$. The group $\bar{G}(\mathbb{Q})$ acts on V by conjugation, giving a representation $\bar{G}(\mathbb{Q}) \to SL(8, \mathbb{Q})$. If we fix one of the $\bar{\Gamma}$'s contained in $\bar{G}(\mathbb{Q})$, then a basis of V can be chosen so $\bar{\Gamma}$ maps into $SL(8, \mathbb{Z})$.

We can then reduce modulo n for various n, and check whether a given $\Pi \leq \bar{\Gamma}$ contains the kernel of

$$\bar{\Gamma} \to SL(8, \mathbb{Z}) \to SL(8, \mathbb{Z}/n\mathbb{Z}).$$

In some of the $(a = 7, p = 2, \emptyset)$ cases, there is no such n. This is shown by reducing to the case n prime.

A matrix group example. Consider the class we call $(\mathcal{C}_{11}, \emptyset)$. It is one of the 5 classes for which we have shown that there is **no** fake projective plane, confirming Gopal and Sai-Kee's conjecture.

The fields k, ℓ involved here are $k = \mathbb{Q}(\sqrt{3})$ and $\ell = \mathbb{Q}(\zeta)$, where $\zeta = \zeta_{12}$ is $e^{2\pi i/12}$. Notice that $2\zeta - \zeta^3 = \sqrt{3}$, so that $k \subset \ell$. Let

$$\bar{\Gamma} = \{\xi \in M_{3\times 3}(\mathbb{Z}[\zeta]) \ : \ \xi^* F \xi = F\}$$

modulo scalars. The possible determinants are the powers of ζ. Here

$$F = \begin{pmatrix} -\sqrt{3}-1 & 1 & 0 \\ 1 & 1-\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

The form here could be replaced by a diagonal one, but then the integrality condition is more complicated.

In [PY], it is shown that $\mu(PU(2,1)/\bar{\Gamma}) = 1/864$. So to exclude fpp's in this case we proved:

Theorem. This $\bar{\Gamma}$ does **not** contain a torsion-free subgroup of index 864 having finite abelianization.

The index 864 is well beyond the capabilities of Magma's `LowIndexSubgroups` command. So we had to use special methods.

Theorem. This $\bar{\Gamma}$ does **not** contain a torsion-free subgroup of index 864 having finite abelianization.

In showing this, we discovered a subgroup Π of $\bar{\Gamma}$ which is the fundamental group of a surface with interesting properties:

Theorem. This $\bar{\Gamma}$ **does** contain a torsion-free subgroup Π of index 864 having abelianization $\mathbb{Z}^2$. It is unique up to conjugation.

So the Euler-Poincaré characteristic of $M = B(\mathbb{C}^2)/\Pi$ is 3, but its first Betti number is 2. So its Betti numbers are 1, 2, 5, 2, 1. For each integer $n \geq 1$, there is a normal subgroup Π_n of index n. So $M_n = B(\mathbb{C}^2)/\Pi_n$ satisfies $c_1(M_n)^2 = 3c_2(M_n) = 9n$.

Example (continued). We first find matrix generators and a presentation for $\bar{\Gamma}$. To find elements in $\bar{\Gamma}$, and finally a short list of generators, we used the Cayley transform search method.

In this case, $\bar{\Gamma}$ is generated by the following four matrices:

$$u = \begin{pmatrix} 1 & 0 & 0 \\ -\zeta^3 - \zeta^2 + \zeta + 1 & \zeta^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} \zeta^3 + 1 & \zeta^3 - \zeta^2 - \zeta + 1 & 0 \\ \zeta^2 + \zeta & -\zeta^3 - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$j = \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \zeta^3 + \zeta^2 & -\zeta^2 & \zeta^2 - 1 \\ \zeta^3 + 2\zeta^2 + \zeta & -\zeta & \zeta^3 + \zeta^2 \\ -\zeta^3 - \zeta^2 + \zeta + 1 & \zeta^3 & -\zeta^3 + \zeta + 1 \end{pmatrix}.$$

The group K generated by u, v and j is finite, of order 288. It has a presentation given by these generators and the relations

$$u^4 = v^8 = [u, j] = [v, j] = j^{-3}v^2 = uvuv^{-1}uv^{-1} = 1.$$

A presentation of $\bar{\Gamma}$ is obtained using these, the extra generator b, and the extra relations (the last four of which are perhaps not needed):

$$bjbju^2v^{-1} = 1,\ [b, vu^2] = 1,\ b^3 = 1,\ \text{and } (bvu^3)^3 = 1,$$
$$(buj)^4 = 1,\ (bu^2buj)^3 = 1,\ (bu^2vuj^{-1}buj)^3 = 1,\ (bu^3vj^{-1}buj)^3.$$

Suppose that $\bar{\Gamma}$ has a torsion-free subgroup Π of index 864. If T is a transversal, there is an action of $\bar{\Gamma}$ on T with the property that if $g \in \bar{\Gamma} \setminus \{1\}$ has finite order, then $\boldsymbol{g}$ **acts on** $\boldsymbol{T}$ **without fixed points**:

$$gt\Pi = t\Pi \quad \Rightarrow \quad t^{-1}gt \in \Pi \text{ has finite order.}$$

In particular, this is true for the elements $g \in K \setminus \{1\}$, and for the element b. We know the action of each $g \in K$ explicitly: we can write $T = Kt_0 \cup Kt_1 \cup Kt_2$, and $k(k't_i) = (kk')t_i$.

We have a homomorphism $\varphi : \bar{\Gamma} \to \mathsf{Perm}(T)$, and it is completely determined by the permutation $B = \lambda(b)$, which must have special properties. In particular, it turns out that bk^{-1} has finite order for 76 of the elements k of K. For no $t \in T$ can $B(t) = kt$ hold for such a k — otherwise $\varphi(bk^{-1})(kt) = \varphi(b)(t) = kt$, and the finite order element bk^{-1} fixes an element of T. All possible B's were found by a back-track search.

Using a specific permutation B of $\{1, \ldots, 864\}$ the following subgroup was found:

$$\Pi = \langle vubju^{-1},\ u^{-1}j^{-1}bj^2,\ u^2vbuj^{-2}\rangle,$$

and checked to have index 864, be torsion-free, and have abelianization $\mathbb{Z}^2$.