QUESTIONS FROM THE PROBLEM SESSION LMS-EPSRC DURHAM SYMPOSIUM 'GEOMETRY AND ARITHMETIC OF LATTICES' JULY 2011

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1. SL(3, $\mathbb{Z}$) (Misha Kapovich)

Question 1.1. *Let* $\Gamma = \mathrm{SL}(3,\mathbb{Z})$*. What are the torsion-free finitely generated subgroups of* Γ*?*

We know that there exist:

(1) plenty of free subgroups (Tits);
(2) plenty of $\mathbb{Z} \times \mathbb{Z}$ (diagonalisable, upper triangular);
(3) plenty of solvable subgroups;
(4) semidirect products of free groups and $\mathbb{Z}^2$;
(5) surface subgroups;
(6) finite-index subgroups.

Are there other subgroups, e.g., $\mathbb{Z}^2 * \mathbb{Z}$?

Question 1.2. *Does there exist an embedding* $\mathbb{Z}^2 * \mathbb{Z} \hookrightarrow \mathrm{SL}(3,\mathbb{Z})$*?*

Question 1.3 (Serre)**.** *Is every finitely generated subgroup* $\Gamma < \mathrm{SL}(3,\mathbb{Z})$ *finitely presented; i.e., is* $\mathrm{SL}(3,\mathbb{Z})$ coherent*? What about the property* FP_∞*?*

Question 1.4 (Alan Reid)**.** *Is there a finite volume hyperbolic* 3*-manifold with a faithful representation into* $\mathrm{SL}(3,\mathbb{Z})$*? (I.e., does there exist a lattice* $\Gamma < \mathrm{SO}(3,1)$ *with an embedding* $\Gamma \hookrightarrow \mathrm{SL}(3,\mathbb{Z})$*?*

We can ask similar questions for $\Gamma = \mathrm{SL}(2,\mathbb{Z}[\sqrt{2}])$ and other Hilbert modular groups: again there does not exist a subgroup $F_2 \times F_2 \hookrightarrow \Gamma$ and we know very few examples of finitely generated subgroups. As above, we can ask for example:

Question 1.5. *Is every finitely generated subgroup* $\Gamma < \mathrm{SL}(2,\mathbb{Z}[1/p])$ *finitely presented? [Et cetera.]*

Remark. Serre asked many of these questions at the 1977 Durham Symposium.

2. Volumes of lattices for real rank at least 2 (Misha Belolipetsky)

Question 2.1 (Nori, cf. [CV])**.** *Let* $\Gamma < G$ *be a Zariski dense discrete subgroup of* G*, where* G *is a simple real Lie group, and suppose that* $H < G$ *is semisimple. Let* $\Delta = \Gamma \cap H$ *and suppose* Δ *is an irreducible lattice. Is* Γ *a lattice in* G*?*

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This question was discussed in the talk by Venkataramana.

Question 2.2. *If the answer to Question 2.1 is yes, can we then estimate (from below)* $\mathrm{vol}(\Gamma\backslash G)$ *in terms of* $\mathrm{vol}(\Delta\backslash H)$*? (Here assume the Borel-Prasad normalisation of the Haar measures on* G *and* H*.)*

For example, we can ask this question for the groups considered by Chatterji and Venkataramana in [CV]. If there is a good estimate, then it would have interesting implications for counting non-uniform lattices in $\mathrm{SL}_n(\mathbb{R})$.

3. Lattices in $\mathrm{PU}(n,1)$, for $n \geqslant 2$ (Misha Kapovich)

Question 3.1. *Is there a (uniform) lattice in* $\mathrm{PU}(n,1)$*, for* $n \geqslant 2$*, that embeds in a Coxeter group?*

Nilpotent subgroups of Coxeter groups are virtually abelian so Γ must be uniform. By Bergeron-Haglund-Wise [BHW] there exist lots of examples of $\Gamma < \mathrm{SO}(n,1)$ ($n \geqslant 3$), arithmetic and nonarithmetic, such that Γ embeds in a right-angled Coxeter group. If G has property (T) then lattices $\Gamma < G$ cannot embed in any Coxeter group.

The receiving Coxeter group must have much larger cohomological dimension than Γ.

4. Gromov-Piatetski-Shapiro lattices (M. S. Raghunathan)

In the Gromov-Piatetski-Shapiro construction [GPS] we get a lattice $\Delta < \mathrm{SO}(n-1,1)$ corresponding to the common hypersurface in two manifolds which correspond to $\Gamma_1, \Gamma_2 < \mathrm{SO}(n,1)$. Additionally we have an arithmetic lattice Φ in $\mathrm{SU}(n-1,1)$ arising from treating the form

$$\sum_{i=1}^{n-1} x_i^2 - x_n^2$$

as a Hermitian form over an imaginary quadratic extension. ("So Φ is the complexification of Δ.") We can also look at the non-arithmetic group $\Gamma < \mathrm{SO}(n,1)$ coming from the Gromov-Piatetski-Shapiro construction as a subgroup of $\mathrm{SU}(n,1)$.

Question 4.1. *Is* $\langle \Gamma, \Phi \rangle$ *discrete in* $\mathrm{SU}(n,1)$*? Does it have finite covolume?*

5. Polysurface groups (Frank Johnson)

Let

$$\Gamma = \Gamma_0 > \Gamma_1 > \cdots > \Gamma_n = \{1\}$$

where Γ_i/Γ_{i+1} are surface groups of genus at least 2. We can realise this topologically so that Γ is a $\mathrm{PD}(\mathbb{Z})$ of dimension $2n$.

Question 5.1.

(1) *Can we realise* $\Gamma = \pi_1($*closed aspherical topological manifold*$)$*?*
(2) *If so, can we then smooth this manifold?*

Note that by passing to a finite index $\Lambda <_{\text{f.i.}} \Gamma$ we can realise this and smooth the manifold [J].

6. Preservation of lattice properties under weak commensurability (Andrei Rapinchuk)

Let G_1 and G_2 be two semisimple groups and let Γ_1 and Γ_2 be weakly commensurable Zariski dense subgroups of G_1 and G_2 respectively.

Question 6.1. *If Γ_1 is discrete, is Γ_2?*

The answer to this is "yes" in the p-adic case. It is also known in the real case when the $\mathbb{R}$-rank of G_1 is equal to the $\mathbb{R}$-rank of K, where K is the maximal compact subgroup of G_1. Otherwise this is unknown.

Question 6.2. *If both Γ_1 and Γ_2 are lattices, and Γ_1 is cocompact, is Γ_2 cocompact?*

The answer to this is "yes" in the arithmetic case.

7. Holomorphic fibrations (Misha Kapovich)

A singular holomorphic fibration is a holomorphic map $f\colon X_1 \to X_2$ that is surjective and has connected fibres.

Question 7.1. *Does there exist torsion free lattices $\Gamma < \mathrm{PU}(n,1)$ (for large enough n) such that $\Gamma\backslash\mathbb{CH}^n$ admits a non-trivial structure of a holomorphic fibration over a compact $\mathbb{C}$-curve of genus at least 2?*

(1) There exist examples in lower dimensions (Deligne-Mostow).
(2) If there exists $\Gamma_0 < \Gamma$ (finite index) with a homomorphism $\varphi\colon \Gamma_0 \to F_2$ then we have such a fibration.

Kapovich: "We don't have much of a picture other than Betti numbers for complex hyperbolic manifolds."

(3) There exist Γ with infinite virtual first Betti number; i.e.,

$$\sup_{\Gamma_i <_{\text{f.i.}} \Gamma} \dim H_1(\Gamma_i;\mathbb{C}) = \infty.$$

(4) It is still unknown whether there exists a finitely presented group G with infinite virtual first Betti number but for which there is no $G_0 <_{\text{f.i.}} G$ such that G_0 surjects onto F_2.
(5) Cf. virtual fibering for $\Gamma < \mathrm{SO}(3,1)$.

8. Bounded Generation (Andrei Rapinchuk)

A group has bounded generation if it can be written as a finite product of cyclic subgroups

Question 8.1.

(1) *Do all higher rank arithmetic lattices have the bounded generation property?*
(2) *Construct one anisotropic example: uniform lattices in $\mathrm{SL}_n(\mathbb{R})$?*

9. Affine crystallographic groups (Misha Belolipetsky)

Let $\Gamma < \mathrm{Aff}(\mathbb{R}^n)$ be an affine crystallographic group. Following Gromov we can define the conformal volume of $\mathcal{O} = \mathbb{R}^n/\Gamma$ by

$$\mathrm{convol}(\mathcal{O}) = \inf_{\phi:|\mathcal{O}|\to\mathbb{R}^N} \sup_{x\in\mathbb{R}^N, r>0} r^{-n}\mathrm{vol}_n(\phi(\mathcal{O})\cap B(x,r)),$$

where ϕ ranges through all possible embeddings of the underlying space $|\mathcal{O}|$ of the orbifold $\mathcal{O}$ into $\mathbb{R}^N$, for sufficiently large N. Another version of the conformal volume, which is closely related to the Li-Yau conformal invariant, can be found in [ABSW]. These definitions are not equivalent but at the same time are closely related to each other (how?).

Question 9.1. *What is the conformal volume of* $\mathbb{R}^n/\Gamma$*?*

If there exist Λ with $\Gamma \leqslant_{\text{f.i.}} \Lambda < \mathrm{Aff}(\mathbb{R}^n)$, where Λ is generated by reflections, then it would allow to obtain a good estimate for the conformal volume. This works well for $n = 2$ but would fail in high dimensions. As a variant of the previous question we can ask:

Question 9.2. *What is the conformal volume of* $\mathbb{R}^{24}/\Lambda$*, where* Λ *is the Leech lattice?*

References

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