Birational Arakelov Geometry
— Durham LMS Symposium —

Atsushi MORIWAKI

Kyoto University

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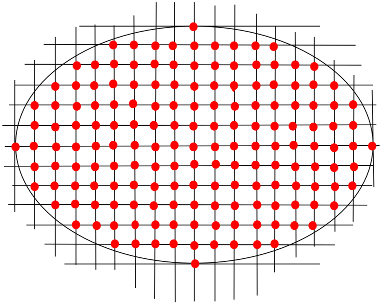
Problem:

For a real number $\lambda > 1$, find an asymptotic estimate of

$$\log \# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \leq \lambda^{2n}\right\}$$

with respect to n.

How many lattice points in the ellipse?



$$a^2 + 2b^2 \leq \lambda^{2n}$$

Considering a shrinking map $(x, y) \mapsto (\lambda^{-n}x, \lambda^{-n}y)$,

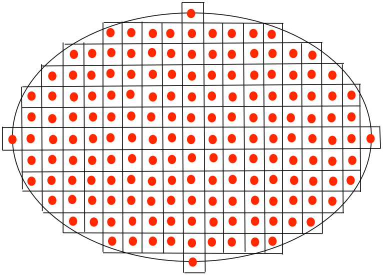
$$\# \left\{(a, b) \in \mathbb{Z}^2 \mid a^2 + 2b^2 \le \lambda^{2n}\right\} = \# \left\{(a', b') \in \left(\mathbb{Z}\lambda^{-n}\right)^2 \mid a'^2 + 2b'^2 \le 1\right\}.$$

We assign a square

$$\left[a' - \frac{\lambda^{-n}}{2}, a' + \frac{\lambda^{-n}}{2}\right] \times \left[b' - \frac{\lambda^{-n}}{2}, b' + \frac{\lambda^{-n}}{2}\right]$$

to each element of

$$\left\{(a', b') \in \left(\mathbb{Z}\lambda^{-n}\right)^2 \mid a'^2 + 2b'^2 \le 1\right\}.$$



$$x^2 + 2y^2 \leq 1$$

$$\sum(\text{the volume of each square}) \sim \text{the volume of the ellipse}$$

Thus

$$\#\left\{(a,b)\in\mathbb{Z}^2 \mid a^2+2b^2\leq\lambda^{2n}\right\}\times(\lambda^{-n})^2$$
$$\sim \text{the volume of } \left\{(x,y)\in\mathbb{R}^2 \mid x^2+2y^2\leq 1\right\} = \frac{\pi}{\sqrt{2}}.$$

Therefore,

$$\boxed{\log\#\left\{(a,b)\in\mathbb{Z}^2 \mid a^2+2b^2\leq\lambda^{2n}\right\}\sim(2\log\lambda)n.}$$

Let K be a number field (i.e. a finite extension of $\mathbb{Q}$) and let $K(\mathbb{C})$ be the set of all embeddings $K \hookrightarrow \mathbb{C}$. Note that $\#(K(\mathbb{C})) = [K : \mathbb{Q}]$ and $K(\mathbb{C})$ is the set of $\mathbb{C}$-valued points of $\operatorname{Spec}(K)$. Let O_K be the ring of integers in K, that is,

$$O_K = \{x \in K \mid x \text{ is integral over } \mathbb{Z}\}.$$

We set $X = \operatorname{Spec}(O_K)$. Let $\operatorname{Div}(X)$ be the group of divisors on X, that is,

$$\operatorname{Div}(X) := \bigoplus_{P \in X \setminus \{0\}} \mathbb{Z}[P].$$

For $D = \sum_P a_P[P]$, $\deg(D)$ is defined by

$$\deg(D) := \sum_P a_P \log \#(O_K/P).$$

$\widehat{\mathrm{Div}}(X)$ is defined by

$$\widehat{\mathrm{Div}}(X) = \mathrm{Div}(X) \times \{\xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_\sigma = \xi_{\bar{\sigma}} \ (\forall \sigma \in K(\mathbb{C}))\},$$

where $\bar{\sigma}$ is the composition of $\sigma : K \hookrightarrow \mathbb{C}$ and the complex conjugation $\mathbb{C} \xrightarrow{-} \mathbb{C}$. An element of $\widehat{\mathrm{Div}}(X)$ is called an arithmetic divisor on X. For simplicity, an element $\xi \in \mathbb{R}^{K(\mathbb{C})}$ is denoted by $\sum_\sigma \xi_\sigma[\sigma]$. For example, if we set

$$\widehat{(x)} := \left(\sum_P \mathrm{ord}_P(x)[P], \sum_\sigma -\log|\sigma(x)|^2[\sigma] \right)$$

for $x \in K^\times$, then $\widehat{(x)} \in \widehat{\mathrm{Div}}(X)$, which is called an arithmetic principal divisor.

The arithmetic degree $\widehat{\deg}(\overline{D})$ for $\overline{D} = (D, \xi)$ is defined by

$$\widehat{\deg}(\overline{D}) := \deg(D) + \frac{1}{2}\sum_{\sigma} \xi_\sigma.$$

Note that $\widehat{\deg}(\widehat{(x)}) = 0$ by the product formula. For

$$\overline{D} = \left(\sum_P n_P[P], \sum_\sigma \xi_\sigma[\sigma]\right),$$

$$\overline{D} \geq 0 \quad \overset{\text{def}}{\iff} \quad n_P \geq 0 \text{ and } \xi_\sigma \geq 0 \text{ for all } P \text{ and } \sigma$$

We set

$$\hat{H}^0(X, \overline{D}) := \{x \in K^\times \mid \overline{D} + \widehat{(x)} \geq 0\} \cup \{0\}.$$

Set $K = \mathbb{Q}(\sqrt{-2})$. Then $O_K = \mathbb{Z} + \mathbb{Z}\sqrt{-2}$ and $K(\mathbb{C}) = \{\sigma_1, \sigma_2\}$ given by $\sigma_1(\sqrt{-2}) = \sqrt{-2}$ and $\sigma_2(\sqrt{-2}) = -\sqrt{-2}$. We set $\overline{D} = (0, \log(\lambda^2)[\sigma_1] + \log(\lambda^2)[\sigma_2])$. Then $\widehat{\deg}(\overline{D}) = 2\log(\lambda)$. Note that, for $x = a + b\sqrt{-2} \in \mathbb{Q}(\sqrt{-2}) \setminus \{0\}$,

$$n\overline{D} + \widehat{(x)} \geq 0 \iff \begin{cases} n\log(\lambda^2) - \log(a^2 + 2b^2) \geq 0 \\ a, b \in \mathbb{Z} \end{cases}$$
$$\iff \begin{cases} a^2 + 2b^2 \leq \lambda^{2n} \\ a, b \in \mathbb{Z} \end{cases}$$

Therefore,

$$\begin{aligned}\hat{H}^0(X, n\overline{D}) &= \left\{x \in K^\times \mid n\overline{D} + \widehat{(x)} \geq 0\right\} \cup \{0\} \\ &= \{a + b\sqrt{-2} \in O_K \mid a^2 + 2b^2 \leq \lambda^{2n}\}.\end{aligned}$$

Thus the previous observation means that

$$\boxed{\log \#\hat{H}^0(X, n\overline{D}) \sim \widehat{\deg}(\overline{D})n.}$$

Theorem (Arithmetic Hilbert-Samuel formula for $\mathrm{Spec}(O_K)$)

If $\widehat{\deg}(\overline{D}) > 0$, *then* $\log \#\hat{H}^0(n\overline{D}) = n\widehat{\deg}(\overline{D}) + O(1)$. *In particular, if* $n \gg 1$, *then there is* $x \in K^{\times}$ *with* $n\overline{D} + \widehat{(x)} \geq 0$. *Moreover,* $\lim_{n\to\infty} \log \#\hat{H}^0(n\overline{D})/n = \widehat{\deg}(\overline{D})$.

Remark

Let r_2 *be the number of complex embeddings* K *into* $\mathbb{C}$ *and let* D_K *be the discriminant of* K *over* $\mathbb{Q}$. *Then, if*

$$\widehat{\deg}(\overline{D}) \geq \log((\pi/2)^{r_2}\sqrt{|D_K|}),$$

then $\hat{H}^0(\overline{D}) \neq \{0\}$.

$$
\begin{cases}
\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \\
\widehat{\mathrm{Div}}(X)_{\mathbb{R}} := \mathrm{Div}(X)_{\mathbb{R}} \times \{\xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_\sigma = \xi_{\bar{\sigma}} \ (\forall \sigma \in K(\mathbb{C}))\}, \\
K^{\times}_{\mathbb{R}} := (K^{\times}, \times) \otimes_{\mathbb{Z}} \mathbb{R}
\end{cases}
$$

For $\overline{D} = (\sum_P x_P[P], \xi) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$, $\widehat{\deg}(D)$ is defined by

$$
\widehat{\deg}(\overline{D}) := \sum_P x_P \log \#(O_K/P) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \xi_\sigma .
$$

For $x = x_1^{a_1} \cdots x_r^{a_r} \in K^{\times}_{\mathbb{R}}$ $(x_1, \ldots, x_r \in K^{\times},\ a_1, \ldots, a_r \in \mathbb{R})$,

$$
\widehat{(x)}_{\mathbb{R}} := \sum a_i \widehat{(x_i)}.
$$

For $\overline{D} = (\sum_P x_P[P], \xi) \in \widehat{\mathrm{Div}}(X)_{\mathbb{R}}$,

$$
\overline{D} \geq 0 \quad \overset{\text{def}}{\Longleftrightarrow} \quad x_P \geq 0 \text{ and } \xi_\sigma \geq 0 \text{ for all } P \text{ and } \sigma
$$

Theorem (Dirichlet's unit theorem)

If $\widehat{\deg}(\overline{D}) \geq 0$*, then there is* $x \in K_{\mathbb{R}}^{\times}$ *such that* $\overline{D} + \widehat{(x)}_{\mathbb{R}} \geq 0$*.*

Remark

The above theorem implies the classical Dirichlet's unit theorem, that is, for any $\xi \in \mathbb{R}^{K(\mathbb{C})}$ *with* $\sum_{\sigma} \xi_{\sigma} = 0$ *and* $\xi_{\sigma} = \xi_{\bar{\sigma}}$*, there are* $x_1, \ldots, x_r \in O_K^{\times}$ *and* $a_1, \ldots, a_r \in \mathbb{R}$ *such that* $\xi_{\sigma} = \sum_i a_i \log|\sigma(x_i)|$ *for all* σ*.*

Let M be an n-equidimensional smooth projective variety over $\mathbb{C}$. Let $\mathrm{Div}(M)$ be the group of Cartier divisors on M and let $\mathrm{Div}(M)_{\mathbb{R}} := \mathrm{Div}(M) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor.
Let us fix $D \in \mathrm{Div}(M)_{\mathbb{R}}$. We set $D = a_1 D_1 + \cdots + a_l D_l$, where $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are prime divisors on M.
Let $g : M \to \mathbb{R} \cup \{\pm\infty\}$ be a locally integrable function on M. We say g is a D-Green function of C^{∞}-type (resp. C^0-type) if, for each point $x \in M$, there are an open neighborhood U_x of x, local equations $f_1, \ldots, f_l$ of $D_1, \ldots, D_l$ respectively and a C^{∞} (resp. C^0) function u_x over U_x such that

$$g = u_x + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad (a.e.)$$

over U_x. The above equation is called a local expression of g.

Let g be a D-Green function of C^0-type on M. Let

$$g = u + \sum(-a_i) \log |f_i|^2 = u' + \sum(-a_i) \log |f_i'|^2 \quad (a.e.)$$

be two local expressions of g. Then, as $\sum(-a_i) \log |f_i/f_i'|^2$ is dd^c-closed, we have $dd^c(u) = dd^c(u')$ as currents, so that it can be defined globally. We denote it by $c_1(D, g)$. Note that $c_1(D, g)$ is a closed $(1, 1)$-current on M. If g is of C^∞-type, then $c_1(D, g)$ is represented by a C^∞-form.

Let X be a d-dimensional, generically smooth normal projective arithmetic variety, that is,

1. X is projective flat integral scheme over $\mathbb{Z}$.
2. If $X_{\mathbb{Q}} = X \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q})$ is the generic fiber of $X \to \mathrm{Spec}(\mathbb{Z})$, then $X_{\mathbb{Q}}$ is smooth over $\mathbb{Q}$.
3. The Krull dimension of X is d, that is, $\dim X_{\mathbb{Q}} = d - 1$.
4. X is normal.

Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X and $\mathrm{Div}(X)_{\mathbb{R}} = \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, whose element is called an $\mathbb{R}$-divisor on X. For $D \in \mathrm{Div}(X)_{\mathbb{R}}$, we set $D = \sum_i a_i D_i$, where $a_i \in \mathbb{R}$ and D_i's are reduced and irreducible subschemes of codimension one. We say D is effective if $a_i \geq 0$ for all i. Moreover, for $D, E \in \mathrm{Div}(X)_{\mathbb{R}}$,

$$D \leq E \text{ (or } E \geq D) \quad \Longleftrightarrow \quad E - D \text{ is effective}$$

Let D be an $\mathbb{R}$-divisor on X and let g be a locally integrable function on $X(\mathbb{C})$. We say a pair $\overline{D} = (D, g)$ is an arithmetic $\mathbb{R}$-divisor on X if $F_\infty^*(g) = g$ $(a.e.)$, where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map, i.e. for $x \in X(\mathbb{C})$, $F_\infty(x)$ is given by the composition $\operatorname{Spec}(\mathbb{C}) \xrightarrow{-} \operatorname{Spec}(\mathbb{C}) \xrightarrow{x} X$. Moreover, we say $\overline{D}$ is of C^∞-type (resp. C^0-type) if g is a D-Green function of C^∞-type (resp. C^0-type). For arithmetic divisors $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$, we define $\overline{D}_1 = \overline{D}_2$ and $\overline{D}_1 \leq \overline{D}_2$ to be

$$\overline{D}_1 = \overline{D}_2 \iff D_1 = D_2 \text{ and } g_1 = g_2 \ (a.e.),$$
$$\overline{D}_1 \leq \overline{D}_2 \iff D_1 \leq D_2 \text{ and } g_1 \leq g_2 \ (a.e.).$$

We say $\overline{D}$ is effective if $\overline{D} \geq (0, 0)$.

Let $\mathrm{Rat}(X)$ be the field of rational functions on X. For $\phi \in \mathrm{Rat}(X)^{\times}$, we set

$$(\phi) := \sum_{\Gamma} \mathrm{ord}_{\Gamma}(\phi)\Gamma \quad \text{and} \quad \widehat{(\phi)} := ((\phi), -\log|\phi|^2).$$

Note that $\widehat{(\phi)}$ is an arithmetic divisor of C^{∞}-type
Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let $\mathrm{Rat}(X)^{\times}_{\mathbb{K}} := \mathrm{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K}$ and let

$$\widehat{(\ )}_{\mathbb{K}} : \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \to \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$$

be the natural extension of the homomorphism

$$\mathrm{Rat}(X)^{\times} \to \widehat{\mathrm{Div}}_{C^0}(X)$$

given by $\phi \mapsto \widehat{(\phi)}$.

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X.

- $H^0(X, D) := \{\phi \in \mathrm{Rat}(X)^\times \mid D + (\phi) \geq 0\} \cup \{0\}$. Note that $H^0(X, D)$ is finitely generated $\mathbb{Z}$-module.
- $H^0_{\mathbb{K}}(X, D) := \{\phi \in \mathrm{Rat}(X)^\times_{\mathbb{K}} \mid D + (\phi)_{\mathbb{K}} \geq 0\} \cup \{0\}$.
- $\hat{H}^0(X, \overline{D}) := \{\phi \in \mathrm{Rat}(X)^\times \mid \overline{D} + \widehat{(\phi)} \geq (0,0)\} \cup \{0\}$. Note that $\hat{H}^0(X, \overline{D})$ is a finite set.
- $\hat{H}^0_{\mathbb{K}}(X, \overline{D}) := \{\phi \in \mathrm{Rat}(X)^\times_{\mathbb{K}} \mid \overline{D} + \widehat{(\phi)}_{\mathbb{K}} \geq (0,0)\} \cup \{0\}$.
- $\hat{h}^0(X, \overline{D}) := \log \#\hat{H}^0(X, \overline{D})$.
- $\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n \to \infty} \dfrac{\log \#\hat{H}^0(X, n\overline{D})}{n^d/d!}$.

Theorem

1. $\widehat{\mathrm{vol}}(\overline{D}) < \infty$.
2. *(H. Chen)* $\widehat{\mathrm{vol}}(\overline{D}) := \lim_{n\to\infty} \frac{\log \#\hat{H}^0(X, n\overline{D})}{n^d/d!}$.
3. $\widehat{\mathrm{vol}}(a\overline{D}) = a^d \widehat{\mathrm{vol}}(\overline{D})$ *for* $a \in \mathbb{R}_{\geq 0}$.
4. *(M.)* $\widehat{\mathrm{vol}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ *is continuous in the following sense: Let* $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_s$ *be arithmetic* $\mathbb{R}$*-divisors of* C^0*-type on* X*. For a compact subset* B *in* $\mathbb{R}^r$ *and a positive number* ϵ*, there are positive numbers* δ *and* δ' *such that*

$$\left| \widehat{\mathrm{vol}} \left(\sum a_i \overline{D}_i + \sum \delta_j \overline{A}_j + (0, \phi) \right) - \widehat{\mathrm{vol}} \left(\sum a_i \overline{D}_i \right) \right| \leq \epsilon$$

for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_s \in \mathbb{R}$ *and* $\phi \in C^0(X)$ *with* $(a_1, \ldots, a_r) \in B$, $|\delta_1| + \cdots + |\delta_s| \leq \delta$ *and* $\|\phi\|_{\sup} \leq \delta'$.

Let C be a reduced and irreducible 1-dimensional closed subscheme of X. We would like to define $\widehat{\deg}(\overline{D}|_C)$. It is characterized by the following properties:

1. $\widehat{\deg}(\overline{D}|_C)$ is linear with respect to $\overline{D}$.
2. If $\phi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, then $\widehat{\deg}(\widehat{(\phi)}_{\mathbb{R}}\big|_C) = 0$.
3. If $C \not\subseteq \mathrm{Supp}(D)$ and C is vertical, then $\widehat{\deg}(\overline{D}|_C) = \log(p)\deg(D|_C)$, where C is contained in the fiber over a prime p.
4. If $C \not\subseteq \mathrm{Supp}(D)$ and C is horizontal, then $\widehat{\deg}(\overline{D}|_C) = \widehat{\deg}(\overline{D}|_{\widetilde{C}})$, where $\widetilde{C}$ is the normalization of C. Note that $\widetilde{C} = \mathrm{Spec}(O_K)$ for some number field K.

- $\overline{D}$ is big $\iff$ $\widehat{\mathrm{vol}}(\overline{D}) > 0$.
- $\overline{D}$ is psedo-effective $\iff$ $\overline{D} + \overline{A}$ is big for any big arithmetic $\mathbb{R}$-divisor $\overline{A}$ of C^0-type.
- $\overline{D} = (D, g)$ is nef $\iff$

  1. $\widehat{\deg}(\overline{D}|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed subschemes C of X.
  2. $c_1(D, g)$ is a positive current.

- $\overline{D} = (D, g)$ is relatively nef $\iff$

  1. $\widehat{\deg}(\overline{D}|_C) \geq 0$ for all vertical reduced and irreducible 1-dimensional closed subschemes C of X.
  2. $c_1(D, g)$ is a positive current.

- $\overline{D} = (D, g)$ is integrable $\iff$ $\overline{D} = \overline{P} - \overline{Q}$ for some nef arithmetic $\mathbb{R}$-divisors $\overline{P}$ and $\overline{Q}$.

Theorem (Arithmetic Hilbert-Samuel formula)

(Gillet-Soulé-Abbes-Bouche-Zhang-M.) If $\overline{D}$ is nef, then

$$\hat{h}^0(X, n\overline{D}) = \frac{\widehat{\deg}(\overline{D}^d)}{d!} n^d + o(n^d).$$

In other words, $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^d)$.

Remark

The above theorem suggests that $\widehat{\deg}(\overline{D}^d)$ *can be defined by* $\widehat{\mathrm{vol}}(\overline{D})$. *Note that*

$$d! X_1 \cdots X_d = \sum_{I \subseteq \{1,\dots,d\}} (-1)^{d-\#(I)} \left(\sum_{i \in I} X_i \right)^d$$

in $\mathbb{Z}[X_1, \dots, X_d]$. *Thus, for nef arithmetic* $\mathbb{R}$*-divisors* $\overline{D}_1, \dots, \overline{D}_d$,

$$d! \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) = \sum_{I \subseteq \{1,\dots,d\}} (-1)^{d-\#(I)} \widehat{\mathrm{vol}} \left(\sum_{i \in I} \overline{D}_i \right).$$

In general, for integrable arithmetic $\mathbb{R}$*-divisors* $\overline{D}_1, \dots, \overline{D}_d$, *we can define* $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ *by linearity.*

Theorem (Generalized Hodge index theorem)

(M.) If $\overline{D}$ is relatively nef, then $\widehat{\mathrm{vol}}(\overline{D}) \geq \widehat{\deg}(\overline{D}^d)$.

Corollary (The existence of small sections)

(Faltings-Gillet-Soulé-Zhang-M.) If $\overline{D}$ is a relatively nef and $\widehat{\deg}(\overline{D}^d) > 0$, *then there are n and* $\phi \in \mathrm{Rat}(X)^{\times}$ *such that* $n\overline{D} + \widehat{(\phi)} \geq 0$.

Corollary (Arithmetic Bogomolov's inequality)

(Miyaoka-Soulé-M.) We assume $d = 2$ and X is regular. Let (E, h) be a C^∞-hermitian locally free sheaf on X. If E is semistable on the generic fiber, then

$$\widehat{\deg}\left(\widehat{c}_2(\overline{E}) - \frac{r-1}{2r}\widehat{c}_1(\overline{E})^2\right) \geq 0,$$

where $r = \operatorname{rk} E$.

Let $\pi : Y = \operatorname{Proj}\left(\bigoplus_{n\geq 0}\operatorname{Sym}^n(E)\right) \to X$ and D the tautological divisor on Y (i.e. $\mathcal{O}_Y(D) = \mathcal{O}(1)$). Roughly speaking, if we give a suitable Green function g to D, then $(D, g) - (1/r)\pi^*(\widehat{c}_1(\overline{E}))$ is relatively nef and its volume is zero, so that

$$\widehat{\deg}\left(((D, g) - (1/r)\pi^*(\widehat{c}_1(\overline{E})))^{r+1}\right) \leq 0$$

by the Generalized Hodge index theorem, which gives the above inequality.

Theorem (Arithmetic Fujita's approximation theorem)

(Chen-Yuan) We assume that $\overline{D}$ *is big. For a given* $\epsilon > 0$*, there are a birational morphism* $\nu_\epsilon : Y_\epsilon \to X$ *of generically smooth, normal projective arithmetic varieties and a nef and big arithmetic* $\mathbb{Q}$*-divisor* $\overline{P}$ *of* C^∞*-type such that* $\nu_\epsilon^*(\overline{D}) \geq \overline{P}$ *and* $\widehat{\mathrm{vol}}(\overline{P}) \geq \widehat{\mathrm{vol}}(\overline{D}) - \epsilon$.

Let S be a non-singular projective surface over an algebraically closed field. Let D be an effective divisor on S. By virtue of Bauer, the positive part of the Zariski decomposition of D is characterized by the greatest element of

$$\{M \mid M \text{ is a nef } \mathbb{R}\text{-divisor on } S \text{ and } M \leq D\}$$

Theorem (Zariski decomposition on arithmetic surfaces)

(M.) We assume that $d = 2$ *and* X *is regular. Let* $\overline{D}$ *be an arithmetic* $\mathbb{R}$*-divisor of* C^0*-type on* X *such that the set*

$$\Upsilon(\overline{D}) = \{\overline{M} \mid \overline{M} \text{ is a nef arithmetic } \mathbb{R}\text{-divisor on } X \text{ and } \overline{M} \leq \overline{D}\}$$

is not empty. Then there is a nef arithmetic $\mathbb{R}$*-divisor* $\overline{P}$ *such that* $\overline{P}$ *gives the greatest element of* $\Upsilon(\overline{D})$*, that is,* $\overline{P} \in \Upsilon(\overline{D})$ *and* $\overline{M} \leq \overline{P}$ *for all* $\overline{M} \in \Upsilon(\overline{D})$*. Moreover, if we set* $\overline{N} = \overline{D} - \overline{P}$*, then the following properties hold:*

1. $\hat{H}^0(X, n\overline{P}) = \hat{H}^0(X, n\overline{D})$ *for all* $n \geq 0$*.*
2. $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2)$*.*
3. $\widehat{\deg}(\overline{P} \cdot \overline{N}) = 0$*.*
4. *If* $\overline{B}$ *is an integrable arithmetic* $\mathbb{R}$*-divisor of* C^0*-type with* $(0,0) \lneq \overline{B} \leq \overline{N}$*, then* $\widehat{\deg}(\overline{B}^2) < 0$*.*

For the proof of the property (3), the following characterization of nef arithmetic $\mathbb{R}$-Cartier is used:

Theorem (Generalized Hodge index theorem on arithmetic surfaces)

(M.) We assume that $d = 2$ and $\overline{D}$ is integrable. If $\deg(D_{\mathbb{Q}}) \geq 0$, then $\widehat{\deg}(\overline{D}^2) \leq \widehat{\mathrm{vol}}(\overline{D})$. Moreover, we have the following:

1. *We assume that $\deg(D_{\mathbb{Q}}) = 0$. The equality holds if and only if there are $\lambda \in \mathbb{R}$ and $\phi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\overline{D} = \widehat{(\phi)}_{\mathbb{R}} + (0, \lambda)$.*
2. *We assume that $\deg(D_{\mathbb{Q}}) > 0$. The equality holds if and only if $\overline{D}$ is nef.*

Let X be a d-dimensional, generically smooth normal projective arithmetic variety and let $\overline{D}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on X. By the above theorem, a decomposition $\overline{D} = \overline{P} + \overline{N}$ is called a *Zariski decomposition of* $\overline{D}$ if

1. $\overline{P}$ is a nef arithmetic $\mathbb{R}$-divisor on X.
2. $\overline{N}$ is an effective arithmetic $\mathbb{R}$-divisor of C^0-type on X.
3. $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P})$.

Let $\mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, T_1, \dots, T_n])$, $D = \{T_0 = 0\}$ and $z_i = T_i/T_0$ for $i = 1, \dots, n$. Let us fix a sequence $\boldsymbol{a} = (a_0, a_1, \dots, a_n)$ of positive numbers. We define a D-Green function $g_{\boldsymbol{a}}$ of C^∞-type on $\mathbb{P}^n(\mathbb{C})$ and an arithmetic divisor $\overline{D}_{\boldsymbol{a}}$ of C^∞-type on $\mathbb{P}^n_{\mathbb{Z}}$ to be

$$g_{\boldsymbol{a}} := \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2) \quad \text{and} \quad \overline{D}_{\boldsymbol{a}} := (D, g_{\boldsymbol{a}}).$$

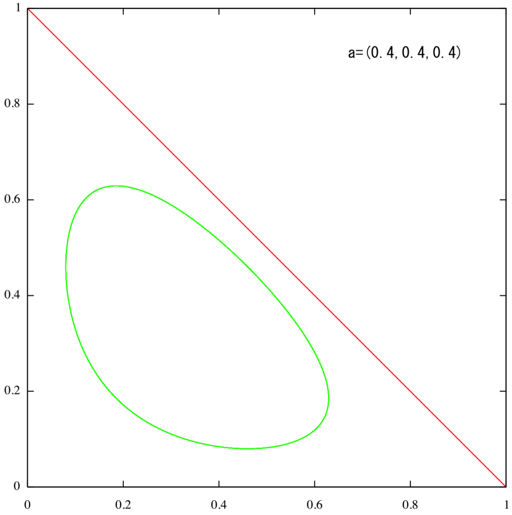
Note that $c_1(\overline{D}_{\boldsymbol{a}})$ is positive. Let $\vartheta_{\boldsymbol{a}} : \mathbb{R}^{n+1}_{\geq 0} \to \mathbb{R}$ be a function given by

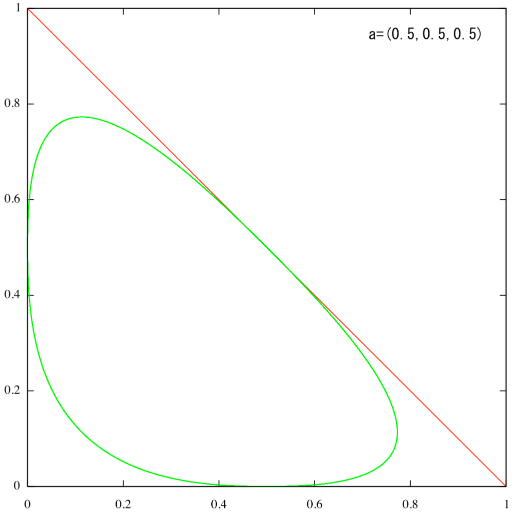
$$\vartheta_{\boldsymbol{a}}(x_0, x_1, \dots, x_n) := \frac{1}{2}\left(-\sum\nolimits_{i=0}^n x_i \log x_i + \sum\nolimits_{i=0}^n x_i \log a_i\right),$$

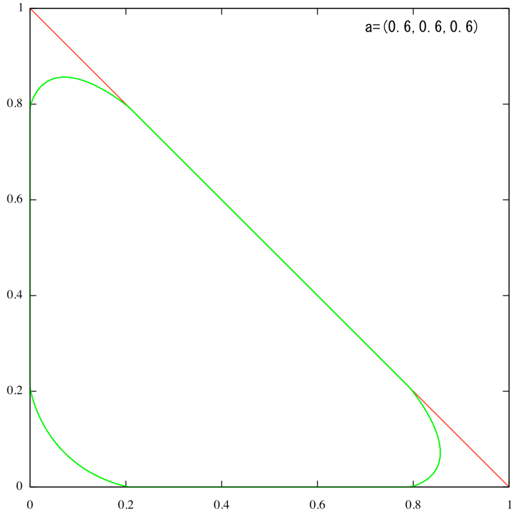
and let

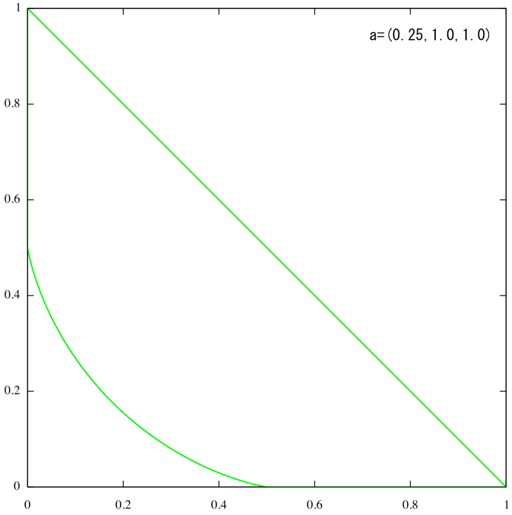
$$\Theta_{\boldsymbol{a}} := \{(x_1, \dots, x_n) \in \Delta_n \mid \vartheta_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \dots, x_n) \geq 0\},$$

where $\Delta_n := \left\{(x_1, \dots, x_n) \in \mathbb{R}^n_{\geq 0} \mid x_1 + \cdots + x_n \leq 1\right\}$.


a= (0. 4, 0. 4, 0. 4)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


a= (0. 5, 0. 5, 0. 5)
0
0.2
0.4
0.6
0.8
1


a= (0. 6, 0. 6, 0. 6)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


a=(0. 25, 1. 0, 1. 0)
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1

The following properties (1) – (6) hold for $\overline{D}_{\boldsymbol{a}}$:
(1) $\overline{D}_{\boldsymbol{a}}$ is ample $\iff$ $a_0 > 1, a_1 > 1, \ldots, a_n > 1$.
(2) $\overline{D}_{\boldsymbol{a}}$ is nef $\iff$ $a_0 \geq 1, a_1 \geq 1, \ldots, a_n \geq 1$.
(3) $\overline{D}_{\boldsymbol{a}}$ is big $\iff$ $a_0 + a_1 + \cdots + a_n > 1$.
(4) $\overline{D}_{\boldsymbol{a}}$ is pseudo-effective $\iff$ $a_0 + a_1 + \cdots + a_n \geq 1$.

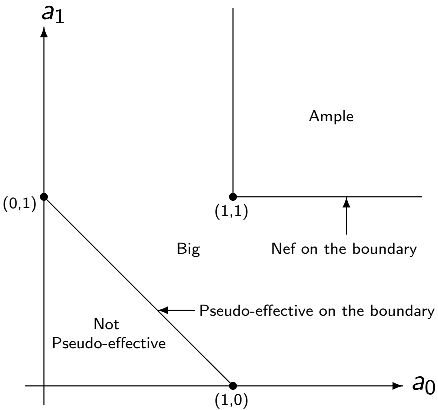


Figure: Geography of $\overline{D}_{\boldsymbol{a}}$ on $\mathbb{P}^1_{\mathbb{Z}}$

(5) (Integral formula) The following formulae hold:

$$\widehat{\mathrm{vol}}(\overline{D}_{\boldsymbol{a}}) = (n+1)! \int_{\Theta_{\boldsymbol{a}}} \vartheta_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) dx_1 \cdots dx_n$$

and

$$\widehat{\deg}(\overline{D}_{\boldsymbol{a}}^{n+1}) = (n+1)! \int_{\Delta_n} \vartheta_{\boldsymbol{a}}(1 - x_1 - \cdots - x_n, x_1, \ldots, x_n) dx_1 \cdots dx_n.$$

Boucksom and H. Chen generalized the above formulae to a general situation by using Okounkov bodies.

(6) (Zariski decomposition for $n = 1$) We assume $n = 1$. The Zariski decomposition of $\overline{D}_{\boldsymbol{a}}$ exists if and only if $a_0 + a_1 \geq 1$. Moreover, the positive part of $\overline{D}_{\boldsymbol{a}}$ is given by $(\theta_{\boldsymbol{a}} H_0 - \theta'_{\boldsymbol{a}} H_1, p_{\boldsymbol{a}})$, where $H_0 = D = \{T_0 = 0\}$, $H_1 = \{T_1 = 0\}$, $\theta'_{\boldsymbol{a}} = \inf \Theta_{\boldsymbol{a}}$, $\theta_{\boldsymbol{a}} = \sup \Theta_{\boldsymbol{a}}$ and

$$p_{\boldsymbol{a}}(z_1) = \begin{cases} \theta'_{\boldsymbol{a}} \log |z_1|^2 & \text{if } |z_1| < \sqrt{\frac{a_0 \theta'_{\boldsymbol{a}}}{a_1(1-\theta'_{\boldsymbol{a}})}}, \\ \log(a_0 + a_1 |z_1|^2) & \text{if } \sqrt{\frac{a_0 \theta'_{\boldsymbol{a}}}{a_1(1-\theta'_{\boldsymbol{a}})}} \leq |z_1| \leq \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}, \\ \theta_{\boldsymbol{a}} \log |z_1|^2 & \text{if } |z_1| > \sqrt{\frac{a_0 \theta_{\boldsymbol{a}}}{a_1(1-\theta_{\boldsymbol{a}})}}. \end{cases}$$

Let $\overline{D}_g = (H_0, g)$ be a big arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on $\mathbb{P}^n_{\mathbb{Z}}$.We assume that

$$g(\exp(2\pi\sqrt{-1}\theta_1)z_1, \dots, \exp(2\pi\sqrt{-1}\theta_n)z_n) = g(z_1, \dots, z_n)$$

for all $\theta_1, \dots, \theta_n \in [0, 1]$. We set

$$\xi_g(y_1, \dots, y_n) := \frac{1}{2} g(\exp(y_1), \dots, \exp(y_n))$$

for $(y_1, \dots, y_n) \in \mathbb{R}^n$.Let ϑ_g be the Legendre transform of ξ_g, that is,

$$\begin{aligned}&\vartheta_g(x_1, \dots, x_n)\\&:= \sup\{x_1 y_1 + \cdots + x_n y_n - \xi_g(y_1, \dots, y_n) \mid (y_1, \dots, y_n) \in \mathbb{R}^n\}\end{aligned}$$

for $(x_1, \dots, x_n) \in \Delta_n$.

Note that if $g = \log(a_0 + a_1|z_1|^2 + \cdots + a_n|z_n|^2)$, then

$$\vartheta_g = \varphi_{\boldsymbol{a}} = \frac{1}{2}\left(-\sum_{i=0}^{n} x_i \log x_i + \sum_{i=0}^{n} x_i \log a_i\right),$$

where $x_0 = 1 - x_1 - \cdots - x_n$.

Theorem (Burgos Gil, M., Philippon and Sombra)

There is a Zariski decomposition of $f^*(\overline{D}_g)$ *for some birational morphism* $f : Y \to \mathbb{P}^n_{\mathbb{Z}}$ *of generically smooth and projective normal arithmetic varieties if and only if*

$$\Theta_g := \{(x_1, \ldots, x_n) \in \Delta_n \mid \vartheta_g(x_1, \ldots, x_n) \geq 0\}$$

is a quasi-rational convex polyhedron, that is, there are $\gamma_1, \ldots, \gamma_l \in \mathbb{Q}^n$ *and* $b_1, \ldots, b_l \in \mathbb{R}$ *such that*

$$\Theta_g = \{x \in \mathbb{R}^n \mid \langle x, \gamma_i \rangle \geq b_i \ \forall i = 1, \ldots, l\},$$

where $\langle \ , \ \rangle$ *is the standard inner product of* $\mathbb{R}^n$.

The above theorem holds for toric varieties.

For example, if $g = \log\max\{a_0, a_1|z_1|^2, a_2|z_2|^2\}$,then $\overline{D}_g$ is big if and only if $\max\{a_0, a_1, a_2\} > 1$.Moreover,

$$\Theta_g = \left\{ (x_1, x_2) \in \Delta_2 \;\middle|\; \log\left(\frac{a_1}{a_0}\right) x_1 + \log\left(\frac{a_2}{a_0}\right) x_2 + \log(a_0) \geq 0 \right\}.$$

Thus there is a Zariski decomposition of $f^*(\overline{D}_g)$ for some birational morphism $f : Y \to \mathbb{P}^2_{\mathbb{Z}}$ of generically smooth and projective normal arithmetic varieties if and only if there is $\lambda \in \mathbb{R}_{>0}$ such that

$$\lambda\left(\log\left(\frac{a_1}{a_0}\right), \log\left(\frac{a_2}{a_0}\right)\right) \in \mathbb{Q}^2.$$

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on X.

Fundamental question

Are the following conditions (1) and (2) equivalent ?

1. $\overline{D}$ *is pseudo-effective.*
2. $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}}$ *is effective for some* $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$.

Obviously (2) implies (1). Moreover, if $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ for some $a \in \mathbb{R}_{>0}$, then (2) holds. Indeed, as we can choose $\phi \in \mathrm{Rat}(X)^{\times}$ with $a\overline{D} + \widehat{(\phi)} \geq 0$, we have $\phi^{1/a} \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ and $\overline{D} + \widehat{(\phi^{1/a})}_{\mathbb{R}} \geq 0$.

As we remarked, it is nothing more than Dirichlet's unit theorem in the case where $d = 1$. Moreover, in the geometric case, (1) does not necessarily imply (2).

We have partial answers to the above question.

Theorem

If $\overline{D}$ is pseudo-effective and D is numerically trivial on $X_{\mathbb{Q}}$, then there exists $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}}$ is effective.

This theorem is a consequence of arithmetic Hodge index theorem and a kind of compactness theorem.

Thank you for your attention.