Polylogs and modular forms in QFT

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Summary: There is growing evidence, from both massless and massive Feynman diagrams, that modular forms first arise in quantum field theory when polylogarithms no longer suffice. In this talk, I aim to
(1) introduce a variety of modular forms,
(2) link enumerations of modular forms and multiple zeta values,
(3) identify modular forms that obstruct evaluations to polylogs,
(4) give a modular form that controls massive and massless diagrams,
(5) use L-functions of modular forms to evaluate Feynman diagrams.
En route, I shall report significant progress, made in recent months by Erik Panzer, Francis Brown and Oliver Schnetz, in many massless cases, and by Spencer Bloch and Pierre Vanhove, in a two-scale massive case.

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1 Dramatis personæ

For $|q| < 1$, let

$$\eta(q) \equiv q^{1/24} \prod_{n>0}(1-q^n) = \sum_{n\in\mathbf{Z}} q^{(6n+1)^2/24}$$

then for $\Im z > 0$,

$$\eta(\exp(2\pi \mathrm{i} z)) = (\mathrm{i}/z)^{1/2}\eta(\exp(-2\pi\mathrm{i}/z)).$$

If $f(z) = (\sqrt{-N}/z)^w f(-N/z)$, we say that f is a modular form of weight w and level N.

$$\Delta(q) \equiv \eta(q)^{24} = \sum_{n>0} A(n)q^n = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - 16744q^7 + \ldots$$

is a modular form of weight 12 and level 1. Moreover its Fourier coefficients $A(n)$ are multiplicative, with $A(mn) = A(m)A(n)$ when $\gcd(m,n) = 1$. Finally for prime p there is a simple rule for obtaining $A(p^n)$ from $A(p)$:

$$L(s) \equiv \sum_{n>0} \frac{A(n)}{n^s} = \prod_p \frac{1}{1 - A(p)p^{-s} + p^{11-2s}}.$$

Note that $-1472 = A(4) = 24^2 - 2^{11}$ and $-6048 = A(6) = -24 \times 252$. This product form leads to the analytic continuation

$$\Lambda(s) \equiv \frac{\Gamma(s)}{(2\pi)^s} L(s) = \sum_{n>0} A(n) \int_1^\infty \mathrm{d}x \left(x^{s-1} + x^{11-s}\right) \exp(-2\pi x) = \Lambda(12-s)$$

with an easy integral for integer $s \in [1, 11]$. Only two of these 11 L-values are independent, since $1620\Lambda(3) = 691\Lambda(1)$, $14\Lambda(5) = 9\Lambda(3)$, $48\Lambda(4) = 25\Lambda(2)$ and $5\Lambda(6) = 4\Lambda(4)$.

1.1 Multiplicative modular forms with level 1

To count these, consider the Eisenstein series defined, for $n > 0$, by

$$E_{2n}(q) \equiv 1 - \frac{4n}{B_{2n}} \sum_{k>0} \frac{k^{2n-1} q^k}{1 - q^k}$$

where the Bernoulli numbers yield $-4n/B_{2n} = -24, 240, -504, 480, -264, 65520/691$, for $n = 1 \ldots 6$. Then E_2 is not modular, since $(q\mathrm{d}/\mathrm{d}q)\Delta = E_2\Delta$ is not. For $n > 1$, E_{2n} is modular, with weight $2n$, but is not multiplicative, since it does not vanish at $q = 0$.

$$\Delta = \frac{E_4^3 - E_6^2}{1728}$$

is both modular and multiplicative. Note that $1728 = 3 \times 240 + 2 \times 504$. For $n > 3$, E_{2n} is a rational polynomial of E_4 and E_6. For example: $E_8 = E_4^2$, $E_{10} = E_4 E_6$,

$$E_{12} = \frac{441 E_4^3 + 250 E_6^2}{691}, \quad E_{14} = E_4^2 E_6, \quad E_{16} = \frac{1617 E_4^4 + 2000 E_4 E_6^2}{3617}.$$

Let M_w be the number of multiplicative modular forms with weight w and level 1. Then

$$\sum_{w>0} M_w x^w = \frac{x^{12}}{(1 - x^4)(1 - x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24} \ldots$$

with Δ at $w = 12$, none at $w = 14$, $E_4\Delta$ at $w = 16$, $E_6\Delta$ at $w = 18$, $E_4^2\Delta$ at $w = 20$, $E_4 E_6 \Delta$ at $w = 22$ and two independent modular forms at $w = 24$, namely $E_4^3\Delta$ and $E_6^2\Delta$.

1.2 Relations between eta values

For brevity, let $\eta_n(q) \equiv \eta(q^n)$. Then η_1 and η_2 are algebraically independent. Yet

$$\eta_2^{24} = \eta_1^8 \eta_4^8 (\eta_1^8 + 4\eta_4^8)$$

relates $\{\eta_1, \eta_2, \eta_4\}$ and is the basis for the process

$$(a_{n+1},\, b_{n+1}) = \left(\frac{a_n + b_n}{2},\, \sqrt{a_n b_n}\right)$$

of the arithmetic–geometric mean (AGM) devised by Gauss for rapid computation of

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{\mathrm{d}\theta}{(a_0 \sin^2\theta + b_0 \cos^2\theta)^{1/2}} = \frac{1}{\mathrm{agm}(a_0,\, b_0)} = \frac{1}{a_\infty} = \frac{1}{b_\infty}.$$

There is a more ornate relation between $\{\eta_1, \eta_2, \eta_3\}$. Let

$$F_2(x) \equiv \frac{(x+8)(x-1)^2}{x}, \qquad F_3(y) \equiv \frac{(y^2+6y-3)^2}{y}.$$

Then, by a method to be explained later, one may obtain the algebraic relation,

$$2^6 F_2\left(2^9(\eta_2/\eta_1)^{24}\right) = 3^3 F_3\left(3^5(\eta_3/\eta_1)^{12}\right).$$

Replacing q by q^2, one may relate $\{\eta_2, \eta_4, \eta_6\}$ and eliminate η_4 in favour of $\{\eta_1, \eta_2\}$. Thus there are two algebraic relations between $\{\eta_1, \eta_2, \eta_3, \eta_6\}$. We shall see that these are perfectly tuned to allow evaluation of a massive Feynman diagram.

1.3 Multiplicative modular forms and eta products

For levels $N < 16$, there are precisely 15 multiplicative modular forms that are products of eta values. Here they are listed with notes on quantum field theory (QFT):

form	weight	level	QFT
$\eta_1^2\eta_{11}^2$	2	11	
$\eta_1\eta_2\eta_7\eta_{14}$	2	14	
$\eta_1\eta_3\eta_5\eta_{15}$	2	15	
$\eta_1^3\eta_7^3$	3	7	BS
$\eta_1^2\eta_2\eta_4\eta_8^2$	3	8	BS
$\eta_2^3\eta_6^3$	3	12	BS + BBBG + BV : Sections 3 and 4
$\eta_1^4\eta_5^4$	4	5	BS
$\eta_1^2\eta_2^2\eta_3^2\eta_6^2$	4	6	BS + BB : Sections 3 and 5
$\eta_2^4\eta_4^4$	4	8	
η_3^8	4	9	
$\eta_1^4\eta_2^2\eta_4^4$	5	4	BS
$\eta_1^6\eta_3^6$	6	3	BS
η_2^{12}	6	4	BS
$\eta_1^8\eta_2^8$	8	2	BS
η_1^{24}	12	1	BK : Section 2

with 10 of these 15 already exposed as participants in QFT, thanks to work by Brown and Schnetz (BS), Bailey, Borwein, Broadhurst and Glasser (BBBG), Bloch and Vanhove (BV), Broadhurst and Brown (BB), Broadhurst and Kreimer (BK). The absence of weight-2 examples is remarkable: does QFT avoid Birch and Swinnerton–Dyer?

1.4 Multiplicative modular forms and lattice sums

Moreover, QFT has links to a pair of multiplicative modular forms that involve lattice sums. With ingenuity one may reduce these to combinations of eta products or quotients.

From BBBG and BB, we find a multiplicative modular form with $w = 3$ and $N = 15$:

$$f_{3,15} \equiv \eta_1\eta_3\eta_5\eta_{15} \sum_{j,k\in\mathbf{Z}} q^{j^2+jk+4k^2} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with a remarkable evaluation as a sum of cubes of eta products.

At $w = 6$ and $N = 6$, QFT led me to a multiplicative modular form

$$f_{6,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2 \sum_{j,k,l,m\in\mathbf{Z}} q^{j^2+jk+k^2} q^{2(l^2+lm+m^2)}$$

with a lattice sum that factorizes. This too may be written as a sum of cubes:

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3 .$$

These two modular forms will be used in Section 5 to evaluate Feynman integrals.

2 Modular forms and multiple zeta values

In 1996, Dirk Kreimer and I (BK) arrived at a conjectural enumeration of irreducible multiple zeta values (MZVs), graded by weight and depth. Let $D_{w,d}$ be the number of MZVs with weight w and depth d that are not reduced by the double–shuffle algebra to MZVs of lesser weight and depth and their products. From extensive data with $d < 4$, and sparser data at higher depths, we conjectured that

$$\prod_{w>2}\prod_{d>0}(1-x^w y^d)^{D_{w,d}} = 1 - y\frac{x^3}{1-x^2} + y^2(1-y^2)\frac{x^{12}}{(1-x^4)(1-x^6)}.$$

It is now proven, at the motivic level, that any difference between the left and right hand sides must be of order y^4. Moreover it must vanish at $y = 1$, where Brown and Zagier (BZ) have proven an enumeration that is blind to depth. Blümlein, Vermaseren and I (BBV) have checked the conjecture for depths $d < 9$ and weights $w < 27$, by laborious methods. Francis Brown indicates that further checking may be done more efficiently.

If the BK conjecture be true, it sets a fine puzzle. Why should a count of modular forms

$$\sum_{w>0} M_w x^w = \frac{x^{12}}{(1-x^4)(1-x^6)} = x^{12} + x^{16} + x^{18} + x^{20} + x^{22} + 2x^{24}\ldots$$

furnish the bizarre final term of our empirical Ansatz? Is this coincidence significant?

3 Polylogs and modular forms in ϕ^4 theory

The standard model of particle physics involves all three of the interactions that are renormalizable, yet not trivially super-renormalizable, in $D = 4$ spacetime dimensions:

1. ϕ^4 self-coupling of the Higgs boson,
2. Yukawa couplings of the Higgs boson to fermions,
3. gauge couplings of vector bosons to the Higgs boson and to fermions.

In 1985, I studied a 6-loop diagram (see whiteboard) that contributes to the 4-point amplitude for Higgs scattering. Its counterterm contributes to the running of the Higgs self-coupling. It is the first 6-loop entry in the recent census by Schnetz (S). I conjectured that the relevant period (B) is:

$$P_{6,1} = 168\zeta(9)$$

and Natalia Ussyukina (U) proved this in 1991. In 2012, BS proved a BK conjecture for all such zigzag diagrams.

3.1 Counterterms reducible to polylogs

In 1995, Dirk Kreimer and I (BK) identified all periods for ϕ^4 primitive divergences up to 6 loops. At 7 loops we lacked three evaluations. Since then I have determined two of these, as follows.

$$
\begin{aligned}
P_{7,8} &= \frac{22383}{20}\zeta(11) - \frac{4572}{5}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - 700\zeta(3)^2\zeta(5) \\
&\quad + 1792\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right] \\
P_{7,9} &= \frac{92943}{160}\zeta(11) - \frac{3381}{20}[\zeta(3)\zeta(5,3) - \zeta(3,5,3)] - \frac{1155}{4}\zeta(3)^2\zeta(5) \\
&\quad + 896\zeta(3)\left[\frac{27}{80}\zeta(5,3) + \frac{45}{64}\zeta(5)\zeta(3) - \frac{261}{320}\zeta(8)\right]
\end{aligned}
$$

with indices of MZVs written in the order adopted by Zagier, by Borwein, Bradley, Broadhurst and Lisonek (BBBL), and in the extensive MZV datamine (BBV):

$$\zeta(5,3) \equiv \sum_{m=2}^{\infty} \frac{1}{m^5} \sum_{n=1}^{m-1} \frac{1}{n^3}.$$

In these two case, the methods of BS allowed the possibility that the periods might involve alternating sums. In fact they do not. One sheep remains lost: the period $P_{7,11}$ in the census has not yet been reduced to MZVs. BS suggest that it might eventually be reduced to polylogs of weight 11 at sixth roots of unity. Such polylogs result from massive diagrams at lesser weights (B).

3.2 Panzer's reductions to MZVs

To calculate counterterms at L loops, it is usually sufficient to obtain the ε expansions of two-point diagrams, at $L-1$ loops in $D = 4 - 2\varepsilon$ dimensions, up to weight $2L-3$.

In May 2013, Erik Panzer (P) showed that the dressing (see whiteboard) of many two-point diagrams by propagator sub-divergences does not take one beyond the realm of MZVs. As a concrete example, consider the 3-loop non-planar two-point diagram, whose ε-expansion was previously known to weight 7. Now it is known up to weight 9:

$$\begin{aligned} N_3(\epsilon) &= 20\zeta_5 + \left(\frac{80}{7}\zeta_2^3 + 68\zeta_3^2\right)\varepsilon + \left(\frac{408}{5}\zeta_3\zeta_2^2 + 450\zeta_7\right)\varepsilon^2 + \left(\frac{102228}{125}\zeta_2^4 - 2448\zeta_3\zeta_5\right. \\ &\quad \left. - \frac{9072}{5}\zeta_{5,3}\right)\varepsilon^3 + \left(\frac{88036}{9}\zeta_9 - \frac{4640}{3}\zeta_3^3 - \frac{10336}{7}\zeta_2^3\zeta_3 + \frac{19872}{5}\zeta_2^2\zeta_5\right)\varepsilon^4 + \ldots \end{aligned}$$

with $\zeta_{5,3} \equiv \sum_{m>n>0} 1/(m^5 n^3)$ appearing at weight 8. Even more impressively, he has shown that at 3 loops no dressing of internal lines by subdivergences can modify the polylogarithmic character of the ε-expansion. Specifically, he proves that the only non-MZV terms that might occur would be alternating Euler sums. As in BS cases at weight 11, no such alternating sum has yet emerged from a massless two-point diagram.

3.3 Brown–Schnetz modular obstructions

In April 2013, Francis Brown and Oliver Schnetz (BS) announced results of a fascinating study that classifies obstructions to polylogarithmic evaluations of ϕ^4 counterterms at 8, 9 and 10 loops. In 16 cases they were able to exhibit a modular form, inferred from study of the Symanzik polynomial, modulo a selection of primes. In 9 cases, listed in Section 1, the modular form was both multiplicative and reducible to an eta product. Here I select for particular attention ϕ^4 diagrams (see whiteboard) that led BS to these modular forms

$$f_{3,12} \equiv (\eta_2\eta_6)^3 \quad \text{and} \quad f_{4,6} \equiv (\eta_1\eta_2\eta_3\eta_6)^2.$$

4 Sunrise in two spacetime dimensions

Here I consider the two-loop massive sunrise diagram in $D = 2$ spacetime dimensions:

$$I(p^2, m_1, m_2, m_3) \equiv \frac{1}{\pi^2} \left(\prod_{k=1}^{3} \int \frac{\mathrm{d}^2 q_k}{q_k^2 - m_k^2 + \mathrm{i}\epsilon} \right) \delta^{(2)}(p - q_1 - q_2 - q_3)$$

with a Minkowski metric: $p^2 \equiv p_0^2 - p_1^2$ where p_0 is the energy and p_1 is the momentum.

4.1 A Bessel moment in configuration space

For $0 < w < m_1 + m_2 + m_3$, configuration space yields BBBG's Bessel moment:

$$I(w^2, m_1, m_2, m_3) = 4 \int_0^\infty I_0(wy) K_0(m_1 y) K_0(m_2 y) K_0(m_3 y) y \mathrm{d}y.$$

4.2 Algebraic geometry in Schwinger parameter space

Algebraic geometers prefer Feynman integrals in parameter space, where Schwinger gives

$$I(w^2, m_1, m_2, m_3) = \int_0^\infty \int_0^\infty \frac{\mathrm{d}x \, \mathrm{d}y}{P(x, y, 1)}$$

with momentum conservation achieved by setting, for example, $z = 1$ in

$$P(x, y, z) = (m_1^2 x + m_2^2 y + m_3^2 z)(xy + yz + zx) - w^2 xyz.$$

I shall not use this representation, here. Yet I respect Spencer Bloch's preference for it.

4.3 Cut constructibility in momentum space

Following BBBG, we obtain an efficient result from the imaginary part on the cut:

$$I(w^2, m_1, m_2, m_3) = 8\pi \int_{m_1+m_2+m_3}^{\infty} \frac{A(x) x \mathrm{d}x}{x^2 - w^2}$$

where

$$A(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(w) - F(-w)}\right)}$$

is the reciprocal of an AGM governed by the quartic

$$F(w) = (w+m_1+m_2+m_3)(w+m_1-m_2-m_3)(w-m_1+m_2-m_3)(w-m_1-m_2+m_3)$$

studied by Davydychev and Delbourgo (DD) and conveniently satisfying

$$F(w) = F(-w) + 16 m_1 m_2 m_3 w.$$

4.4 Wronskian from Legendre

The sunrise integral satisfies an inhomogeneous second–order differential equation whose homogeneous form is satisfied by $A(w)$. The complementary solution is

$$B(w) = \frac{1}{\mathrm{agm}\left(\sqrt{F(w)}, \sqrt{F(-w)}\right)}.$$

$$W(w) = A'(w)B(w) - A(w)B'(w) = \frac{N_1(w) + N_2(w) + N_3(w)}{\pi w F(w) F(-w)}$$

is the Wronskian of the homogeneous equation, easily found by using Legendre's relation between elliptic integrals. The Wronskian of $F(w)$ with $F(-w)$ yields

$$N_1(w) = (w^2 - m_1^2)^2 - (m_2^2 - m_3^2)^2$$

with $N_2(w)$ and $N_3(w)$ obtained by cyclic permutation of masses. Müller–Stach, Weinzierl and Zayadeh (MWZ) have determined the inhomogeneous term.

4.5 Bloch–Vanhove q-series in the equal–mass case

From now on, we assume that $m_1 = m_2 = m_3 = 1$. Then $F(w) = (w+3)(w-1)^3$ and the Wronskian is $W(w) = 3/(\pi w(w^2-1)(w^2-9))$. We define $q(w) \equiv \exp(-\pi B(w)/A(w))$, which is the nome of the elliptic integral resulting from the Dalitz plot (in this case a Dalitz line). Then the inhomogeneous differential equation, found with Jochem Fleischer and Oleg Tarasov (BFT) in 1993, may be written as

$$-\left(\frac{q(w)}{q'(w)}\frac{\mathrm{d}}{\mathrm{d}w}\right)^2 \left(\frac{I(w^2,1,1,1)}{24\sqrt{3}A(w)}\right) = \frac{w^2(w^2-1)(w^2-9)A(w)^3}{9\sqrt{3}}.$$

At a seminar on 6 June 2013, in Berlin, Spencer Bloch announced the stunning result that he and Pierre Vanhove (BV) had solved this BFT equation, using q-series. I now show how to recover the BV result, without using the algebraic geometry that inspired it.

Regarding w and $A(w)$ as functions of q, we obtain from Maier (M) the parametric solution

$$\frac{w}{3}=\left(\frac{\eta_3}{\eta_1}\right)^4\left(\frac{\eta_2}{\eta_6}\right)^2,\quad 4\sqrt{3}A=\frac{\eta_1^6\eta_6}{\eta_2^3\eta_3^2}.$$

Moreover, the two algebraic relations between $\{\eta_1,\eta_2,\eta_3,\eta_6\}$ may be written as

$$\frac{w^2-1}{8}=\left(\frac{\eta_2}{\eta_1}\right)^9\left(\frac{\eta_3}{\eta_6}\right)^3,\quad \frac{w^2-9}{72}=\left(\frac{\eta_6}{\eta_1}\right)^5\frac{\eta_2}{\eta_3},$$

whose resultant w.r.t. η_6 was given in Section 1. Hence the BFT equation reduces to

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{I}{24\sqrt{3}A}\right)=\frac{w}{3}f_{3,12}=\left(\frac{\eta_3^3}{\eta_1}\right)^3+\left(\frac{\eta_6^3}{\eta_2}\right)^3$$

where $f_{3,12}\equiv(\eta_2\eta_6)^3$ is the weight-3 level-12 modular form found in ϕ^4 theory by BS and the sum of cubes yields Lambert q-series given by Borwein and Borwein (B&B) in 1991.

Now define a character with $\chi(n)=\pm1$ for $n=\pm1$ mod 6 and $\chi(n)=0$ otherwise. Then

$$-\left(q\frac{\mathrm{d}}{\mathrm{d}q}\right)^2\left(\frac{I}{24\sqrt{3}A}\right)=\sum_{n>0}\frac{n^2(q^n-q^{5n})}{1-q^{6n}}=\sum_{n>0}\sum_{k>0}n^2\chi(k)q^{nk}.$$

Integrating twice and using the known imaginary part on the cut, I recover the BV result

$$\frac{I(w^2,1,1,1)}{4A(w)}=C(-1)-C\left(\mathrm{e}^{-\pi B(w)/A(w)}\right),\quad C(q)=\pi\log(-q)+\sum_{k>0}\frac{6\sqrt{3}\chi(k)q^k}{k^2(1-q^k)},$$

where the Clausen value $C(-1)=-5\,\mathrm{Cl}_2(\pi/3)$ makes $I(1,1,1,1)$ finite. So we are done.

5 Modular forms and higher–loop on–shell sunrises

In two dimensions, equal–mass on–shell sunrise diagrams and massive vacuum banana diagrams (see whiteboard) are examples of Bessel moments studied by BBBG:

$$S_{N,L} \equiv 2^L \int_0^\infty I_0(y)^{N-L-1} K_0(y)^{L+1} y \mathrm{d}y$$

where N is the total number of Bessel functions and L is the number of loops. For convergence, we require that $L < N \le 2L+2$. With $N = 2L+2$ we require that $L > 1$. BBBG proved that:

$$S_{1,0} = S_{2,1} = 1, \quad S_{3,1} = \frac{2\pi}{3\sqrt{3}}, \quad S_{3,2} = \frac{4\,\mathrm{Cl}_2(\pi/3)}{\sqrt{3}}, \quad S_{4,2} = \frac{\pi^2}{4}, \quad S_{4,3} = 7\zeta(3),$$

$$S_{5,2} = \frac{\pi^2}{8}\left(\sqrt{15}-\sqrt{3}\right)\left(\sum_{n\in\mathbf{Z}} \mathrm{e}^{-\sqrt{15}\pi n^2}\right)^4 = \frac{\sqrt{3}}{120\pi}\prod_{k=0}^{3}\Gamma\left(\frac{2^k}{15}\right)$$

where the final product of Gamma values results from the Chowla–Selberg theorem. We also conjectured (and checked to 1000 digits) that

$$S_{5,3} = \frac{4\pi}{\sqrt{15}} S_{5,2}, \quad S_{6,4} = \frac{4\pi^2}{3} S_{6,2}, \quad S_{8,5} = \frac{18\pi^2}{7} S_{8,3}.$$

5.1 Sunrise at 3 loops from a modular form of weight 3

Let $L_{3,15}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{3,15} = (\eta_3\eta_5)^3 + (\eta_1\eta_{15})^3$$

with weight 3 and level 15. Then I conjecture (and have checked to 1000 digits) that

$$S_{5,2} = 3L_{3,15}(2), \quad S_{5,3} = \frac{8\pi^2}{15} L_{3,15}(1),$$

where $S_{5,3}$ is the 5-Bessel moment giving the on-shell 3-loop sunrise diagram.

5.2 Sunrise at 4 loops from a modular form of weight 4

Let $L_{4,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{4,6} = (\eta_1\eta_2\eta_3\eta_6)^2$$

with weight 4 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{6,2} = 6L_{4,6}(2), \quad S_{6,3} = 12L_{4,6}(3), \quad S_{6,4} = 8\pi^2 L_{4,6}(2),$$

where $S_{6,4}$ is the 6-Bessel moment giving the on-shell 4-loop sunrise diagram.

5.3 Almost sunrise at 6 loops from a modular form of weight 6

Let $L_{6,6}(s)$ be the Dirichlet L-function defined by the multiplicative modular form

$$f_{6,6} = \left(\frac{\eta_2^3\eta_3^3}{\eta_1\eta_6}\right)^3 + \left(\frac{\eta_1^3\eta_6^3}{\eta_2\eta_3}\right)^3$$

with weight 6 and level 6. Then I conjecture (and have checked to 1000 digits) that

$$S_{8,3} = 8L_{6,6}(3), \quad S_{8,4} = 36L_{4,6}(4), \quad S_{8,5} = 216L_{4,6}(5),$$

but lack a result for $S_{8,6}$, the 8-Bessel moment giving the on-shell 6-loop sunrise diagram.

Epilogue

Thus far, with rough and all–unable pen,
Our bending author hath pursued the story,
In little room confining mighty men,
Mangling by starts the full course of their glory.

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