Proof of the 1-factorization and Hamilton decomposition conjectures

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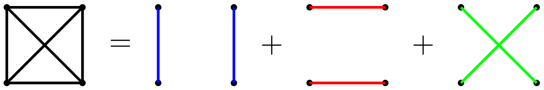
LMS–EPSRC Durham Symposium
22 July 2013

Joint work with
Béla Csaba (Szeged), Daniela Kühn (Birmingham),
Deryk Osthus (Birmingham) and Andrew Treglown (QMUL)

A 1-factorization of a graph G is a decomposition into edge-disjoint perfect matchings.



If G contains a 1-factorization, then $|G|$ is even and G is D-regular.
$\Rightarrow \chi'(G) = D$.

1-factorization

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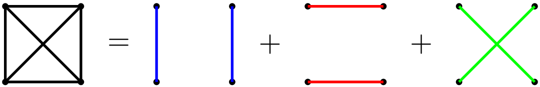

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Question

Does every D-regular graph G with $|G| = n$ even and $D \geq n/2$ contain a 1-factorization?

1-factorization conjecture (Chetwynd and Hilton 1985, Dirac 1950s)

Every D-regular graph G with $|G| = n$ even and

$$D \geq \begin{cases} n/2 - 1 & \text{if } n = 0 \pmod 4 \\ n/2 & \text{if } n = 2 \pmod 4, \end{cases}$$

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The bound is best possible.

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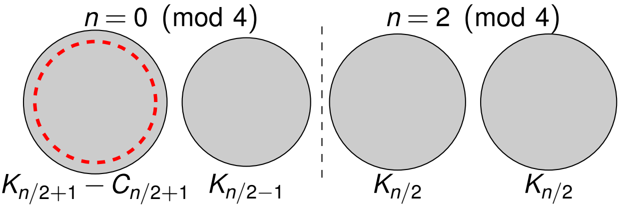

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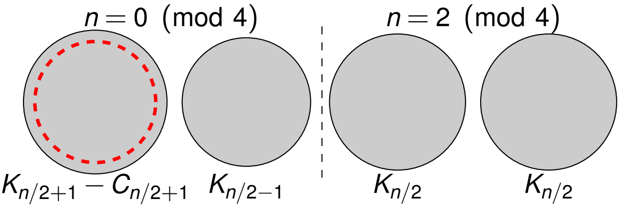

Same bound for the existence of a single perfect matching in D-regular graphs.

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Some partial results:

- True for $D = n - 1$, i.e. complete graphs.
- Chetwynd and Hilton (1989), and independently Niessen and Volkmann (1990), for $D \geq (\sqrt{7} - 1)n/2 \approx 0.82n$.
- Perkovic and Reed (1997) for $D \geq (1/2 + \varepsilon)n$ with $\varepsilon > 0$.

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Theorem (Csaba, Kühn, L, Osthus, Treglown 2013^{+})

The 1-factorization conjecture holds for large n.

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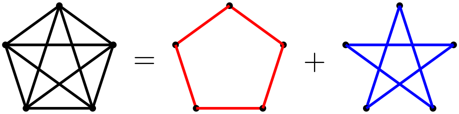

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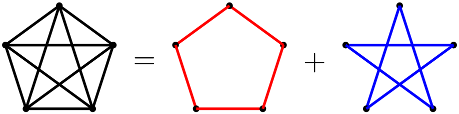

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Does every D-regular graph on n vertices with $D \geq n/2$ even contain a Hamilton decomposition?

Hamilton decomposition conjecture

Hamilton decomposition conjecture (Nash-Williams 1970)

Every D-regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$ can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.

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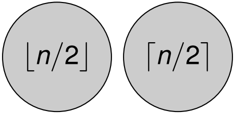

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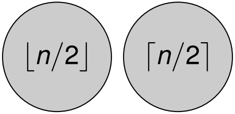

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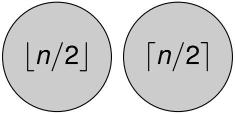

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Observation

If n is even, then ‘a Hamilton cycle = two perfect matchings’.
Hamilton decomposition $\Rightarrow$ 1-factorization.

Hamilton decomposition conjecture (Nash-Williams 1970)

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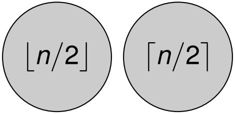

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If n is even, then 'a Hamilton cycle = two perfect matchings'.
Hamilton decomposition $\Rightarrow$ 1-factorization.
But Hamilton decomposition conjecture $\not\Rightarrow$ 1-factorization conjecture.

Hamilton decomposition conjecture

Hamilton decomposition conjecture (Nash-Williams 1970)

Every D-regular graph on n vertices with $D \geq \lfloor n/2 \rfloor$ can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.

Some partial results:

- Walecki (1890) K_n has a Hamilton decomposition.
- Nash-Williams (1969), $D \geq \lfloor n/2 \rfloor$ implies a Hamilton cycle.
- Jackson (1979), $D/2 - n/6$ edge-disjoint Hamilton cycles
- Christofides, Kühn and Osthus (2012) if $D \geq n/2 + \varepsilon n$, then G contains $(1 - \varepsilon')D/2$ edge-disjoint Hamilton cycles
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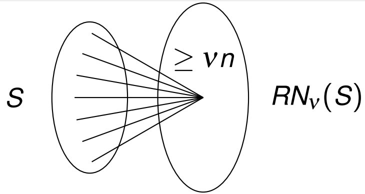
Theorem (Csaba, Kühn, L, Osthus, Treglown 2013^{+})

The Hamilton decomposition conjecture holds for large n.

Definition

Given $0 < \nu < \tau < 1$, we say that a graph G on n vertices is a robust (ν, τ)-expander, if for all $S \subseteq V(G)$ with $\tau n \leq |S| \leq (1-\tau)n$ the number of vertices that have at least νn neighbours in S is at least $|S| + \nu n$.



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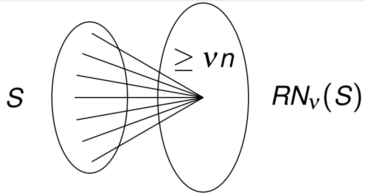


"*G* is still an expander after removing a sparse subgraph."

Theorem (Kühn and Osthus 2013)

For $1/n \ll \nu \ll \tau \ll \alpha$, every αn-regular robust (ν, τ)-expander G on n vertices can be decomposed into edge-disjoint Hamilton cycles and at most one perfect matching.

Extremal structure

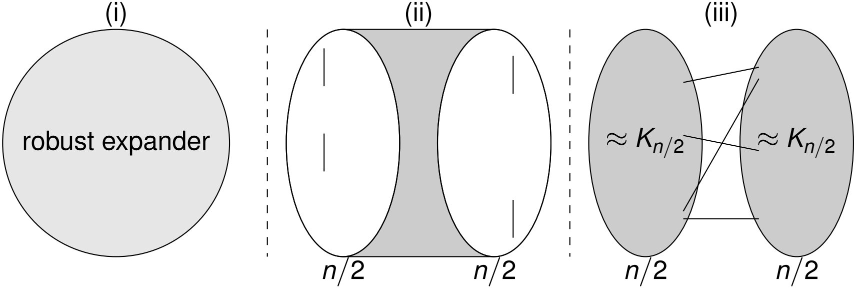
Structural Lemma

Let G be a D-regular graph with $|G| = n$ and $D \geq n/2 - 1$. Then either

(i) G is a robust expander;

(ii) G is ε-close to complete bipartite graph $K_{n/2,n/2}$;

(iii) G is ε-close to union of two complete graphs $K_{n/2}$.



G is ε-close to H if G can be transformed to H by adding/removing at most εn^2 edges.

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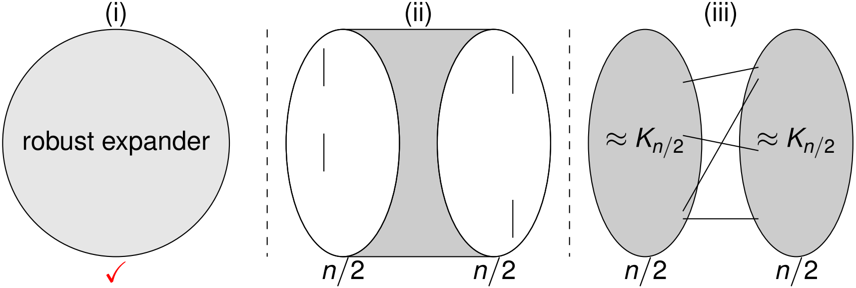

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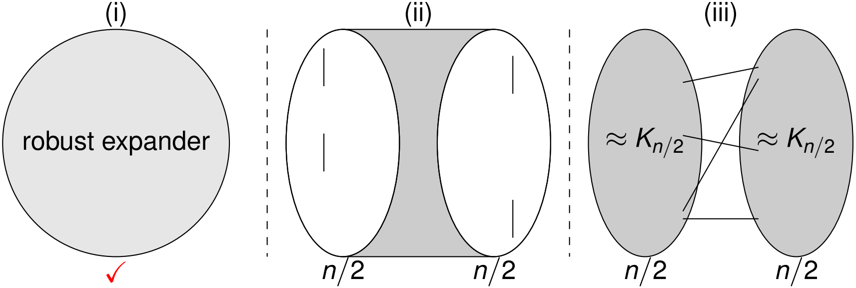

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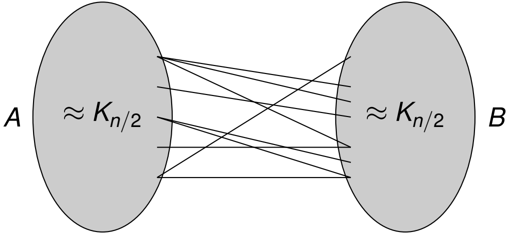

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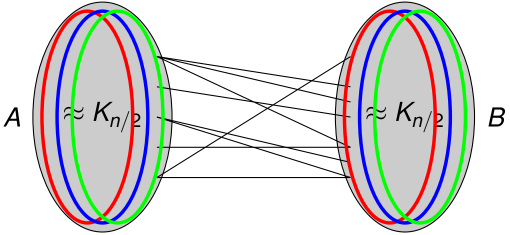

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   'Easy', because $G[A]$ and $G[B]$ are almost complete.

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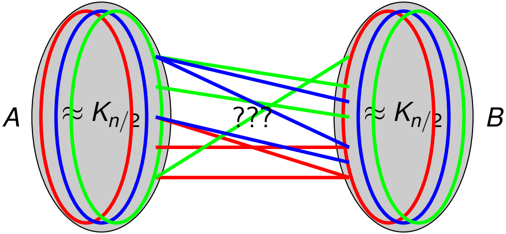

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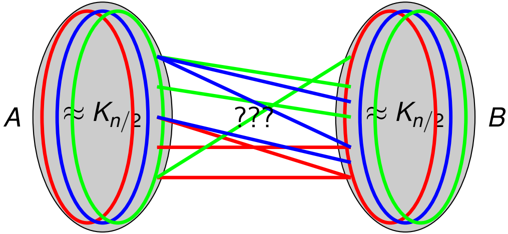

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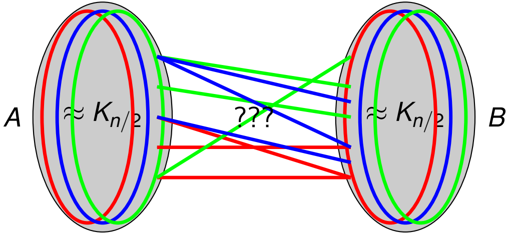

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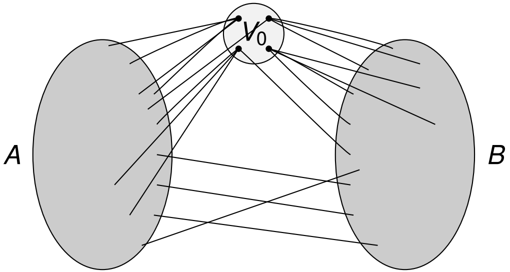
Actually, we need to first construct the connections, then extend each connection into a Hamilton cycle.

Vertex partition



- $G[A]$ and $G[B]$ are almost complete.
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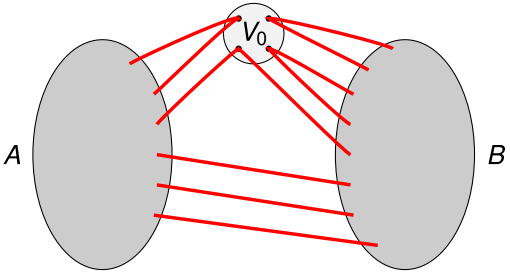

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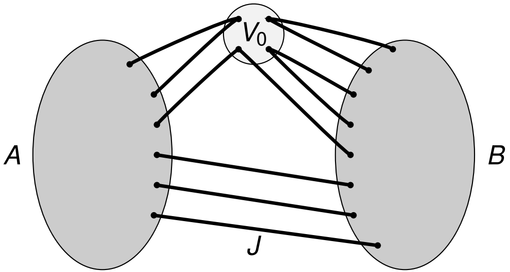
A connecting subgraph J will cover V_0, connect A and B, has endpoints in $A \cup B$.

A key property of connecting subgraphs



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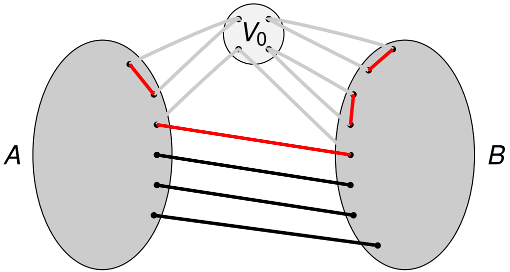

- ‘Contract’ all paths incident with V_0.

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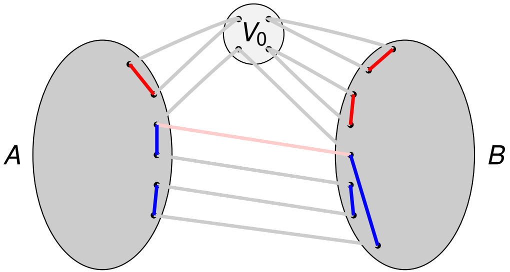

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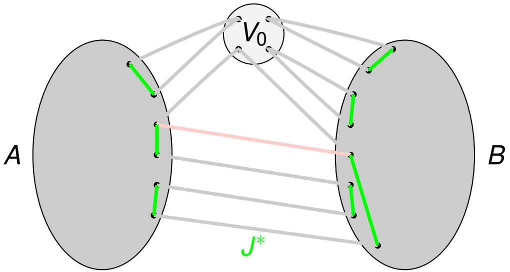

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- Call resulting green graph J^*.

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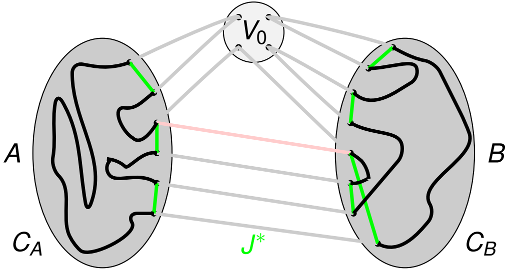

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Properties of J^*

Let C_A be a spanning cycle on A and C_B be a spanning cycle on B.
Suppose that $J^* \subseteq C_A + C_B$ (in a suitable order).
If we replace J^* with J, then we get a Hamilton cycle.

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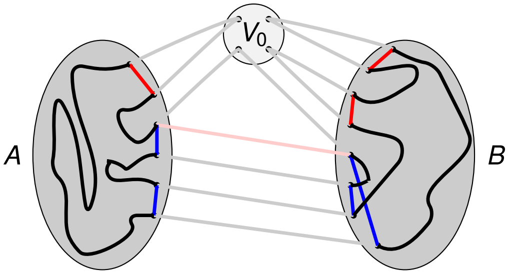

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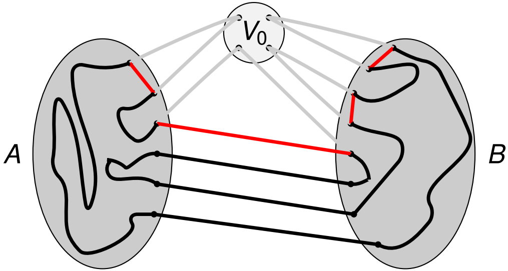

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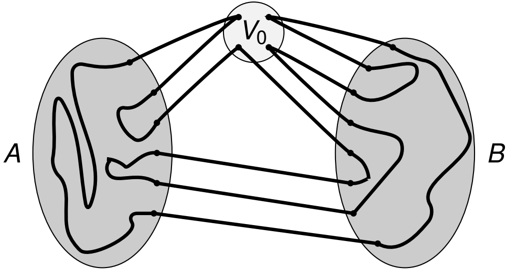

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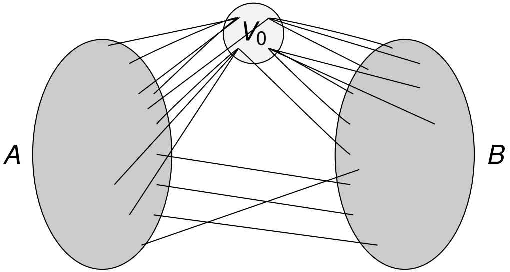
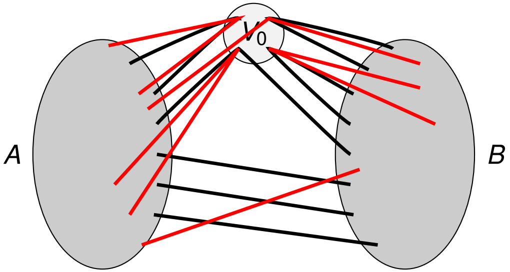
Finding a Hamilton decomposition





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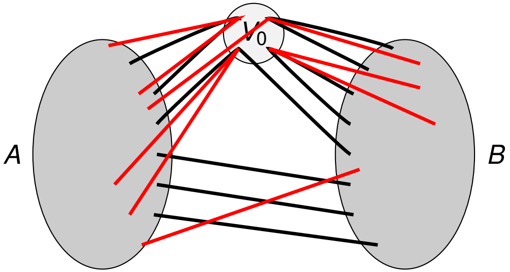

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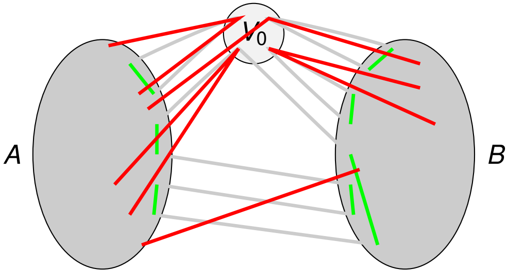

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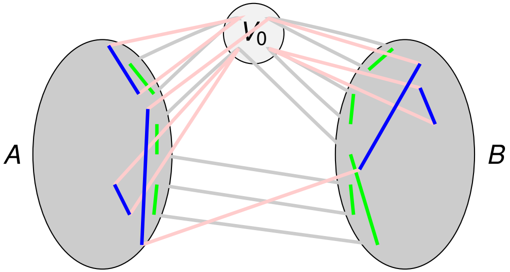

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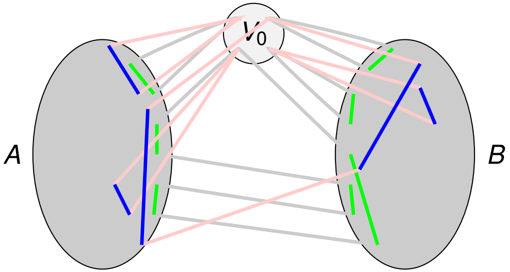

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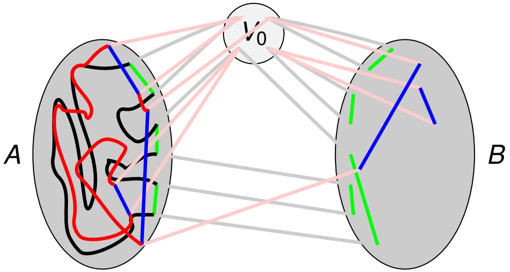
   Let $G^* = G[A] + G[B] + J_1^* + \cdots + J_{D/2}^*$.
   Note that $G^*[A]$ and $G^*[B]$ are D-regular multigraphs.

Finding a Hamilton decomposition



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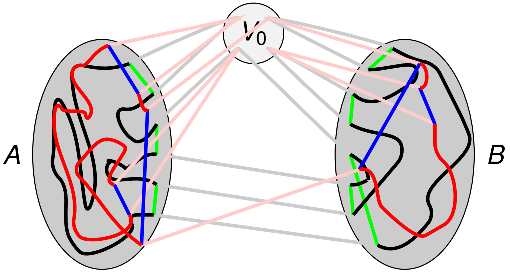
Finding a Hamilton decomposition



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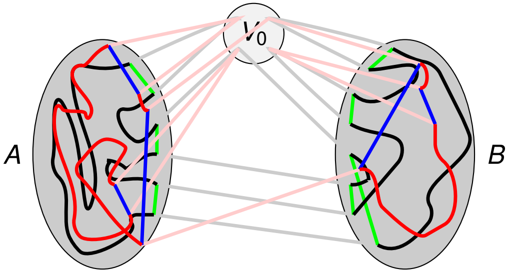

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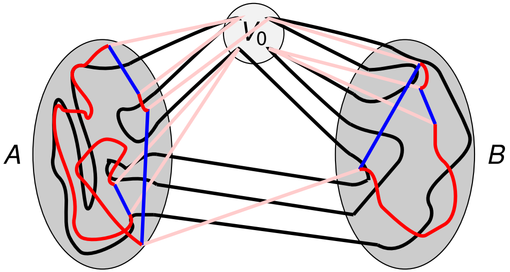

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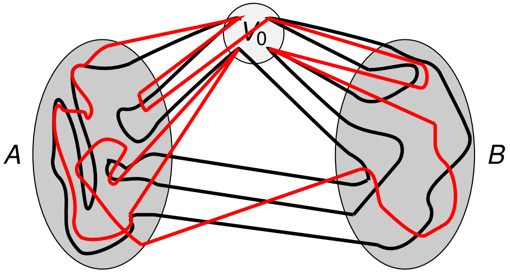

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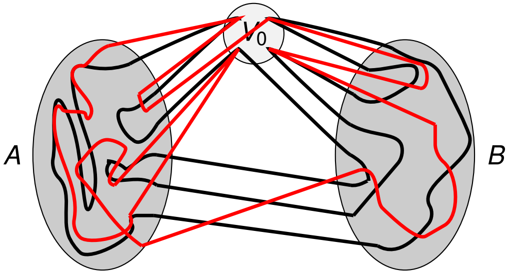

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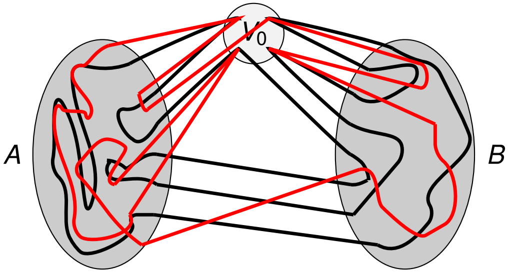

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Packing Hamilton cycles with large $\delta(G)$

Question

Suppose that G is a graph on n vertices with $\delta(G) \geq n/2$ (not necessarily regular). How many edge-disjoint Hamilton cycles are contained in G?

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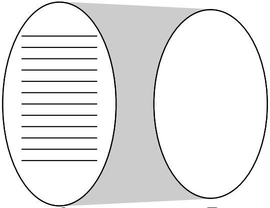
Conjecture (Nash-Williams 1971)

Every graph G on n vertices with $\delta(G) \geq n/2$ contains $(n-2)/8$ edge-disjoint Hamilton cycles.

Babai's construction:



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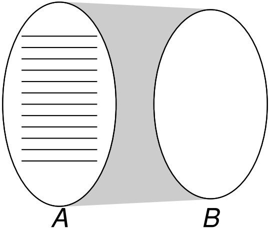

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Observation

Every Hamilton cycles contains at least 2 edges from A
$\Rightarrow$ G has at most $e(A)/2$ edge-disjoint Hamilton cycles
$\Rightarrow$ G has $\leq k$ edge-disjoint Hamilton cycles

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Theorem (Csaba, Kühn, Lapinskas, L, Osthus, Treglown 2013^+)

The Nash-Williams conjecture is true for large n.

Open problem

Let $\text{reg}_{\text{even}}(G)$ be the degree of the largest even-regular spanning subgraph in G.

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Some partial results by

- Ferber, Krivelevich and Sudakov (2013^{+})
- Kühn and Osthus (2013^{+})