Ricci curvature and spectra estimates on graphs

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Joint work with Frank Bauer and Jürgen Jost

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MAX-PLANCK-GESELLSCHAFT

Max Planck Institute for Mathematics in the Sciences

Introduction: two ingredients

Graph Laplace operator

Settings: an undirected, simple, finite, connected graph $G = (V, E)$.

- Combinatorial Laplace operator L

$$Lf(x) = \sum_{y, y \sim x} f(y) - d_x f(x), \;\; \forall f : V \to \mathbb{R};$$

- Normalized Laplace operator Δ

$$\Delta f(x) = \frac{1}{d_x} \sum_{y, y \sim x} f(y) - f(x), \;\; \forall f : V \to \mathbb{R}.$$

We call λ an eigenvalue of Δ if there exists some $f \not\equiv 0$ such that

$$\Delta f = -\lambda f.$$

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$$\Delta f(x) = \sum_{y \in V} f(y) m_x(y) - f(x), \;\; \forall f : V \to \mathbb{R}.$$

$$m_x(y) = \begin{cases} \frac{1}{d_x}, & \text{if } y \sim x; \\ 0, & \text{otherwise.} \end{cases}$$

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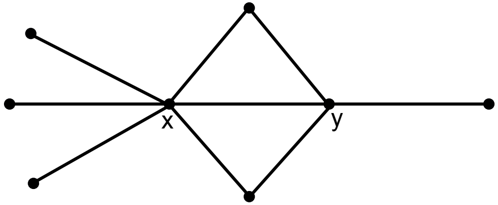
Number of common neighbors and Ricci curvature

Number of common neighbors of $x \sim y$,

$$\sharp(x,y) := \sum_{z, z\sim x, z\sim y} 1.$$



$$\sharp(x,y) = 2$$

Overlaps of two distance balls ←----→ lower Ricci curvature bounds

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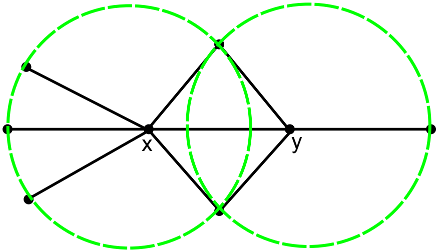

Overlaps of two distance balls ←- - - -→ lower Ricci curvature bounds

Largest eigenvalue and number of common neighbors

The largest eigenvalue

Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{N-1}$ be eigenvalues of L.

- Anderson-Morley, 1985

$$\lambda_{N-1} \leq \max_{x \sim y} \{d_x + d_y\};$$

- Rojo-Soto-Rojo, 2000

$$\lambda_{N=1} \leq \max_{x \neq y} \{d_x + d_y - \sharp(x, y)\};$$

- Das, 2003

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- Bauer-Jost-L. 2012

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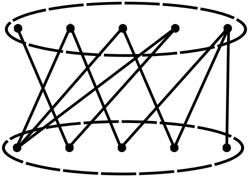
- $\lambda_{N-1} = 2$ iff G is bipartite (with out any odd-length cycles)



Iterated operator

Consider the iterated operator $\Delta[2] = -I + (I + \Delta)^2$. We have

$$\Delta[2]f(x) = \frac{1}{d_x} \sum_{y, y \sim_{[2]} x} \left(\sum_{\substack{z, z \sim x, \\ z \sim y}} \frac{1}{d_z} \right) f(y) - f(x).$$

Proof: For u s.t. $\Delta u = -\lambda_{N-1} u$, we have

$$2 - \lambda_{N-1} = \frac{(u, \Delta[2]u)}{(u, \Delta u)} = \frac{\sum_{x \sim_{[2]} y} \left(\sum_{\substack{z, z \sim x, \\ z \sim y}} \frac{1}{d_z} \right) (u(x) - u(y))^2}{\sum_{x \sim y} (u(x) - u(y))^2}$$

$$\geq \min_{x \sim y} \sum_{\substack{z, z \sim x, \\ z \sim y}} \frac{1}{d_z} \geq \frac{\min_{x \sim y} \sharp(x, y)}{\max_x d_x},$$

where we used

- $(f, g) = \sum_x f(x) g(x) d_x$;
- $\min_{x \sim y} \sharp(x, y) \neq 0$ ensures $x \sim y \Rightarrow x \sim_{[2]} y$.

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$$\begin{aligned} 2-\lambda_{N-1} = \frac{(u, \Delta[2]u)}{(u, \Delta u)} &= \frac{\sum_{x\sim_{[2]}y}\left(\sum_{\substack{z, z\sim x,\\ z\sim y}} \frac{1}{d_z}\right)(u(x)-u(y))^2}{\sum_{x\sim y}(u(x)-u(y))^2} \\ &\geq \min_{x\sim y} \sum_{\substack{z, z\sim x,\\ z\sim y}} \frac{1}{d_z} \geq \frac{\min_{x\sim y}\sharp(x,y)}{\max_x d_x}, \end{aligned}$$

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Neighborhood graphs

Definition (Bauer-Jost 2010)

The neighborhood graph $G[t] = (V, E[t])$ of the graph $G = (V, E)$ of order $t \geq 1$ is defined as

- V: unchanged;
- $E[t]$ changed:

$$w_{xy}[t] := \delta_x P^t(y) d_x = \sum_{\substack{x_1, \ldots, x_{t-1} \\ x \sim x_1 \sim \cdots \sim x_{t-1} \sim y}} \frac{1}{d_{x_1}} \cdots \frac{1}{d_{x_{t-1}}}.$$

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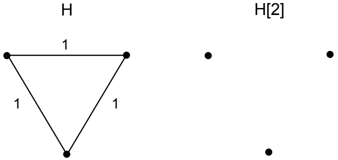

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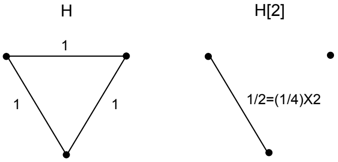

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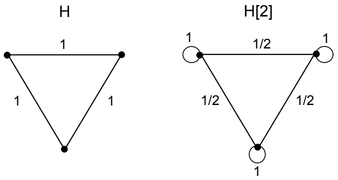

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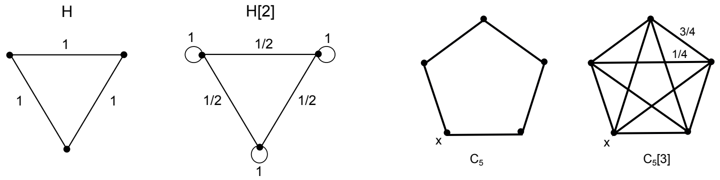

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- $x \sim_{[t]} y$ in $G[t]$ iff $\exists$ a path of length t between x and y in G;
- $$d_x[t] := \sum_y w_{xy}[t] = d_x;$$
- $$\Delta[t] = -I + (I + \Delta)^t.$$

Eigenvalues and number of common neighbors

Eigenvalues and coarse Ricci curvature

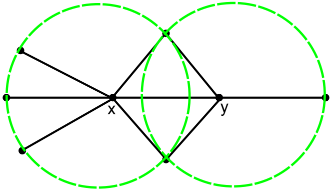
Ollivier's Ricci curvature notion

Definition (Ollivier, 2009)

For any two distinct points $x, y \in X$, the (Ollivier-) Ricci curvature of G along (xy) is defined as

$$\kappa(x, y) := 1 - \frac{W_1(m_x, m_y)}{d(x, y)},$$

- $W_1(m_x, m_y)$ is the optimal transportation distance between the two probability measures m_x and m_y using the graph distance as cost function.



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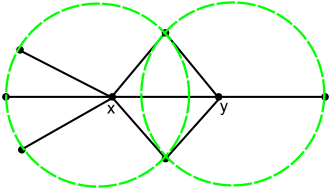
- $W_1(m_x, m_y)$ is the optimal transportation distance between the two probability measures m_x and m_y using the graph distance as cost function.
- Earlier ideas of defining Ricci curvature on graphs, Dodziuk-Karp 1988, Chung-Yau 1996.



Ricci curvature and common neighbors

Theorem (Jost-L., 2011)

For any pair of neighboring vertices x, y,

$$\frac{\sharp(x,y)}{d_x \vee d_y} \geq \kappa(x,y) \geq -\left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \wedge d_y}\right)_+ - \left(1 - \frac{1}{d_x} - \frac{1}{d_y} - \frac{\sharp(x,y)}{d_x \vee d_y}\right)_+ + \frac{\sharp(x,y)}{d_x \vee d_y}.$$

Moreover, this inequality is sharp for certain graphs.

Notations:

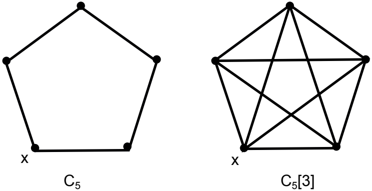
$a_+ := \max\{a, 0\}$, $a \wedge b := \min\{a, b\}$, and $a \vee b := \max\{a, b\}$.

- Lower bound improves the estimate of Lin-Yau 2010.

Ricci curvature $\kappa[t]$

$\kappa[t]$ capture the information of number of 3-cycles on $G[t]$ which may come from

- 3-cycles are preserved from G to $G[t]$.
- $x \in$ an odd cycle in $G \to x \in$ 3-cycles in $G[t]$.



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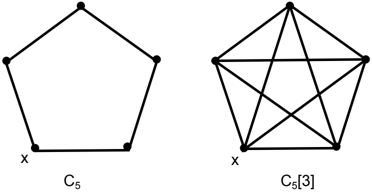

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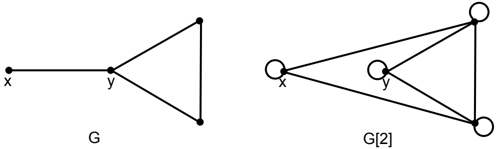

Eigenvalues and Curvature

Theorem (Bauer-Jost-L. 2012)

Let $k[t]$ be a lower bound of Ollivier-Ricci curvature of the neighborhood graph $G[t]$. Then for all $t \geq 1$ the eigenvalues of Δ on G satisfy

$$1 - (1 - k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1 - k[t])^{\frac{1}{t}}.$$

Moreover, if G is not bipartite, then there exists a $t' \geq 1$ such that for all $t \geq t'$ the eigenvalues of Δ on G satisfy

$$0 < 1 - (1 - k[t])^{\frac{1}{t}} \leq \lambda_1 \leq \cdots \leq \lambda_{N-1} \leq 1 + (1 - k[t])^{\frac{1}{t}} < 2.$$

- $t = 1$ case follows directly from Ollivier.

$$k \leq \lambda_1 \leq \cdots \lambda_{N-1} \leq 2 - k$$

is nontrivial only when $k > 0$.

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An Example

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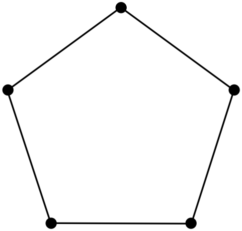

C_5

$$\lambda_1 = 1 - \cos\frac{2\pi}{5} \doteq 0.6910,\ \ \lambda_4 = 1 - \cos\frac{4\pi}{5} \doteq 1.8090.\ k = 0.$$

$$t = 1 : 0 \leq \lambda_1 \leq \lambda_4 \leq 2.$$

$$t = 2 : 0.1340 \leq \lambda_1 \leq \lambda_4 \leq 1.8660.$$

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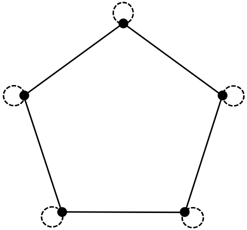

$C_5[2]$

$$\lambda_1 = 1 - \cos\frac{2\pi}{5} \doteq 0.6910,\ \ \lambda_4 = 1 - \cos\frac{4\pi}{5} \doteq 1.8090.\ k[2] = 1/4.$$

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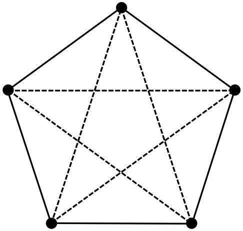

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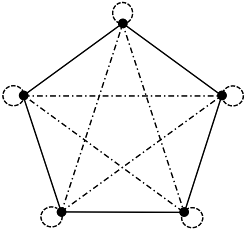

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Exponential decay

Theorem (Bauer-Jost-L. 2012)

If G is not bipartite, the limit

$$\lim_{t\to\infty} \frac{\log(1-k[t])}{t} := -a$$

exists with $a \in (0, +\infty]$. *That means,* $k[t]$ *behaves like* $1 - P(t)e^{-at}$ *as* $t \to \infty$ *where* $P(t)$ *is a polynomial in* t.

Proof:

- Subadditivity implies existence of the limit $-a$ (for $s, t \geq t'$, $1 - k[t+s] \leq (1-k[s])(1-k[t])$);
- Further estimates implies $a > 0$.

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Thank you for your attentions!