Euclidean Artin groups I: factoring euclidean isometries

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Coxeter groups

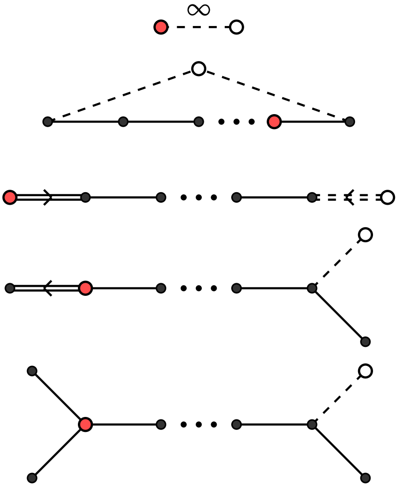
The spherical and euclidean Coxeter groups are reflection groups that act geometrically on spheres and euclidean space. They arise in the study of regular polytopes and Lie theory.

Their classification is classical and their presentations are encoded in the well-known Dynkin diagrams and extended Dynkin diagrams, respectively, using conventions sufficient for these groups, but not for general Coxeter groups.

The extended Dynkin diagrams consist of:

Four infinite families

- $\widetilde{A}_1$
- $\widetilde{A}_n$
- $\widetilde{C}_n$
- $\widetilde{B}_n$
- $\widetilde{D}_n$



Five sporadic examples

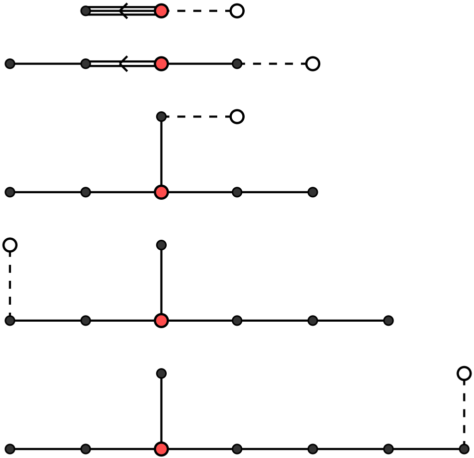
$\widetilde{G}_2$

$\widetilde{F}_4$

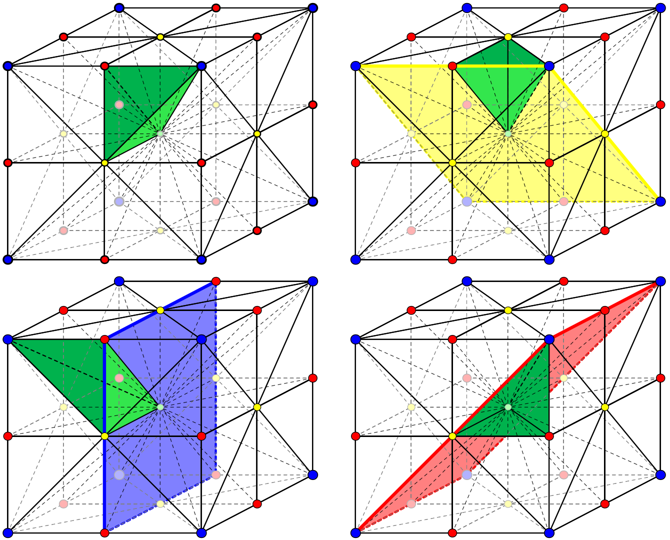
$\widetilde{E}_6$

$\widetilde{E}_7$

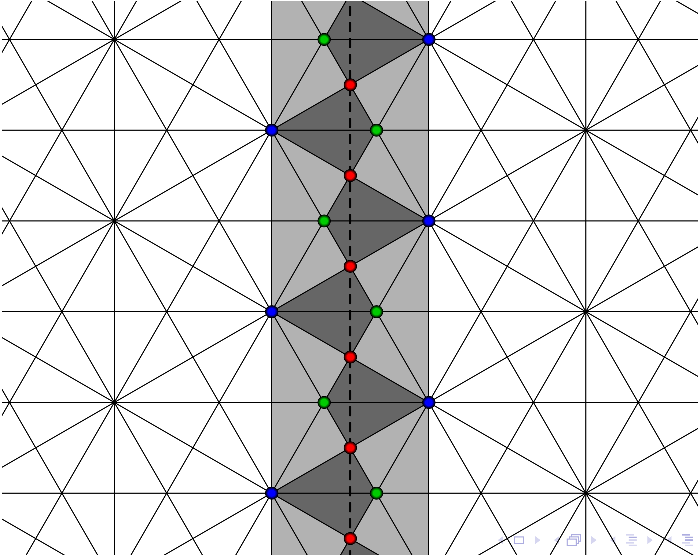
$\widetilde{E}_8$



The spherical Coxeter group $\mathrm{Cox}(B_3)$



The euclidean Coxeter Group $\mathrm{Cox}(\widetilde{G}_2)$



General Coxeter groups

Spherical and euclidean Coxeter groups are key examples that motivate the general theory introduced by Jacques Tits in the early 1960s. All Coxeter groups are defined by simple presentations encoded in diagrams.

In that first (unpublished) paper, Tits proved that every Coxeter group has a faithful linear representation preserving a symmetric bilinear form and thus has a solvable word problem.

Coxeter groups can be coarsely classified by the signature of the symmetric bilinear forms they preserve. The spherical and euclidean groups are those which have positive definite and positive semi-definite forms.

General Artin groups

Artin groups first appear in print in 1972 (Brieskorn and Saito, Deligne). General Artin groups are defined by simple presentations that can be encoded in the same diagrams as Coxeter groups and then coarsely classified in the same way.

Those early papers connected spherical Artin groups to the fundamental groups of spaces derived from complexified hyperplane complements and successfully analyzed their structure.

Given the centrality of euclidean Coxeter groups and the elegance of their structure, one might have expected euclidean Artin groups to be well understood shortly thereafter. It is now 40 years later and these groups are still revealing their secrets.

Basic Questions

In a recent article Eddy Godelle and Luis Paris highlight four basic conjectures about Artin groups:

Conjectures

A) *Every Artin group is torsion-free*

B) *Every non-spherical Artin group has trivial center*

C) *Every Artin group has a solvable word problem*

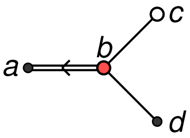
D) *Artin groups satisfy the* $K(\pi, 1)$ *conjecture*

They also remark:

> *"A challenging question in the domain is to prove Conjectures A, B, C, and D for the so-called Artin-Tits groups of affine type, that is, those Artin-Tits groups for which the associated Coxeter group is affine."*

Example: $\textsc{Art}(\widetilde{B}_3)$

The group $\textsc{Art}(\widetilde{B}_3)$ has diagram



and presentation

$$\left\langle a, b, c, d \;\middle|\; \begin{array}{cc} abab = baba & cd = dc \\ bcb = cbc & ad = da \\ bdb = dbd & ac = ca \end{array} \right\rangle$$

The basic questions were open for this group until very recently.

Known: planar Artin groups

The few previously known results about euclidean Artin groups are easy to review.

In 1987 Craig Squier successfully analyzed the structure of the three irreducible euclidean Artin groups $\mathrm{ART}(\widetilde{A}_2)$, $\mathrm{ART}(\widetilde{C}_2)$ and $\mathrm{ART}(\widetilde{G}_2)$ that correspond to the three irreducible euclidean Coxeter groups acting on the euclidean plane.

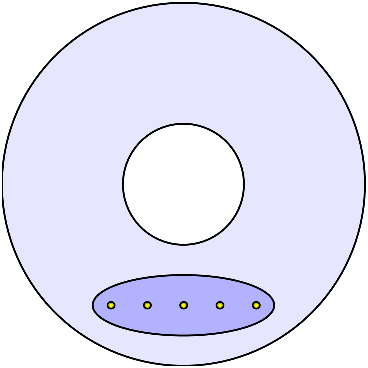
He worked directly with the presentations and analyzed them as amalgamated products and HNN extensions of well-known groups.

His techniques do not appear to generalize to higher dimensions.

Known: euclidean braid groups

The euclidean braid group $\text{ART}(\widetilde{A}_n)$ embeds into the annular braid group $\text{ART}(B_{n+1})$, and this makes its structure clear. In fact, there is a short exact sequence

$$\text{ART}(\widetilde{A}_n) \hookrightarrow \text{ART}(B_{n+1}) \twoheadrightarrow \mathbb{Z}$$



Known: types A and C

Finally, there are recent results due to François Digne.

Theorem (Digne)

The groups $\text{ART}(\widetilde{A}_n)$ and $\text{ART}(\widetilde{C}_n)$ have Garside structures.

Digne uses the embedding $\text{ART}(\widetilde{A}_n) \hookrightarrow \text{ART}(B_{n+1})$ to show that type A has a Garside structure and then an embedding of type C into type A to show the same for type C.

To my knowledge, these are the only euclidean Artin groups that were previously fully understood, and they did not include simple examples such as $\text{ART}(\widetilde{B}_3)$.

New: all euclidean Artin groups

Robert Sulway and I provide positive solutions to Conjectures *A*, *B* and *C* for all euclidean Artin groups and we also make progress on Conjecture *D*. In particular, we prove the following:

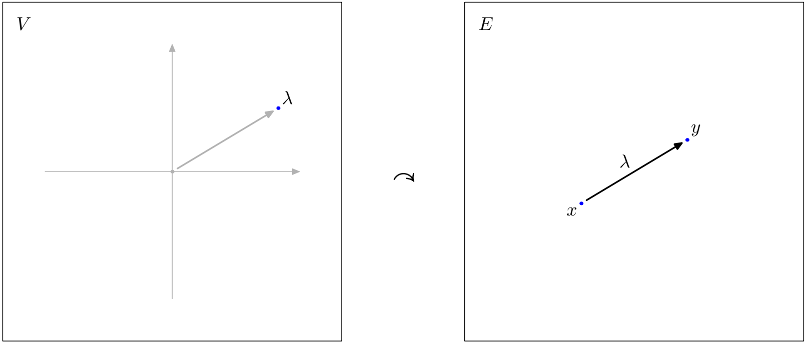
Theorem (M-Sulway)

Every irreducible euclidean Artin group $\textsc{Art}(\widetilde{X}_n)$ *is a torsion-free centerless group with a solvable word problem and a finite-dimensional classifying space.*

The proof uses the structure of intervals in euclidean Coxeter groups and other euclidean groups generated by reflections.

Points and Vectors

We distinguish points and vectors. Let V be a vector space with a simple transitive action on a set E. Elements of E are *points* and elements of V are *vectors*. E has no distinguished origin.



Linear and affine subspaces

Definition (Subspaces)

The vector space V has *linear subspaces* through the origin and other *affine subspaces*. E only has affine subspaces. For any affine subspace $B \subset E$, vectors between points in B form a linear subspace $\textsc{Dir}(B) \subset V$ called its *space of directions*.

Remark (Poset structure)

The linear subspaces of V ordered by inclusion are the poset $\textsc{Lin}(V)$, a graded, bounded, self-dual lattice. The affine subspaces of E ordered by inclusion are the poset $\textsc{Aff}(E)$. It is graded and bounded above, but not bounded below, not self-dual and not a lattice.

There is a well-defined rank-preserving map $\textsc{Aff}(E) \twoheadrightarrow \textsc{Lin}(V)$ that sends B to $\textsc{Dir}(B)$.

Elliptic and hyperbolic

If we add a positive definite inner product to V we get a euclidean metric on E and we can discuss isometries of E.

Definition (Basic invariants)

The *move-set* of an isometry w is the affine subspace $\textsc{Mov}(w) \subset V$ that collects all the motions that its points undergo. And if μ is the unique vector in $\textsc{Mov}(w)$ closest to the origin, the points in E that undergo the motion μ are an affine subspace $\textsc{Min}(w) \subset E$ called the *min-set* of w. These are the *basic invariants* of w.

Definition (Elliptic and hyperbolic)

If $\textsc{Mov}(w)$ includes the origin, μ is trivial, $\textsc{Min}(w) = \textsc{Fix}(w)$ and w is *elliptic*. Otherwise, w is *hyperbolic*.

Reflections and translations

The basic invariants for some elementary isometries:

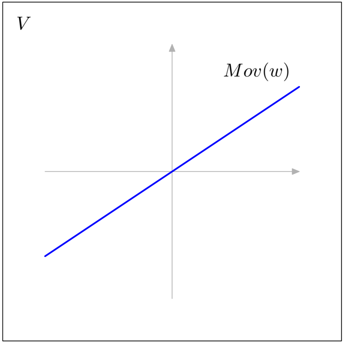
Definition (Translations)

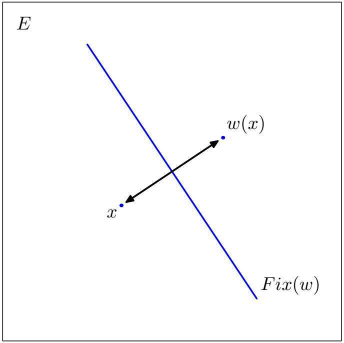
For each vector $\lambda \in V$ there is a *translation* isometry t_λ which is hyperbolic with min-set E and move-set $\{\lambda\}$.

Definition (Reflections)

For each *hyperplane* H in E (an affine subspace of codimension 1) there is a unique nontrivial isometry r fixing H called a *reflection*. It is elliptic with fix-set H and move-set a line through the origin in V. We call any nontrivial vector α in this line a *root* of r.

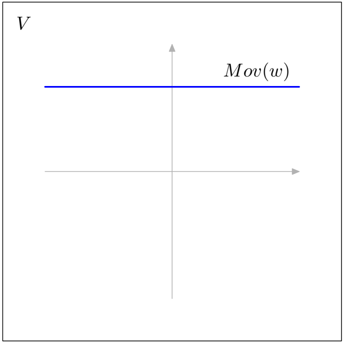
Example: Reflections

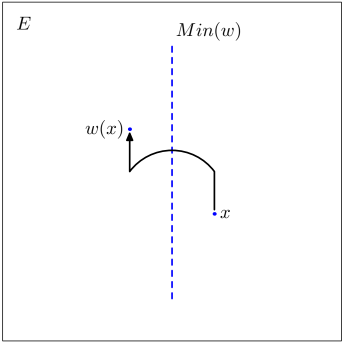




The basic invariants of a reflection. The move-set is linear and the isometry is elliptic.

Example: Glide reflections



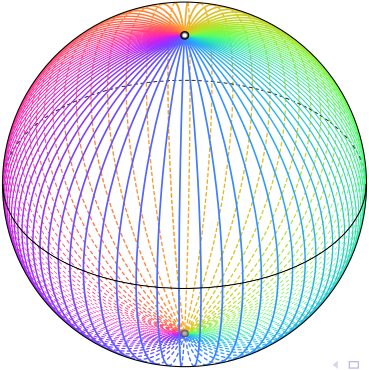


The basic invariants of a glide reflection. The move-set is a nonlinear affine subspace and the isometry is hyperbolic.

Intervals

Definition

z is between x and y when $d(x,z) + d(z,y) = d(x,y)$. All points between x and y form the interval $[x,y]$. Intervals are posets with $z \le w$ iff $d(x,z) + d(z,w) + d(w,y) = d(x,y)$.



Groups ⇒ Intervals

Definition (Group intervals)

A group G with a fixed discretely weighted generating set is a metric space, and thus it has intervals. Let $[g,h]^G$ denote the portion of the Cayley graph between g and h, by which I mean the union of all the minimal length directed paths from v_g to v_h. And note that this edge-labeled directed graph also encodes the poset structure.

Remark

Cayley graphs are homogeneous so the interval $[g,h]^G$ is isomorphic (as an edge-labeled directed graph) to the interval $[1,g^{-1}h]^G$. Thus we can restrict to intervals of the form $[1,g]^G$.

Euclidean intervals

The Lie group $L = \textsc{Isom}(\mathbb{R}^n)$ is generated by the set R of all reflections and its Cayley graph with respect to R has diameter $n+1$. In fact, Scherk's theorem identifies the minimal reflection length of an isometry from its basic geometric attributes.

Question

What about the order structure of the interval $[1, w]^L$?
Is this poset a lattice? What if w is loxodromic?

As a key first step towards understanding euclidean Artin groups, Noel Brady and I completely characterized the poset structure of these intervals.

Scherk's theorem

To understand intervals in $L = \textsc{Isom}(\mathbb{R}^n)$ we need to understand reflection length, and this is the content of Scherk's theorem.

Proposition (Factorizations)

An elliptic isometry with a k-dimensional move-set has a length k reflection factorization. A hyperbolic isometry with a k-dimensional move-set has a length $k+2$ reflection factorization.

In fact, these factorizations have minimal length.

Theorem (Scherk)

Let w be an isometry with a k-dimensional move-set. If w is elliptic, its reflection length is k. If w is hyperbolic, its reflection length is $k+2$.

Reflections and invariants

Basic facts about the invariants of a product of an isometry and a reflection are determined by the relations between their basic invariants. Here is a sample lemma.

Lemma

Suppose w is hyperbolic with $\ell_R(w) = k$ and $\textsc{Mov}(w) = U + \mu$ in standard form, r is a reflection with root α and let U_α denote the span of $U \cup \{\alpha\}$.

- *If $\alpha \in U$ then rw is hyperbolic with $\ell_R(rw) = k - 1$.*
- *If $\alpha \notin U$ and $\mu \in U_\alpha$ then rw is elliptic and $\ell_R(rw) = k - 1$.*
- *If $\alpha \notin U$ and $\mu \notin U_\alpha$ then rw is hyperbolic and $\ell_R(rw) = k + 1$.*

Elliptic intervals

Proposition (Elliptic intervals)

Let w be an elliptic isometry with $\textsc{Mov}(w) = U \subset V$. *The map* $u \mapsto \textsc{Mov}(u)$ *creates a poset isomorphism* $[1, w]^L \cong \textsc{Lin}(U)$. *In particular,* $[1, w]^L$ *is a lattice.*

Alternatively the map $u \mapsto \textsc{Fix}(u)$ gives a poset isomorphism with the affine subspaces containing $\textsc{Fix}(w)$ under reverse inclusion, i.e. the fixed points determine the motion!

Remark

The most remarkable aspect of this result is that the structure of the interval only depends on the codimension of the fix-set and is otherwise independent of w.

Poset

To describe the intervals for hyperbolic w, we first define an abstract poset that mimics the basic invariants of euclidean isometries.

Definition

The poset P has two types of elements:

- an element $h^M \in P$ for each nonlinear affine $M \subset V$
- an element $e^B \in P$ for each affine $B \subset E$

Its order relations are:

$$\begin{array}{cc} h^M \geq h^{M'} \text{ iff } M \supset M' & h^M > e^B \text{ iff } M^\perp \subset \textsc{Dir}(B) \\ e^B \geq e^{B'} \text{ iff } B \subset B' & \text{no } e^B \text{ is ever above } h^M \end{array}$$

Invariant map

Next we define a map that sends each isometry in $L = \textsc{Isom}(E)$ to an element of P based on its type and its invariants:

Definition (Invariant map)

Define a map $\textsc{inv} : L \to P$ by setting $\textsc{inv}(u) = h^{\textsc{Mov}(u)}$ when u is hyperbolic and $\textsc{inv}(u) = e^{\textsc{Fix}(u)}$ when u is elliptic.

Remark

There is a ordering on L based on distance to the identity that turns L into a poset. Under this ordering the invariant map is a rank-preserving poset map, but it is far from injective.

Model posets

Definition (Model posets)

For each affine subspace $B \subset E$, let P^B denote the poset of elements below e^B in P. For each nonlinear affine subspace $M \subset V$ let P^M denote the poset of elements below h^M in P.

Since the invariant map is a poset map, the image of the interval $[1, w]^L$ is contained in $P^{\mathrm{INV}(w)}$. In fact:

Theorem (Brady-M)

For each isometry w, the map $u \mapsto \mathrm{INV}(u)$ gives a poset isomorphism $[1, w]^L \cong P^{\mathrm{INV}(w)}$.

Thus we can forget about the isometries themselves and focus on a poset defined in terms of affine subspaces of V and E.

Maximal hyperbolic isometries

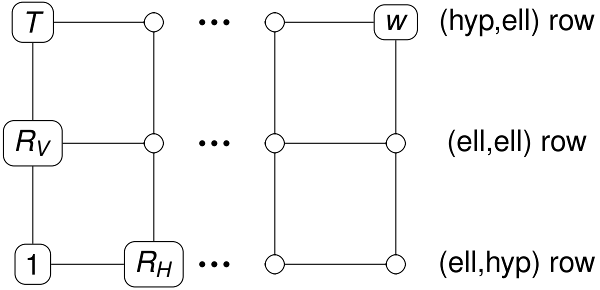
When w is a hyperbolic isometry of maximal reflection length its min-set is a line and its move-set is a nonlinear affine hyperplane.

We call the direction of its min-set *vertical* and the orthogonal directions *horizontal*. More generally we call any motion with a non-trivial vertical component vertical.

For every $u \in [1, w]$ there is a v such that $uv = w$. We split $[1, w]$ into 3 rows based on the types of u and v. When one is hyperbolic, the other is a purely horizontal elliptic. When both are elliptic, both motions have vertical components. Within each row we grade based on the dimensions of the basic invariants.

Coarse structure

When w is a maximal hyperbolic isometry, $[1, w]^L$ has the following coarse structure:



A unique hyperbolic element for each subspace of $\text{Mov}(w) \subset V$ and a unique elliptic for each subspace of E.

Lattice failure

We understand the structure of the intervals and precisely where the lattice property fails.

Theorem (Brady-M)

If w is a hyperbolic isometry whose move-set is at least 2-dimensional, then the interval $[1, w]^L$ is NOT a lattice.

In the paper we give an explicit characterization of these failures and where they occur.

Corollary

The interval for a loxodromic isometry in $\mathbb{R}^3$ is NOT a lattice.

References

These talks are based on three papers. Rough drafts are available from my preprints page; they are not yet on the arXiv.

• Noel Brady and Jon McCammond, “Factoring euclidean isometries”.

• (John Crisp and) Jon McCammond, “Dual euclidean Artin groups and the failure of the lattice property”.

• Jon McCammond and Robert Sulway, “Artin groups of euclidean type”