Practical generalisations of small cancellation

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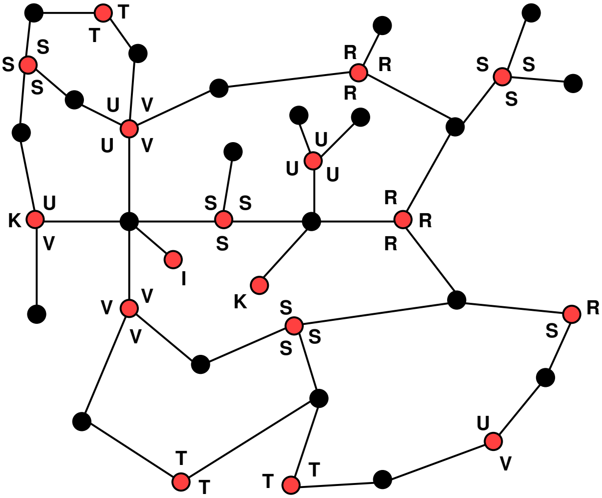
We draw connected finite plane bipartite graphs:



Labels are on the red corners. Faces are oriented clockwise.

What are these labels?

Definition

An infrastructure is a semigroup S and two subsets $S_+, S_L \subseteq S$, such that:

$$\text{if } xy \in S_+ \text{ for } x, y \in S, \text{then } yx \in S_+.$$

The elements in S_+ are acceptors. The elements in S_L are labels.

If $0 \in S$ then we usually insist that $0 \notin S_+$, $0 \notin S_L$, and for all $x \in S \setminus \{0\}$ there is a $y \in S$ with $xy \in S_+$.

Examples of infrastructures

- Let G be a group. Let $S = S_L = G$ and $S_+ := \{1\}$.
- Let $S^{(2)} := \{A, 1, 0\}$ with $A \cdot A = 1$ and all other products 0. Set $S_+^{(2)} := \{1\}$ and $S_L^{(2)} := \{A\}$.
- Let $S^{(3)} := \{A, A^{-1}, 1, 0\}$ with $A \cdot A^{-1} = A^{-1} \cdot A = 1$ and all other products 0.
Set $S_+^{(3)} := \{1\}$ and $S_L^{(3)} := \{A, A^{-1}\}$.
- Take any groupoid, adjoin a 0 and set undefined products to 0. Let all identities accept.

Lemma

The zero direct product of infrastructures (with unions of labels and accepters) is an infrastructure.

Diagrams

S – infrastructure. $\mathcal{R}$ – set of cyclic words in S_L.

Definition (Valid diagram)

A valid diagram is: a finite set X, permutations R, G, B of X and a function $\ell : X \to S$, such that

- the product $RGB = 1$,
- the group $\langle R, G, B\rangle$ is transitive on X,
- the total number of cycles of R, G and B on X is $|X| + 2$,
- for every R-cycle $x, xR, \ldots, xR^k$ the product $\ell(x) \cdot \ell(xR^{-1}) \cdot \cdots \cdot \ell(xR^{-k}) \in S_+$, and
- for all but maybe one (the boundary) G-cycle $x, xG, \ldots, xG^k$ the word $(\ell(x), \ell(xG), \ldots, \ell(xG^k))^{\circlearrowleft} \in \mathcal{R}$.

There is a bijection between plane bipartite graphs and such triples R, G, B, up to appropriate equivalence.

Two fundamental problems

S – infrastructure. Let $\mathcal{R}$ be a finite set of cyclic words in S_L.

Problem (Diagram boundary problem)

Algorithmically devise a procedure that decides for any cyclic word $w^{\circlearrowleft}$ in S_L whether or not there is a reduced diagram such that the external face is labelled by w.

Problem (Isoperimetric inequality problem)

Algorithmically find and prove a function $\mathcal{D} : \mathbb{N} \to \mathbb{N}$, s.t. for every cyclic word w in S_L of length k that is the boundary label of a diagram, there is one with at most $\mathcal{D}(k)$ internal faces.

If there is a linear $\mathcal{D}$, we call $\langle S \mid \mathcal{R} \rangle$ hyperbolic.

Applications

To encode classical van Kampen diagrams in our diagrams:

- Let S be a zero-direct product of copies of $S^{(3)} = \{A, A^{-1}, 1, 0\}$, one for each free generator. (This enforces that all red vertices have valency two.)
- Map from classical vKD to ours: original vertices become black vertices; replace each labelled edge with path length two containing a red vertex, with labels in each corner.

We also "know" how to encode:

- Diagrams for relative presentations;
- Diagrams for quotients of free products of free and finite groups;
- Diagrams for cancellative monoids;
- Computations of non-deterministic Turing machines.

Pre-processing: first gear

Input: $S, \mathcal{R}$.
First find pieces,

- store elts of $\mathcal{R}$ rotated to lex minimum,
- read forwards through R_1, backwards through R_2, maintaining validity of internal vertices,
- also bound valency of end vertices: $3, 4, 5, 6$, more?

For nice S, we "have" $O(n^2)$ algorithms for edges, where n is sum of relator lengths.

Now diagrams have all edges labelled by pieces:

- edges now have different lengths,
- vertices have valency at least three,
- denote the new set of sides of edges in a diagram by $\hat{E}$.

Combinatorial curvature

Given a diagram, we endow

- each vertex with $+1$ unit of combinatorial curvature,
- each edge with -1 unit of combinatorial curvature and
- each internal face with $+1$ unit of combinatorial curvature.

Euler's formula

The total sum of the combinatorial curvature in a diagram is $+1$.

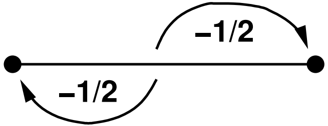
Curvature redistribution

Idea (Officers)

We redistribute the curvature locally in a conservative way.
We call a curvature redistribution scheme an officer.

"Officer Tom":

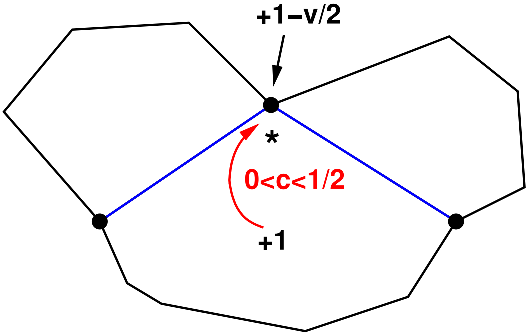
Phase 1: Tom moves the negative curvature to the vertices:



Any vertex in any diagram with valency v (≥ 3) now has curvature $+1 - \frac{v}{2} < 0$.
All internal faces still have $+1$, all edges now have 0.

Phase 2 of Tom

Tom now moves the positive curvature from faces to vertices:



Corner values for Tom

Corner value c depends on two edges that are adjacent on a face. Tom moves c units of curvature to the vertex v.
Default values for c: $1/6$ if v might have valency 3, and $1/4$ otherwise. (Tweak for external faces).

What do officers achieve?

Officers try to redistribute the curvature, such that for all permitted diagrams, after redistribution

- every edge has 0 curvature,
- every vertex has ≤ 0 curvature,
- every internal face has $< -\varepsilon$ curvature (for some explicit $\varepsilon > 0$),
- every face with more than one external edge has ≤ 0 curvature.

Consequence:

All positive curvature is on faces touching the boundary once.

Enables proof of a Greendlinger-type lemmma.

Using curvature

All curvature has been left on the faces.

- The total positive curvature $\leq n$ (boundary length).
- Let $F := \#$internal faces, then

$$1 < n - F \cdot \varepsilon \quad \Longrightarrow \quad F < \varepsilon^{-1} \cdot (n - 1) \quad \Longrightarrow \quad \text{hyperbolic}$$

(Can improve the constant by analysing faces that touch boundary once: One-dimensional analysis).

An example: Classical Small Cancellation

Consider a classical $C'(1/4) - T(4)$ small cancellation presentation $\mathcal{P} = \langle X | \mathcal{R} \rangle$, relators freely cyclically cancelled, inverse closed.

Set all corner values to be $1/4$, unless face is external, in which case $1/3$ for external corners.

Each permitted internal vertex has valency $v \geq 4$, so finishes with curvature $1 - v/2 + v/4 = 1 - v/4 \leq 0$.

Each internal face has at least 5 edges, so finishes with curvature $\leq -1/4$.
Each external face that touches the boundary more than once has at least 2 external corners, so finishes with curvature ≤ 0.

All our officers generalise all small cancellation conditions that imply hyperbolicity.

Analysing curvature

Don't want to analyse all ways of bounding each face by cyclic sequence of edges.
We can't analyse all vertices!

Let $L := \{1, 2, \dots, \ell\}$ and $a_1, a_2, \dots, a_\ell \in \mathbb{R}$ and $T := \sum_{m \in L} a_m$.
Define $\pi_L : \mathbb{Z} \to L$ such that $z \equiv \pi_L(z) \pmod{\ell}$.

Lemma (Goes up and stays up)

If $T \geq 0$ then $\exists\, j \in L$ s.t. for all $i \in \mathbb{N}$ the partial sum

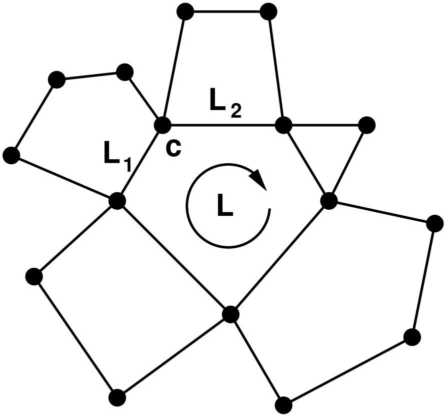
$$t_{j,i} := \sum_{m=0}^{i-1} a_{\pi_L(j+m)} \geq 0.$$

Corollary

Assume that there are $k \in \mathbb{N}$ and $\varepsilon \geq 0$ such that for all $j \in L$ there is an $i \leq k$ with $t_{j,i} < -\varepsilon$, then $T < -\varepsilon \cdot \ell / k$.

Sunflower

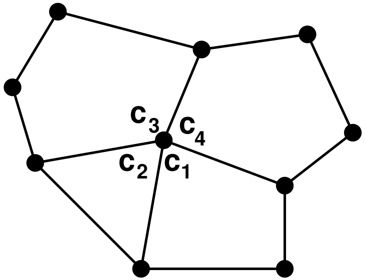
To show that every internal face has curvature $< -\varepsilon$:



Use corollary of Goes up and stays up on $\frac{L_1+L_2}{2L} - c$: only need to consider next corner if this is positive.

Poppy

To show that every internal vertex has curvature ≤ 0:



Use Goes up and stays up on $c_1 + \frac{1 - v/2}{v} = c_1 + \frac{2-v}{v}$.
Only consider next corner if this is positive.

This terminates: higher valencies tend to be negatively curved.

What does Tom achieve?

If Tom found no bad sunflowers or poppies, we have

- determined an explicit ϵ s.t. all internal faces have curvature $\leq \epsilon$,
- proved an explicit isoperimetric inequality, and
- can in principle solve the diagram boundary problem.

If we did find bad sunflowers or poppies, we can

- improve our choices for the corner values (can lead to difficult optimisation/linear program problems),
- forbid more diagrams (if possible) (need to show that every boundary is proved by a permitted one),
- or switch to a more powerful officer (with further sight or redistribution), . . .

and try again.