Groups associated with dynamical systems generalizing Higman-Thompson groups

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Partial self-coverings

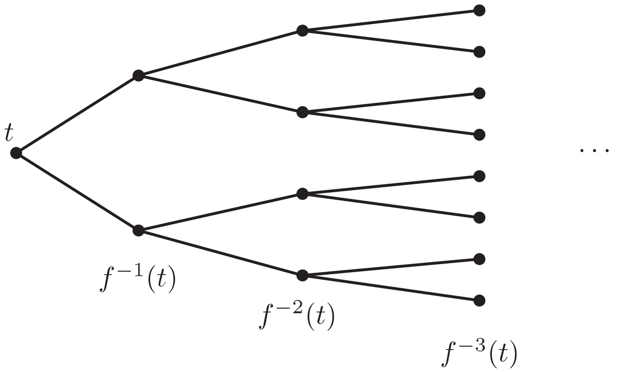
Let $\mathcal{M}$ be a topological space, and let $\mathcal{M}_1 \subseteq \mathcal{M}$. Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be a finite degree covering map.

Examples: $z \mapsto z^2$ acting on the unit circle ($\mathcal{M}_1 = \mathcal{M}$).

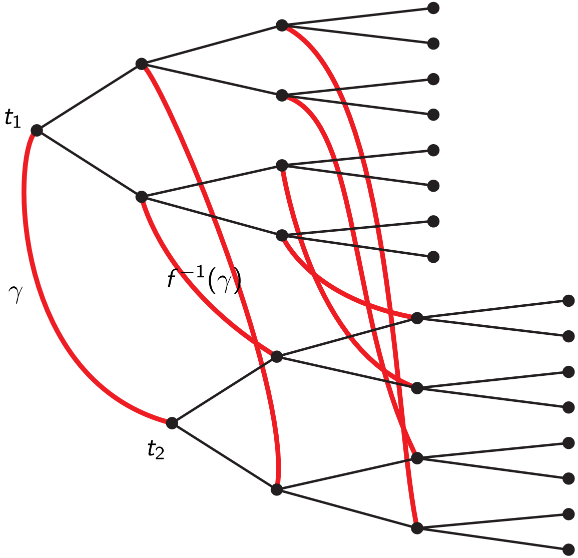
$z \mapsto z^2 - 1$ for $\mathcal{M}_1 = \mathbb{C} \setminus \{0, 1, -1\}$ and $\mathcal{M} = \mathbb{C} \setminus \{0, -1\}$.

Let $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ be as above. For a point $t \in \mathcal{M}$, define the *tree of preimages*

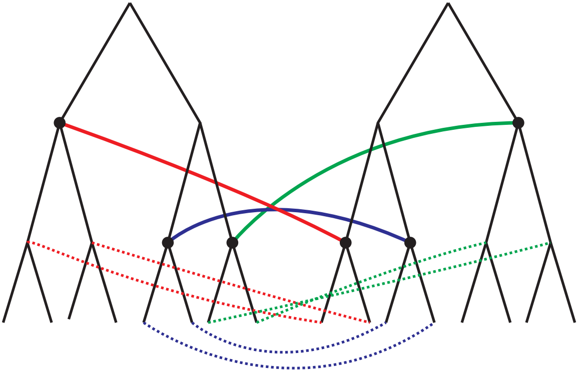
$$T_t = \bigsqcup_{n \geq 0} f^{-n}(t).$$



Let γ be a path from t_1 to t_2. Lifting γ by iterations of f we get an isomorphism $S_\gamma : T_{t_1} \longrightarrow T_{t_2}$.



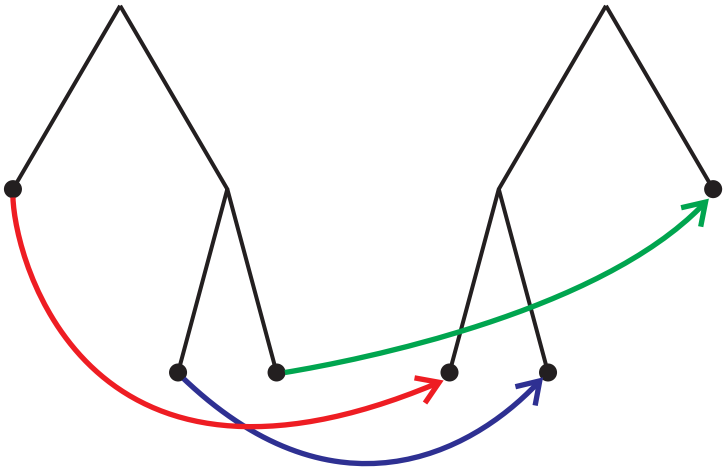
Choose a basepoint $t \in \mathcal{M}$. Consider two maximal antichains $A_1, A_2 \subset T_t$ of equal cardinalities. Choose a bijection $\alpha : A_1 \longrightarrow A_2$ and a collection of paths γ_a from $a \in A_1$ to $\alpha(a) \in A_2$. These paths define isomorphisms $S_{\gamma_a} : T_a \longrightarrow T_{\alpha(a)}$. The union of the maps S_{γ_a} is an "almost automorphism" $S_{\{\gamma_a \,:\, a \in A_1\}}$ of the tree T_t.

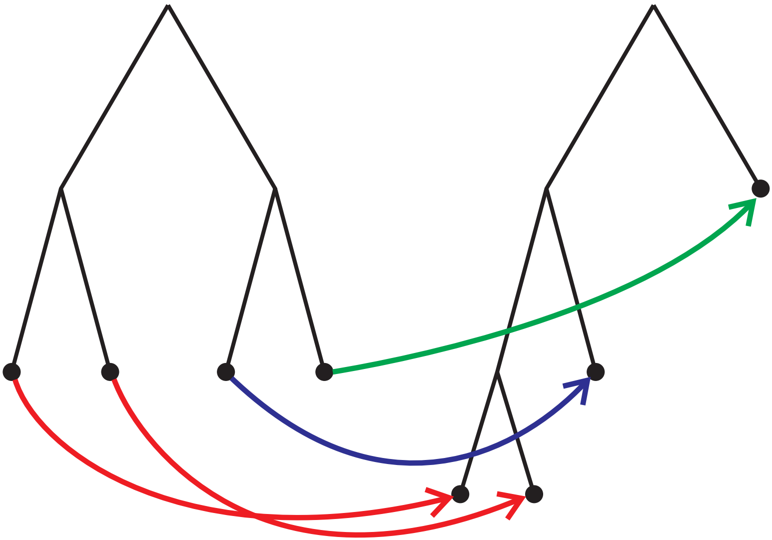


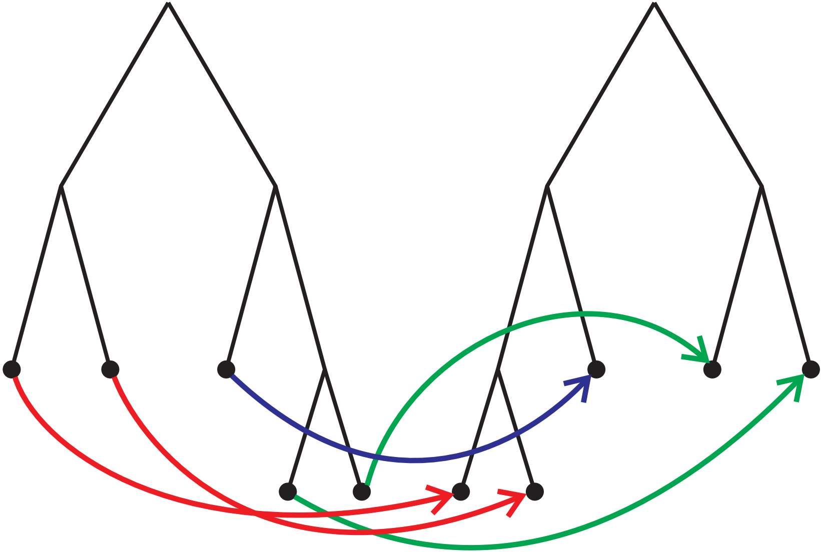
Groups V_f

The set of all such almost automorphisms of T_t is a group, if we identify two almost automorphisms that agree on all but a finite number of vertices. Denote it by V_f.

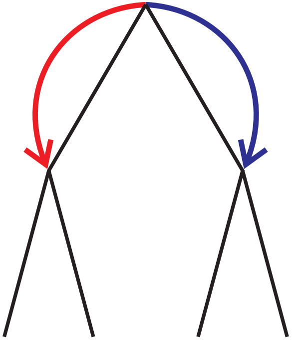
Similarly, we can consider the inverse semigroup P_f generated by finite unions of maps of the form S_γ.



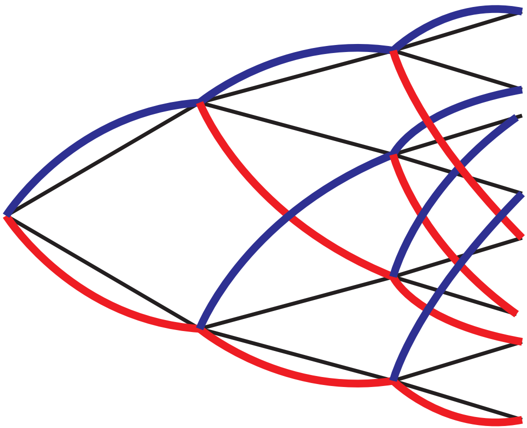




For every $z \in f^{-1}(t)$ choose a path γ_z from t to z. Let $S_z = S_{\gamma_z}$ be the corresponding element of P_f. The elements S_z satisfy the *Cuntz algebra relations*: $S_z^* S_z = 1$, $\sum_{z \in f^{-1}(t)} S_z S_z^* = 1$.



A product $S_{z_1}S_{z_2}\cdots S_{z_n}$ is equal to S_γ for a path γ starting at t and ending in a vertex of the nth level of T_t. We get an isomorphism between the tree of finite words X^* over the alphabet $X = f^{-1}(t)$ and T_t. Denote $S_{z_1 z_2 \ldots z_n} = S_{z_1}S_{z_2}\cdots S_{z_n}$.



Higman-Thompson subgroup

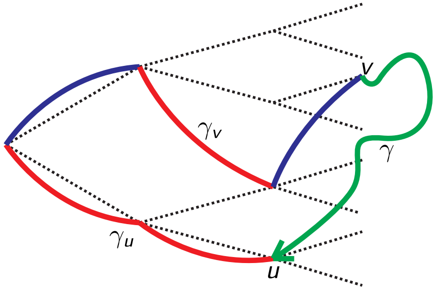
The tree X^* consists of finite words over X, where a word v is connected to vx for $x \in X^*$. The transformations S_x are the *creation operators*: $S_x(v) = xv$. Their inverses are the *annihilation operators*: $S_x^*(xv) = v$.

If $A_1, A_2 \subset X^*$ are maximal antichains, and $\alpha : A_1 \longrightarrow A_2$ is a bijection, then the union $g = \sum_{v \in A_1} S_{\alpha(v)} S_v^*$ is an element of V_f. It acts on (finite or infinite) words by the rule

$$g(vw) = \alpha(v)w.$$

The set of such elements is naturally isomorphic to the Higman-Thompson group $G_{d,1}$ for $d = \deg f$.

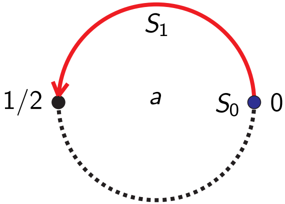
Let γ be a path in $\mathcal{M}$ from v to $u \in X^*$. Let γ_v and γ_u be such that $S_v = S_{\gamma_v}$ and $S_u = S_{\gamma_u}$. Then $\gamma' = \gamma_u^{-1}\gamma\gamma_v \in \pi_1(\mathcal{M}, t)$, and $S_\gamma = S_u S_{\gamma'} S_v^*$. It follows that V_f is generated by $G_{n,1}$ and an image of $\pi_1(\mathcal{M}, t)$.



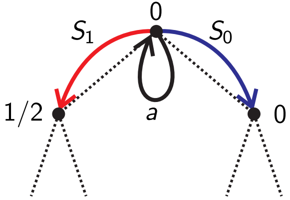
Examples

Let f be the map $x \mapsto 2x$ acting on the circle $\mathbb{R}/\mathbb{Z}$. Then V_f is generated by a copy of $V = G_{2,1}$ and a copy of $\mathbb{Z} = \pi_1(\mathbb{R}/\mathbb{Z})$.

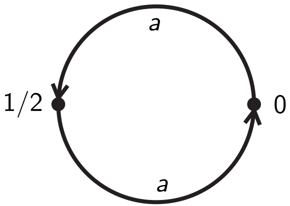
Choose the basepoint $t = 0$. Then $f^{-1}(t) = \{0, 1/2\}$. Let S_0 be the trivial path at 0, and let S_1 be the image of the interval $[0, 1/2]$. Let a be the generator of $\pi_1(\mathbb{R}/\mathbb{Z}, 0)$ equal to the image of $[0, 1]$.

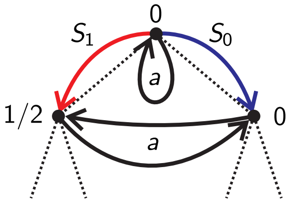


S_1
1/2
a
S_0
0



0
S_1
S_0
1/2
a
0





$aS_0 = S_1$, $aS_1 = S_0 a$, $a = S_1 S_0^* + S_0 a S_1^*$.

If f is $z \mapsto z^2 - 1$ seen as a map $\mathbb{C} \setminus \{0, \pm 1\} \longrightarrow \mathbb{C} \setminus \{0, -1\}$, then V_f is generated by V and two elements a, b satisfying

$$a = S_1 S_0^* + S_0 b S_1^*, \qquad b = S_0 S_0^* + S_1 a S_1^*.$$

Iterated monodromy group

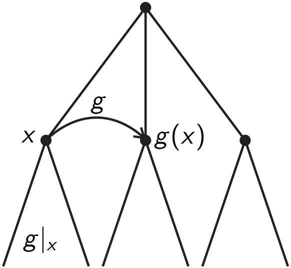
The subgroup of V_f generated by $\pi_1(\mathcal{M}, t)$ is called the *iterated monodromy group* of f, denoted $\mathrm{IMG}\,(f)$.

Typically, it is infinitely presented, and often has exotic properties (e.g., intermediate growth, non-elementary amenable).

For example, $\mathrm{IMG}\,(z^2 - 1)$ is the first example of an amenable group that can not be constructed from groups of sub-exponential growth (R. Grigorchuk-A.Žuk, L. Bartholdi-B. Virag). $\mathrm{IMG}\,(z^2 + i)$ has intermediate growth (K.-U. Bux, R. Perez).

For every $g \in \mathrm{IMG}(f)$ and $x \in X$ there exist unique $y \in X$ and $h \in \mathrm{IMG}(f)$ such that $gS_x = S_y h$. We denote $y = g(x)$ and $h = g|_x$. Then

$$g = \sum_{x \in X} S_{g(x)} g|_x S_x^*.$$



Definition

A group G of automorphisms of the tree X^* is said to be *self-similar* if for every $x \in X$ there exists $h \in G$ such that

$$g(xw) = g(x)h(w)$$

for all $w \in X^*$.

Let G be a self-similar group acting on X^*. Denote by V_G the group generated by G and the Higman-Thompson group $G_{|X|,1}$ naturally almost acting on X^*.

Theorem (C. Röver, 2002)

Let G be the Grigorchuk group. Then V_G is isomorphic to the commensurizer of G, is finitely presented, and simple.

For every self-similar group G, the group V_G has simple commutator subgroup.

Expanding maps

The map $f : \mathcal{M}_1 \longrightarrow \mathcal{M}$ is said to be *expanding* (or *hyperbolic*) if there exists a compact set $J \subset \mathcal{M}_1$ (its *Julia set*) such that $f(J) = f^{-1}(J) = J$, and a metric d on J such that f is uniformly locally expanding on J. (There exist $\epsilon, C > 0$ and $L > 1$ such that $d(f^{-n}(x), f^{-n}(y)) \leq CL^{-n}d(x, y)$ for all x, y such that $d(x, y) < \epsilon$.)

If f is hyperbolic, then $\mathrm{IMG}\,(f)$ is *contracting*, i.e., there exists a finite set $N \subset G$ such that for every element $g \in \mathrm{IMG}\,(f)$ there exists n_0 such that $g = \sum_{v \in X^n} S_{g(v)} g|_v S_v^*$ for $g|_v \in N$ for all $n \geq n_0$.

Theorem

Suppose that G is a contracting self-similar group. Then V_G is finitely presented. In particular, if f is hyperbolic, then V_f is finitely presented.

Open questions: are contracting groups amenable? It is known that $\mathrm{IMG}(f)$, for f a post-critically finite polynomial, are amenable (L. Bartholdi, V. Kaimanovich, V.N.).

When are contracting groups finitely presented? (Conjecturally, only when they are virtually nilpotent.)

Theorem

Let f_1, f_2 be hyperbolic maps with connected Julia sets. Then V_{f_1} and V_{f_2} are isomorphic if and only if (f_1, J_1) and (f_2, J_2) are topologically conjugate.

Ideas of the proof: It follows from M. Rubin's theorem that every isomorphism between V_{f_i} comes from a homeomorphism between the boundaries X^ω of the corresponding trees. Action of V_{f_i} on X^ω generates a *hyperbolic groupoid*. In particular, the graphs of the action on V_{f_i} on generic orbits are Gromov-hyperbolic. Their boundaries are one-point compactification of an infinite "zoom" of J_i. A naturally defined *dual groupoid* acting on the boundary is equivalent to the groupoid generated by $f_i : J_i \longrightarrow J_i$.

Rabbit and Airplane

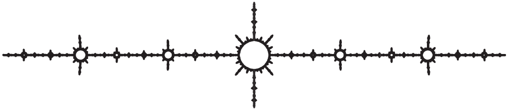
Consider the groups V_f associated with complex quadratic polynomials whose critical point belongs to a cycle of length three. There exists two such groups. Rabbit:

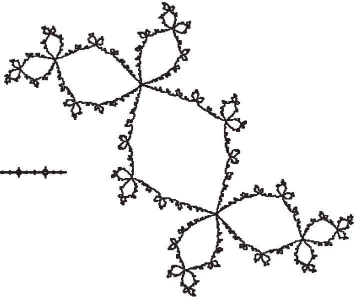
$$a = S_1 S_0^* + S_0 b S_1^*, \quad b = S_0 S_0^* + S_1 c S_1^*, \quad c = S_0 S_0^* + S_1 a S_1^*.$$

Airplane:

$$a = S_1 S_0^* + S_0 b S_1^*, \quad b = S_0 S_0^* + S_1 c S_1^*, \quad c = S_0 a S_0^* + S_1 S_1^*.$$

They are not isomorphic, since the corresponding Julia sets are not homeomorphic.





Cuntz-Pimsner algebras

C^*-algebras $\mathcal{O}_f$ of operators on a Hilbert space generated by operators S_x, unitaries $g \in \mathrm{IMG}(f)$, subject to Cuntz algebra relations $S_x^* S_x = 1$, $\sum_{x \in X} S_x S_x^* = 1$, and relations of the form $gS_x = S_y h$, are examples of *Cuntz-Pimsner algebras*.

Theorem

Let f be hyperbolic. Then $\mathcal{O}_f$ is simple, nuclear (amenable), finitely presented, and are classified by their K-theory (belong to the class of Kirchberg-Phillips).

One can show, by computing the K-theory, that $\mathcal{O}_f$ are pairwise isomorphic for all hyperbolic complex quadratic polynomials f.