An introduction to the Virtual Element Method

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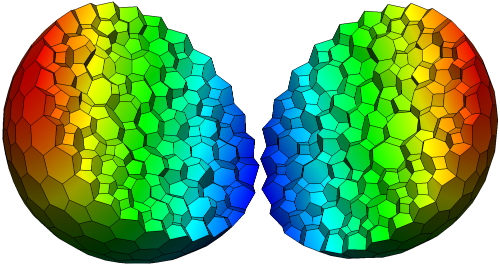
The Virtual Element Method

The Virtual Element Method (VEM) is a generalization of the Finite Element Method that takes inspiration from modern Mimetic Finite Difference schemes.



- VEM allow to use very general polygonal and polyhedral meshes, also for high polynomial degrees and guaranteeing the patch test.
- The flexibility of VEM is not limited to the mesh: an example will be shown later.

Why polygons/polyhedrons?

The interest (and use in commercial codes*) for polygons/polyhedra is recently growing.

- Immediate combination of tets and hexahedrons
- Easier/better meshing of domain (and data) features
- Automatic inclusion of "hanging nodes"
- Adaptivity: more efficient mesh refinement/coarsening
- Generate meshes with more local rotational simmetries
- Robustness to distortion
- ....

$\star$ for example CD-ADAPCO and ANSYS.

Some polytopal methods

- Mimetic F.D. Shashkov, Lipnikov, Brezzi, Manzini, BdV, ...
- HMM: Eymard, Droniou, ...
- Polygonal FEM: Sukumar, Paulino, ...
- Weak Galerkin FEM: Wang, ...
- HHO: Ern, di Pietro
- Polygonal DG: Cangiani, Houston, Georgoulis, ...
- VEM: this talk !!!
- ....

The model problem

We consider the Poisson problem in two dimensions

$$-\Delta u \; = \; f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where

- $\Omega \subset \mathrm{R}^2$ is a polygonal domain;
- the loading f is assumed in $L^2(\Omega)$.

Variational formulation:

$$\begin{cases} \text{find } u \in V := H^1_0(\Omega) \text{ such that} \\ a(u,v) \; = \; \int_\Omega f \, v \, \mathrm{d}x \quad \forall v \in V, \end{cases}$$

where

$$a(v,w) = \int_\Omega \nabla v \cdot \nabla w \, \mathrm{d}x \; , \qquad \forall v, w \in V.$$

A Virtual Element Method

We will build a discrete problem in following form

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) =< f_h, v_h > \qquad \forall\, v_h \in V_h, \end{cases}$$

where

- $V_h \subset V$ is a finite dimensional space;
- $a_h(\cdot,\cdot)\ :\ V_h \times V_h \to \mathbb{R}$ is a discrete bilinear form approximating the continuous form $a(\cdot,\cdot)$;
- $< \mathbf{f}_h, \mathbf{v}_h >$ is a right hand side term approximating the load

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Let $m \geq 1$ be a fixed integer index. Such index will represent the degree of accuracy of the method.

The local spaces $V_{h|E}$

Let Ω_h be a simple polygonal mesh on Ω. This can be any decomposition of Ω in non overlapping polygons E with straight faces.

The space V_h will be defined element-wise, by introducing

- local spaces $V_{h|E}$;
- the associated local degrees of freedom.

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For all $E \in \Omega_h$:

$$V_{h|E} = \Big\{ v \in H^1(E) \ : \ -\Delta v \in \mathbb{P}_{m-2}(E), \\ v|_e \in \mathbb{P}_m(e) \quad \forall e \in \partial E \Big\}.$$

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- the functions $v \in V_{h|E}$ are continuous (and known) on ∂E;
- the functions $v \in V_{h|E}$ are unknown inside the element E!
- it holds $\mathbb{P}_m(E) \subseteq V_{h|E}$

Degrees of freedom for $V_{h|E}$

The dimension of the space $V_{h|E}$ is clearly

$$\dim(V_{h|E}) = N_e\, m + m(m-1)/2,$$

with N_e number of edges of E.

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with N_e number of edges of E.

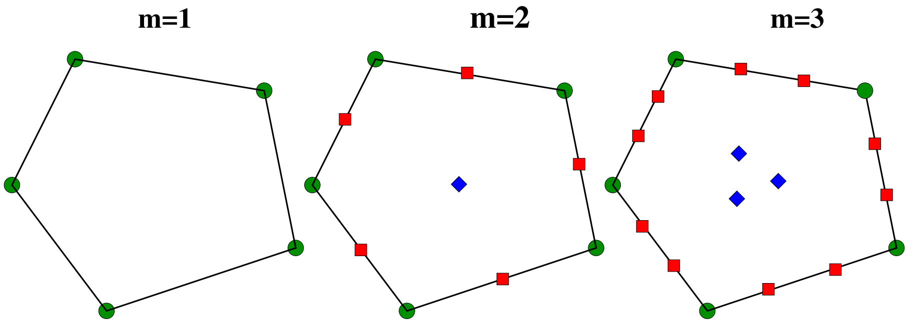
- pointwise values $v_h(\nu)$ at all corners ν of E;
- $(m-1)$ pointwise values on each edge:

$$v_h(x_i^e)\ , \quad \{x_i^e\}_{i=1}^{m-1} \text{ distinct points on edge } e\,;$$

- volume moments:

$$\int_E v_h \cdot p_{m-2} \qquad \forall p_{m-2} \in \mathbb{P}_{m-2}(E)\,.$$

Depiction of the degrees of freedom for $V_{h|E}$



Green dots stand for vertex pointwise values

Red squares represent edge pointwise values

Blue squares represent internal (volume) moments

Degrees of freedom for $V_{h|E}$

The following holds
[BdV,Brezzi,Cangiani,Manzini,Marini,Russo, M3AS 2013].

Proposition

The proposed collection of operators $V_{h|E} \to \mathbb{R}$ constitutes a set of degrees of freedom for $V_{h|E}$, $\forall E \in V_{h|E}$.

- We already know $\#dofs = \dim(V_{h|E})$.
- If $v_h \in V_{h|E}$ is null on all the d.o.f.s, then it clearly vanishes on the boundary.
- The function $\Delta v_h \in \mathbb{P}_{m-2}$ and thus

$$0 = \int_E v_h \cdot \Delta v_h = \int_E (\nabla v_h) \cdot (\nabla v_h).$$

The space V_h

The global space V_h is built by assembling the local spaces $V_{h|E}$ as usual:

$$V_h = \{v \in H^1_0(\Omega) \ : \ v|_E \in V_{h|E} \, \forall E \in \Omega_h\}$$

The total d.o.f.s are one per internal vertex, $m-1$ per internal edge and $m(m-1)/2$ per element.

The choice of degrees of freedom guarantees the global continuity of the functions in V_h.

The bilinear form $a_h(\cdot,\cdot)$

The bilinear form $a_h(\cdot,\cdot)$ is built element by element

$$a_h(v_h, w_h) = \sum_{E\in\Omega_h} a_h^E(v_h, w_h) \quad \forall\, v_h,\, w_h \in V_h,$$

where

$$a_h^E(\cdot,\cdot) \;:\; V_{h|E} \times V_{h|E} \longrightarrow \mathbf{R}$$

are symmetric bilinear forms that mimic

$$a_h^E(\cdot,\cdot) \;\simeq\; a(\cdot,\cdot)|_E$$

by satisfying a stability and a consistency condition.

The stability property

Stability

There exist two positive constants α_* and α^*, independent of h and of E, such that

$$\alpha_* \, a^E(v_h, v_h) \leq a_h^E(v_h, v_h) \leq \alpha^* \, a^E(v_h, v_h) \qquad \forall v_h \in V_{h|E}.$$

- The stability property guarantees that $a_h(\cdot, \cdot)$ is uniformly coercive and continuous;
- clearly, it is sufficient for the (uniform) well posedness of the discrete problem, but not for convergence.

The consistency property

Consistency

For all h and for all $E \in \Omega_h$ it holds

$$a_h^E(p, v_h) = a^E(p, v_h) \qquad \forall p \in \mathbb{P}_m(E), v_h \in V_{h|E}.$$

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For all h and for all $E \in \Omega_h$ it holds

$$a_h^E(p, v_h) = a^E(p, v_h) \qquad \forall p \in \mathbb{P}_m(E), v_h \in V_{h|E}.$$

NOTE: an integration by parts gives

$$\begin{aligned} a^E(p, v_h) &= \int_E \nabla p \cdot \nabla v_h \, \mathrm{d}x \\ &= -\int_E (\Delta p) \, v_h \, \mathrm{d}x + \int_{\partial E} (\nabla p \cdot \mathbf{n}_E) \, v_h \, \mathrm{d}s. \end{aligned}$$

for all $p \in \mathbb{P}_m(E), v_h \in V_{h|E}$.

Therefore the right hand side above is explicitly computable even if we ignore v_h inside E.

The discrete load term

We consider $m \geq 2$ first. Let, for all $E \in \Omega_h$, the approximated load $f_h|_E$ be the L_2-projection of $f|_E$ on $\mathbb{P}_{m-2}(E)$.

Then

$$(f_h, v_h)_h := \sum_{E \in \Omega_h} \int_E f_h v_h \, \mathrm{d}x$$

that is computable due to the internal dofs of $V_{h|E}$.

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that is computable due to the internal dofs of $V_{h|E}$.

In the case $m = 1$ a simple integration rule based on the vertex values of the polygon can be used, for instance

$$(f_h, v_h)_h := \sum_{E \in \Omega_h} \left(\int_E f \, \mathrm{d}x \right) \frac{1}{N_E} \sum_{\nu \in \partial E} v_h(\nu) \, \mathrm{d}x.$$

Note: for $m = 2$ better choices can be made [Ahmad, Alsaedi, Brezzi, Marini, Russo, CMA 2013], [BdV, Brezzi, Marini, SINUM 2013].

A convergence result

Let the sequence $\{\Omega_h\}_h$ satisfy the following mesh assumptions:

- each element E in Ω_h is star-shaped with respect to a ball of uniform radius $($or suitable union of$)$;
- for each element E in Ω_h, the lenght of all edges is comparable with its diameter h_E $($not needed by paying $|log(h_e)|)$.

Then the following holds [... volley team ..., M3AS 2013].

Theorem

Let the stability and consistency assumptions hold. Then, if $f \in H^s(\Omega_h)$ and $u \in H^{s+1}(\Omega_h)$, we have

$$|u - u_h|_{H^1(\Omega)} \leq C\, h^s \left(|u|_{H^{s+1}(\Omega_h)} + |f|_{H^s(\Omega_h)}\right)$$

for $0 \leq s \leq m$ and with C independent of h.

k-plain

$$-\mathrm{div}(K\nabla u) + \alpha u = f$$

exact solution:

$$u_e(x,y) = y - x + \log\left(y^3 + x + 1\right) - x\,y - x\,y^2 + x^2\,y + x^2 + x^3 + \sin(5\,x)\,\sin(7\,y) - 1$$

diffusion:

$$K(x,y) = 1$$

zero-order term:

$$\alpha(x,y) = 1$$

right-hand-side:

$$f(x,y) = \log\left(y^3 + x + 1\right) - y - 5\,x + \frac{1}{\left(y^3 + x + 1\right)^2} + \frac{9\,y^4}{\left(y^3 + x + 1\right)^2} - x\,y - x\,y^2 + x^2\,y + x^2 + x^3 - \frac{6\,y}{y^3 + x + 1} + 75\,\sin(5\,x)\,\sin(7\,y) - 3$$

L2 norm of the exact solution:

$$||u_e||_{0,\Omega} = 0.6431084584$$

H1 seminorm of the exact solution:

$$|u_e|_{1,\Omega} = 5.031264492$$

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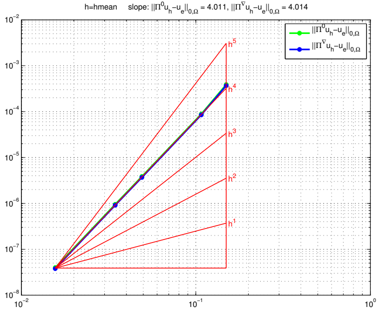
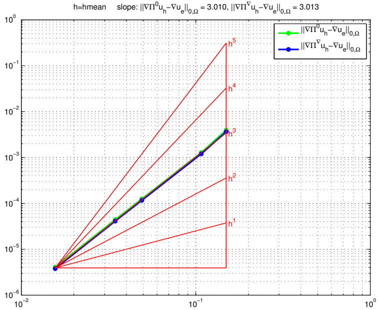
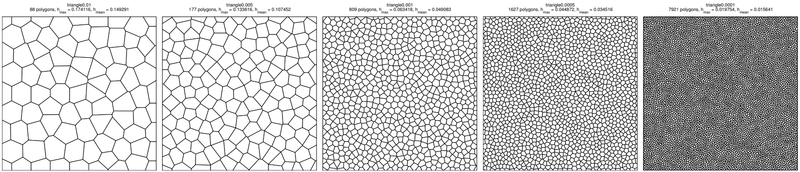


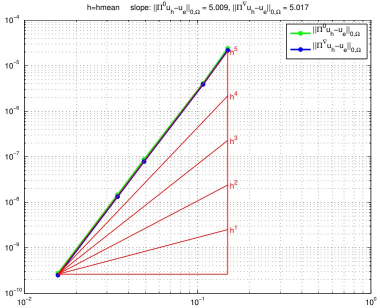
L^2 **error**

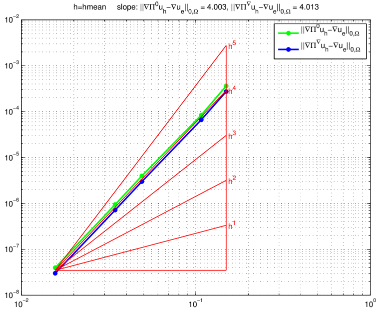
$k = 3$



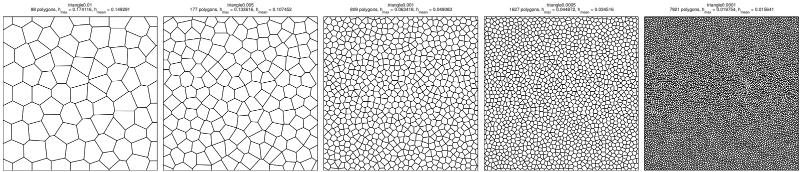
H^1 **error**

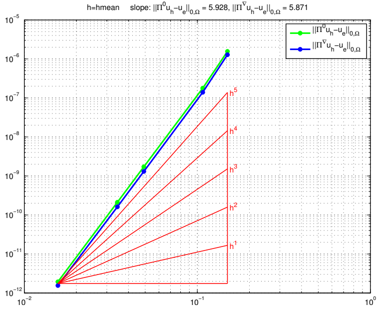






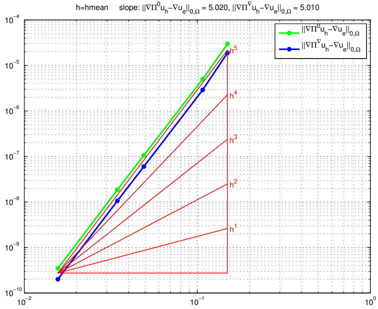
L^2 **error** $k = 4$ H^1 **error**



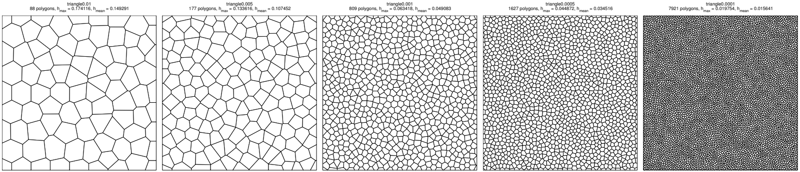


L^2 **error**

$k = 5$



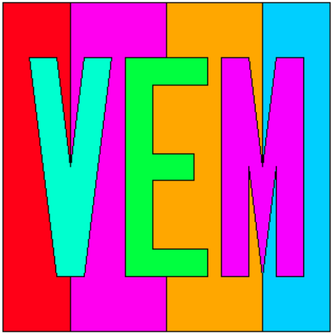
H^1 **error**



Just for fun ...

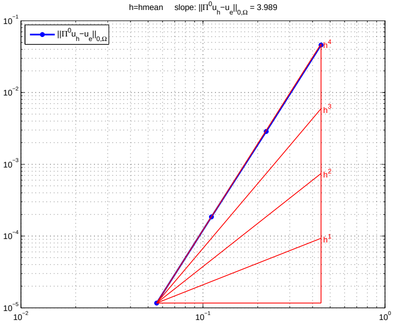
We consider a family of meshes based on the following pattern:

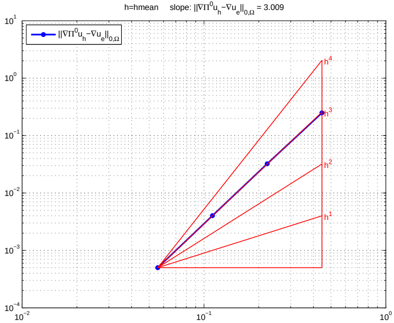
7 polygons



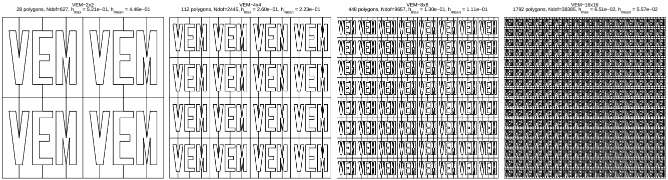
Courtesy of A. Russo !!

$$-\Delta u = f$$





L^2 **error** $k = 3$ H^1 **error**



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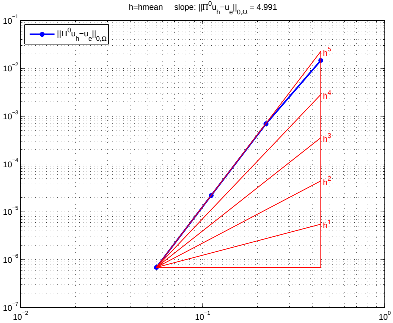
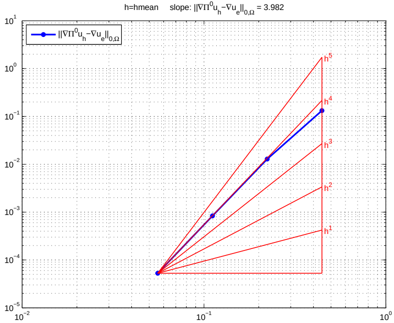
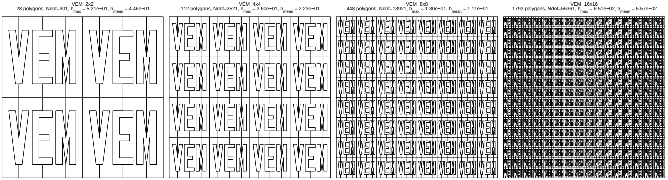
$$-\Delta u = f$$





L^2 **error** $k = 4$ H^1 **error**



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Construction of the stiffness matrix

We introduce the following energy projector

$$\Pi : V_{h|E} \longrightarrow \mathbb{P}_m(E)$$

defined by, for all $v_h \in V_{h|E}$,

$$\begin{cases} a^E(\Pi v_h, p) = a^E(v_h, p) & \forall p \in \mathbb{P}_m(E)/\mathbb{R} \\ P_0(\Pi v_h) = P_0(v_h) \end{cases}$$

The operator P_0 is a projection on constants, that for $m \geq 2$ is simply the average, and for $m = 1$ the vertex value average.

NOTE: due to the consistency assumption, the projection operator above is computable (more later).

Construction of the stiffness matrix

It is immediate to check that $\forall\ v_h, w_h \in V_{h|E}$

$$a^E(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + a^E((I - \Pi)v_h, (I - \Pi)w_h).$$

Then the bilinear form

$$a_h^E(v_h, w_h) = a^E(\Pi v_h, \Pi w_h) + s^E((I - \Pi)v_h, (I - \Pi)w_h)$$

is consistent and stable, provided the positive bilinear form $s^E : V_{h|E} \times V_{h|E} \to \mathbb{R}$ scales like the original bilinear form *a*.

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Let now the local stiffness matrix M^E

$$\mathrm{M}_{ij}^E := a_h^E(\phi_i, \phi_j) \qquad \forall i, j = 1, 2, ..., N$$

with $\{\phi_i\}$ the canonical basis associated to the degrees of freedom.

Construction of the stiffness matrix

We introduce also $\{m_\alpha\}_{\alpha=1}^n$ a basis for the polynomial space

$$\mathbb{P}_m = \mathrm{span}\{m_\alpha\}_{\alpha=1}^n.$$

Since $\mathbb{P}_m \subseteq V_{h|E}$, the matrix

$$\mathsf{D} = \begin{bmatrix} \mathrm{dof}_1(m_1) & \mathrm{dof}_1(m_2) & \dots & \mathrm{dof}_1(m_n) \\ \mathrm{dof}_2(m_1) & \mathrm{dof}_2(m_2) & \dots & \mathrm{dof}_2(m_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{dof}_N(m_1) & \mathrm{dof}_N(m_2) & \dots & \mathrm{dof}_N(m_n) \end{bmatrix}.$$

expresses the $\{m_\alpha\}$ in terms of the $V_{h|E}$ basis.

Construction of the stiffness matrix

We have also a matrix

$$\mathsf{B} = \begin{bmatrix} P_0\phi_1 & \dots & P_0\phi_N \\ a^E(\phi_1, m_2) & \dots & a^E(\phi_N, m_2) \\ \vdots & \ddots & \vdots \\ a^E(\phi_1, m_n) & \dots & a^E(\phi_N, m_n) \end{bmatrix}$$

expressing (in terms of the bases) the right hand side in the definition

$$\begin{cases} a^E(\Pi v_h, p) = a^E(v_h, p) & \forall p \in \mathbb{P}_m(E)/\mathbb{R} \\ P_0(\Pi v_h) = P_0(v_h) \end{cases}$$

NOTE: such matrix is computable! (integration by parts and d.o.f.s definition)

Construction of the stiffness matrix

Let

$$\mathbf{G} = \mathrm{BD} = \begin{bmatrix} P_0 m_1 & P_0 m_2 & \dots & P_0 m_n \\ 0 & (\nabla m_2, \nabla m_2)_{0,\mathrm{P}} & \dots & (\nabla m_2, \nabla m_n)_{0,\mathrm{P}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (\nabla m_n, \nabla m_2)_{0,\mathrm{P}} & \dots & (\nabla m_n, \nabla m_n)_{0,\mathrm{P}} \end{bmatrix}.$$

Construction of the stiffness matrix

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Compute the matrices corresponding to the projection operator:

$$\mathbf{\Pi}_\star = (\mathrm{G})^{-1}\mathrm{B}\,, \qquad \mathbf{\Pi} = \mathrm{D}\mathbf{\Pi}_\star.$$

Construction of the stiffness matrix

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Compute the matrices corresponding to the projection operator:

$$\mathbf{\Pi}_\star = (\mathrm{G})^{-1}\mathrm{B}\,, \qquad \mathbf{\Pi} = \mathrm{D}\mathbf{\Pi}_\star.$$

Finally compute the local stiffness matrix

$$\mathrm{M}^E = \mathbf{\Pi}_\star^T \,\widetilde{\mathrm{G}}\, \mathbf{\Pi}_\star + (\mathbf{I} - \mathbf{\Pi})^T (\mathbf{I} - \mathbf{\Pi}).$$

The three dimensional case

In principle the 3D case is analogous; the degrees of freedom are

- one point value per vertex
- $(m-1)$ point values per edge
- $M_0^f, M_1^f, .., M_{m-1}^f$ moments per face
- $M_0^E, M_1^E, .., M_{m-2}^E$ moments per element.

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- $M_0^f, M_1^f, .., M_{m-1}^f$ moments per face
- $M_0^E, M_1^E, .., M_{m-2}^E$ moments per element.

A more efficient formulation with less degrees of freedom per face

- $M_0^f, M_1^f, .., M_{m-2}^f$ moments per face

can be built using the L^2 projector in [Ahmed, Alsaedi, Brezzi, Marini, Russo, CMA 2013].

Virtual Elements for H_{div}: introduction

Consider the diffusion problem in mixed form

$$\begin{cases} \text{Find } F \in V := H_{\mathrm{div}}(\Omega), p \in Q := L^2(\Omega) : \\ \displaystyle\int_\Omega F \cdot G + \int_\Omega (\mathrm{div} G) p = 0 \qquad \forall G \in V, \\ \displaystyle\int_\Omega (\mathrm{div} F) q = -\int_\Omega f\, q \qquad \forall q \in Q. \end{cases}$$

We introduce the VEM spaces:

$$Q_h = \{ q \in L^2(\Omega) \; : \; q|_E \in \mathbb{P}_{k-1}(E) \, \forall E \in \Omega_h \} \subset Q,$$
$$V_h = \{ G \in H_{\mathrm{div}}(\Omega) \; : \; G|_E \in V_{h|E} \, \forall E \in \Omega_h \} \subset V.$$

What follows taken from [BdV, Brezzi, Marini, Russo, submitted].
(se also [Brezzi, Falk, Marini, M2AN, 2014])

Virtual Elements for $H_{\rm div}$: local spaces

Let $E \in \Omega_h$. We introduce the local VEM space

$$V_{h|E} = \Big\{ G \in H_{\rm div}(E) \cap H_{\rm rot}(E) \ : {\rm div} G \in \mathbb{P}_{k-1}(E),\ {\rm rot} G \in \mathbb{P}_{k-1}(E),$$
$$G_{|e} \cdot \mathbf{n}_E^e \in \mathbb{P}_k(e)\ \forall e \in \partial E \Big\}.$$

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This space is associated to the problem

$$\begin{cases} {\rm div} G = f_1 \ , \quad {\rm rot} G = f_2 & \text{on } E, \\ G \cdot \mathbf{n} = f_\partial & \text{on } \partial E \ . \end{cases}$$

that is well posed if $\int_E f_1 = \int_{\partial E} f_\partial$.

Thus

$$\dim(V_{h|E}) \ = \ 2 \dim(P_{k-1}(E)) + N_e \dim(P_k(e)) - 1 \ .$$

Virtual Elements for H_{div}: local degrees of freedom

- edge moments

$$\int_e (G \cdot \mathbf{n}_E^e) p_k \quad \forall p_k \in \mathbb{P}_k(e), \ \forall e \in \partial E \, .$$

- div volume moments:

$$\int_E (\mathrm{div} G) p_{k-1} \qquad \forall p_{k-1} \in \mathbb{P}_{k-1}(E)/\mathbb{R} \, .$$

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- additional volume moments:

$$\int_E G \cdot \mathbf{p}_k \qquad \forall \mathbf{p}_k \in \mathcal{G}_k^{\perp}(E)\,.$$

The space

$$\mathcal{G}_k^{\perp} = \{\mathbf{p} \in \mathbb{P}_k \ : \ \int_E \mathbf{p} \cdot \nabla q = 0 \ \forall q \in \mathbb{P}_{k+1}(E)\}.$$

has dimension equal to $\mathbb{P}_{k-1}(E)$.

Virtual Elements for H_{div}: local degrees of freedom

Unisolvence. Let $G \in V_{h|E}$, null on all dofs. Since

$$\bullet \, \mathrm{rot} : \ \mathcal{G}_k^{\perp} \to \mathbb{P}_{k-1} \text{ is a bijection,}$$
$$\bullet \, \mathrm{rot} G \in \mathbb{P}_{k-1},$$

it exists $\varphi \in \mathcal{G}_k^{\perp}$ such that

$$0 = \mathrm{rot}(G - \varphi) \quad \Longrightarrow \quad G = \nabla\psi + \varphi \text{ with } \psi \in H^1(E).$$

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Thus

$$||G||^2_{L^2(E)} = \int_E G\,(\nabla\psi + \varphi)$$
$$= -\int_E (\text{div} G)\psi + \int_{\partial E} (G \cdot \mathbf{n}_E)\psi + \int_E G\,\varphi = 0.$$

Virtual Elements for H_{div}: computing the L^2 projection

- the first set of dofs determines $G \cdot \mathbf{n}$ on ∂E;
- since $\mathrm{div} G \in \mathbb{P}_{k-1}(E)$, the second set of dofs determines $\mathrm{div} G$.
- therefore we can compute

$$\int_E G \, \nabla\psi = - \int_E (\mathrm{div} G)\psi + \int_{\partial E} (G \cdot \mathbf{n}_E)\psi \quad \forall \psi \text{ polynomial;}$$

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$$\int_E G \, \nabla\psi = -\int_E ({\rm div} G)\psi + \int_{\partial E} (G \cdot \mathbf{n}_E)\psi \quad \forall \psi \text{ polynomial};$$

- any $\mathbf{q} \in [\mathbb{P}_k(E)]^2$ can be written as

$$\mathbf{q} = \mathbf{p} + \nabla\psi \,, \quad \mathbf{p} \in \mathcal{G}_k^{\perp}, \; \psi \in \mathbb{P}_{k+1}(E).$$

Thus we can compute

$$\int_E G \cdot \mathbf{q} = \int_E G \cdot \mathbf{p} + \int_E G \cdot \nabla\psi.$$

H_{div} Virtual Elements : final observations

The proposed Virtual spaces (V_h, Q_h) satisfy a commuting diagram property.

Thus are suitable for the approximation of the problem:

$$\begin{cases} \text{Find } F_h \in V_h,\ p \in Q_h : \\ \int_\Omega F_h \cdot G_h + \int_\Omega (\mathrm{div} G_h) p_h = 0 & \forall G_h \in V_h, \\ \int_\Omega (\mathrm{div} F_h) q_h = \int_\Omega f\, q_h & \forall q_h \in Q_h. \end{cases}$$

NOTE: with the choices that we made, everything above is computable (up to the usual VEM construction and using the local L^2 projections).

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VEM exact sequences

A full "Safari" of VEM to appear in [BdV, Brezzi, Marini, Russo].

An application: the Cahn-Hilliard equation

- With standard finite elements it is very complicated to build spaces with global regularity higher than C^0.
- With VEM, this is instead easy to achieve. We can build elements with arbitrary C^k regularity:

  [Brezzi and Marini, CMAME, 2013]: C^1 VEM for Kirchhoff plates

  [BdV, Manzini, IMA J. Num. An. 2013]: C^k VEM for diffusion.
- We will here show some "spoiler" from a paper in collaboration with Antonietti, Scacchi, Verani for applications to the Cahn-Hilliard equation for phase transition.

The Cahn-Hilliard equation

We search for $u : \Omega \times [0, T] \to \mathbb{R}$ such that:

$$\begin{cases} \partial_t u - \Delta\big(\phi'(u) - \gamma^2 \Delta u(t)\big) = 0 & \text{in } \Omega \times [0, T] \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega \\ \partial_n u = \partial_n\big(\phi'(u) - \gamma^2 \Delta u(t)\big) = 0 & \text{on } \partial\Omega \times [0, T], \end{cases}$$

where the function $\phi(x) = (1 - x^2)^2/4$ and $\gamma \in \mathbb{R}^+$ "small".

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where the function $\phi(x) = (1 - x^2)^2/4$ and $\gamma \in \mathbb{R}^+$ "small".

The natural variational space is $H^2(\Omega)$, thus a C^1 regularity is needed for a conforming method.

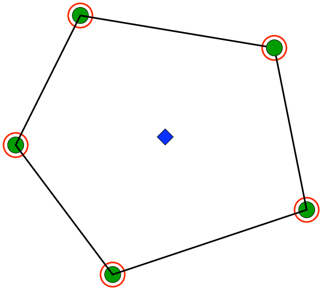
- Mixed DG (Kay, Styles, Suli)
- Morley element (Elliott)
- Isogeometric Analysis (Hughes and co-workers)
- ....

C^1 VEM elements (of minimal degree)

$$V_{h|E} = \Big\{ v \in H^2(E) \ : \ -\Delta^2 v \in \mathbb{P}_0(E),$$
$$v_{|e} \in \mathbb{P}_3(e),\ \partial_{\mathbf{n}} v_{|e} \in \mathbb{P}_1(e) \quad \forall e \in \partial E \Big\}.$$

Degrees of freedom:

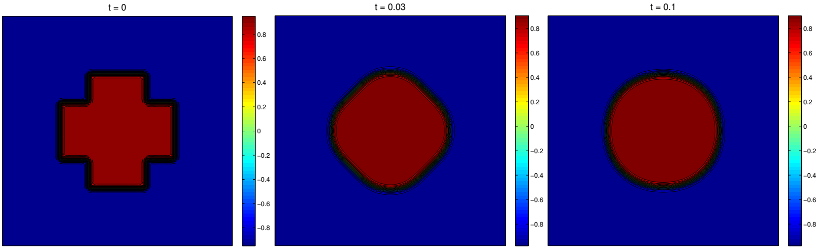
k=1, p=2



Some numerical result

We apply a primal VEM C^1 discretization to the problem:

- it involves Π^0, Π^∇ and Π^Δ projections;
- it grants a conforming solution and accepts general polygons;
- theoretical convergence estimates hold;
- initial numerical tests are encouraging.



More VEM literature

More VEM literature not mentioned in the previous slides:

- Virtual Elements for linear elasticity problems
  [BdV, Brezzi, and Marini, SINUM, 2013]
- A stream function formulation for Stokes
  [Antonietti, BdV, Mora, Verani, SINUM, 2014]
- Three dimensional compressible elasticity
  [A.L. Gain, C. Talischi, G.H. Paulino, CMAME, 2014]
- IN PROGRESS: nonconforming elements (Ayuso, Lipnikov, Manzini), eigenvalue problems (Mora, Rodriguez), discrete fracture network (Berrone at al.), contact problems (Wriggers et al.), topology optimization (Paulino et al.), etc..

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Regarding VEM implementation:

On M3AS: The hitchhikers guide to VEM, a paper all about VEM implementation.

Conclusions

- The Virtual Element Method is a generalization of FEM that takes inspiration from modern mimetic schemes
- The freedom that is left to the local spaces allows for a large flexibility, for instance in terms of meshes (polygons, “hanging nodes”), global regularity of the discrete space, definition of the local matrixes (M-optimization), etc ..
- A lot of development is still to be done in VEM, and we believe it can be a very interesting new field of research.
- Morever, more complex coding and problems need to be challenged in order to assess the impact of VEM in applications (various nonlinear problems already under development ...).