Isogeometric mortaring

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IGA is based on spline theory

B-Splines are defined by the Cox-DeBoor formulae:

$$N_{i,0}(\zeta) = \begin{cases} 1 & \text{if } \xi_i \leq \zeta < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{i,p}(\zeta) = \frac{\zeta - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\zeta) + \frac{\xi_{i+p+1} - \zeta}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\zeta).$$

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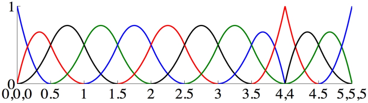

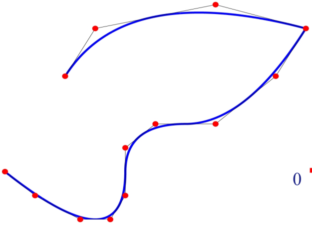

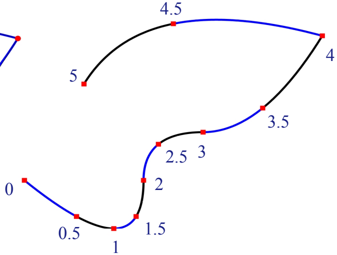

• - control points ▪ - knots

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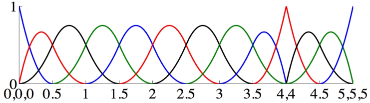

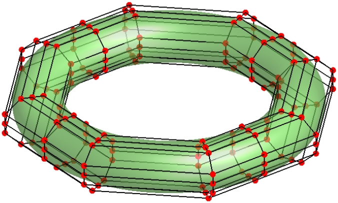

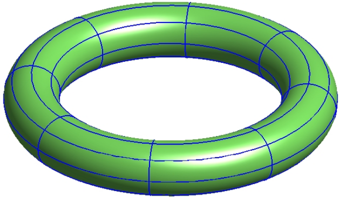

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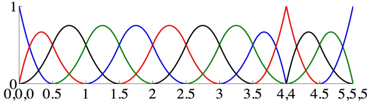
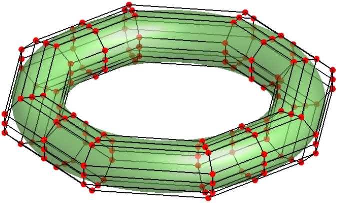
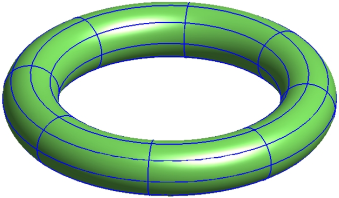
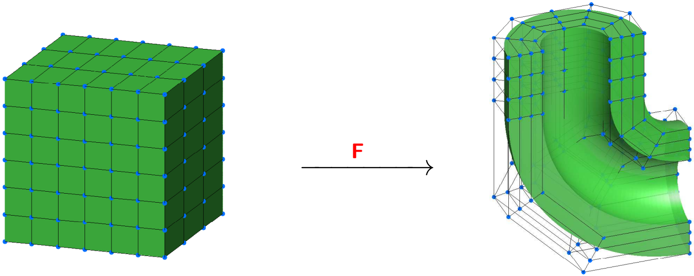






NURBS: projection of splines in $\mathbf{R}^{d+1}$ ... no need for this talk.

Construction of approximation spaces



- The geometry Ω and its NURBS parametrization **F** is "given" by CAD
  general geometry: unstructured collection of "patches".

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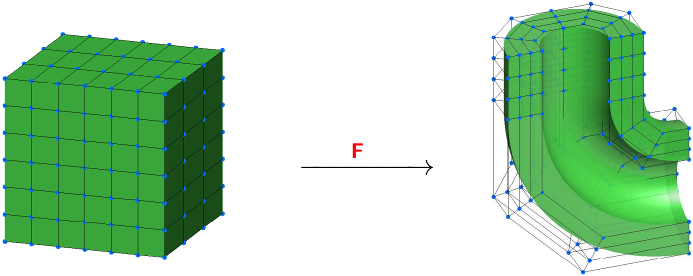

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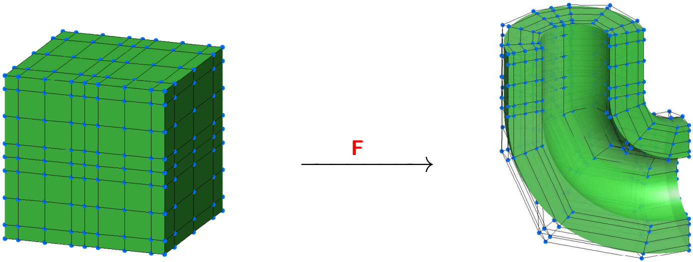

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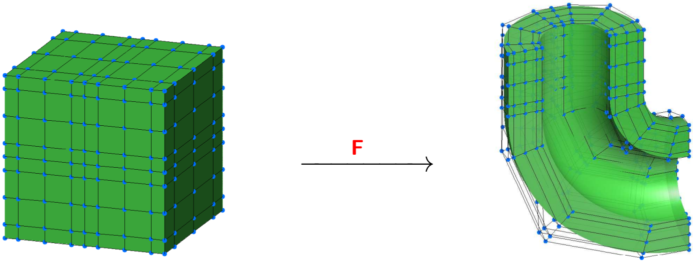

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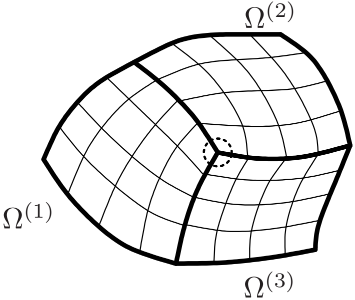

Globally unstructured
Locally structured

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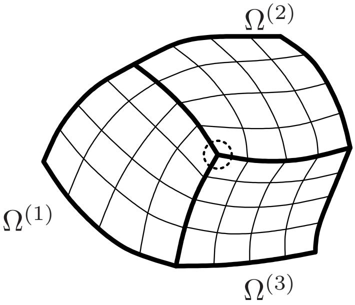

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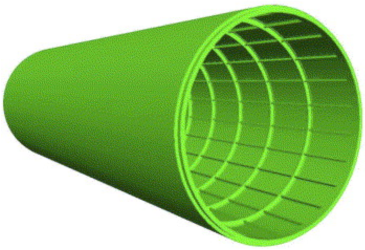

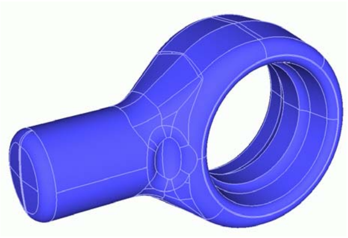

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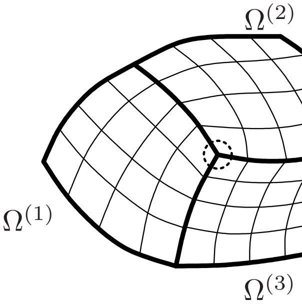
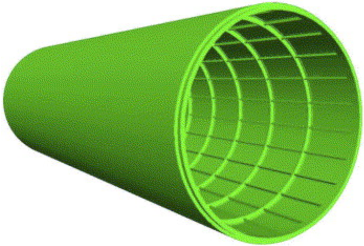
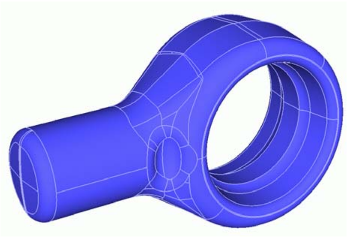






Globally unstructured
Locally structured

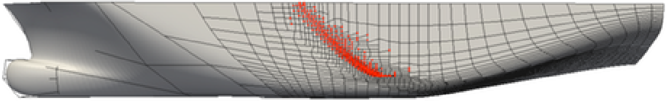
Question: How to enhance flexibility?
Question: How to treat non conforming interfaces?

Perfect setting for adaptivity

if splines can support local refinement

- **T-splines** : Sederberg et al 2004, Hughes, Scott, Evans, Li, Zhang, ... Pavia team

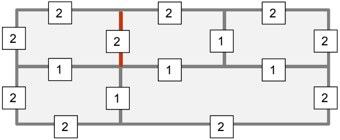
  Pure geometric modeling approach:



courtesy of M. Scott

- **LR-splines** Dokken et al. 2013, Bressan 2013

  Definition $\mathbb{R}^n$, spline theory of LR-Splines

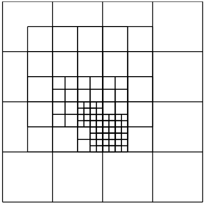


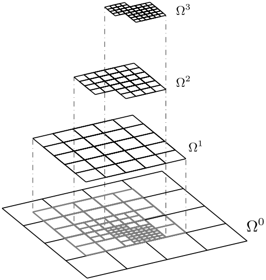
- **Hierarchical splines** Kraft 1998, ...

  the closest to adaptive finite elements on quadrangular meshes

Hierarchical splines

Kraft 1998, Giannelli, Jüttler, Simeon, Speelers, Voung 2010–2013, B.-Giannelli 2014





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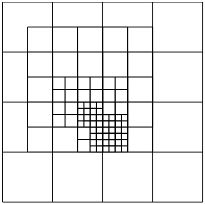

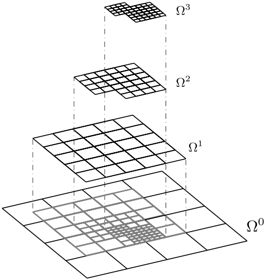

- $V_0 \subset V_1 \subset V_2 ... \subset V_J$
- Refinement has to contain at least one function : $(p+1)^2$ elements
- Definition of **truncated-basis** ensuring good spline properties

Giannelli, Juettler, Speelers

Adaptivity with hierarchical splines

B. and Giannelli, 2014

- Residual based estimator: $\eta_Q = h_Q \|Au_h - f\|_{L^2(Q)}$ (no jumps)
- Dörfler marking $\mathcal{E}(u_h, \mathcal{M}) \geq \theta \mathcal{E}(u_h, \mathcal{T})$, $0 < \theta < 1$;
- Suitable local quasi-interpolant and their approximation properties

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- The theory of adaptive methods

Morin, Nochetto, Siebert –Binev, Dahmen, DeVore 2002–2005

$\Rightarrow$ **Convergence and optimality** !

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. . . but this is another story . . .

Non conforming interfaces and mortaring

Non conforming interfaces and mortaring

Let Ω be a computational domain in $\mathbb{R}^n$, we want to solve the Laplace problem (or linear elasticity with minor changes)

$$-\operatorname{div}(\mathbf{A}\nabla u) = f$$

with boundary conditions $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$.

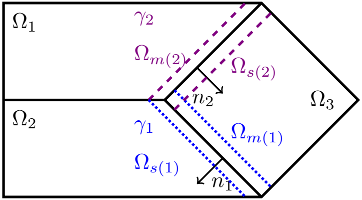
$$u = 0 \text{ on } \Gamma_D \text{ and } (\mathbf{A}\nabla u) \cdot \mathbf{n} = h \text{ on } \Gamma_N$$

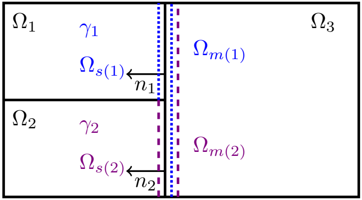
We suppose that

$\Omega = \bigcup_i^N \Omega_i$, $\Omega_i = \mathbf{F}_i(\widehat{\Omega})$, $\Gamma_{ij} = \partial\Omega_i \cap \Omega_j$,

- $\mathbf{F}_i$ are splines (or NURBS)
- non compatible meshes at the interfaces Γ_{ij}

About the admissible partition of the domain





- Decomposition can be conforming or non-conforming
- We can handle the case when Γ_{ij} is a face of either Ω_i or Ω_j.
- There is the need for **cross-point treatment/reduction**

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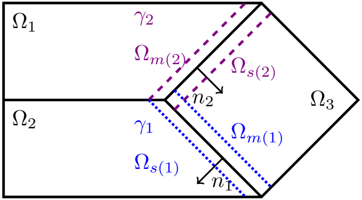

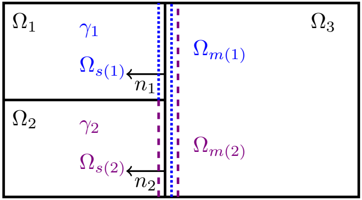

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- Non compatible geometries interfaces at the interfaces Γ_{ij} (?)

Non conforming interfaces and mortaring

Let $S_p(\widehat{\mathcal{T}}_j)$ be the space of tensor product splines/NURBS of degree p, on the knot mesh $\widehat{\mathcal{T}}_j$.

- in each subdomain Ω_j,

$$V_j = \{v_j \in H^1(\Omega_j) \ : \ v \circ \mathbf{F}_j \in S_p(\widehat{\mathcal{T}}_j)\}$$

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$$V = \{v \in L^2(\Omega) \ : v_{|\Omega_j} \in V_j \, , \ v_{|\Gamma_D} = 0\} \quad \|v\|_V^2 = \sum_{i=1}^{N} \|v\|_{H^1(\Omega_j)}^2.$$

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Interface numbering and spaces

$$\Sigma_0 = \bigcup_{\ell=1}^{n_I} \Gamma_\ell\,, \qquad \forall \ell \quad \exists (i_\ell, j_\ell) \ : \Gamma_\ell = \partial\Omega_{i_\ell} \cap \Omega_{j_\ell}.$$

Continuity across Σ_0 imposed via Lagrange multipliers:

$$M = \{\lambda \in L^2(\Sigma_0) \ : \ \lambda_\ell = \lambda_{|\Gamma_\ell} \in M_\ell\}$$

M_ℓ to be chosen properly!

Variational formulation of the problem

Find $u_h \in V$, $\lambda_h \in M$ such that

$$
\begin{aligned}
a(u_h, v_h) + b(\lambda_h, v_h) &= R(v_h) && \forall v_h \in V \\
b(\mu_h, u_h) &= 0 && \forall \mu_h \in M
\end{aligned}
$$

where

$$
a(u,v) = \int_\Omega \mathbf{A} \nabla u \cdot \nabla v \qquad b(\lambda, v) = \sum_\ell \int_{\Gamma_\ell} \lambda_\ell [u] \qquad [u] = u_{i_\ell} - u_{j_\ell}
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$R(v)$ is the RHS taking into account also Neumann BC...

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Choice of the Langrange multiplier space

- Topology for M_ℓ is $H^{1/2}_{00}(\Gamma_\ell)$...
- ... I want to have the **largest possible set of multipliers** such that the form $b(\lambda, v) = \int_{\Gamma_\ell} \lambda_\ell [u]$ remains uniformly stable

Favorite choice: if i_ℓ is the slave side, we want $M_\ell \sim V_{i_\ell}|_{\Gamma_\ell}$!
It contraints **all** slave functions.

But it is known that stability fails with this choice, and there is a need for **cross point degree reduction**..

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$$\dim(M_\ell) \leq \dim\{v \in V_{i_\ell}|_{\Gamma_\ell} \ : \ v|_{\partial\Gamma_\ell} = 0\}$$

Choice of the Langrange multiplier space

Each Γ_ℓ is a face of a subdomain Ω_i (the slave side)

- Γ_ℓ inherits a spline mapping $\mathbf{F}_\ell : (0,1)^{d-1} \to \Gamma_\ell$
- and a parametric mesh on $\widehat{\Gamma} = (0,1)^{d-1}$ denoted as $\widehat{\mathcal{T}}_\ell$.

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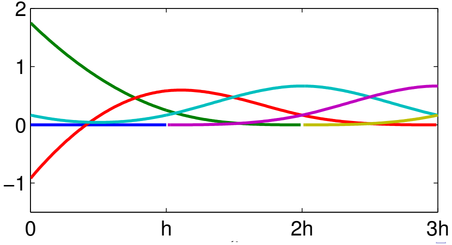
Choice 1: same degree, cross point reduction

$$\widehat{M}^1_\ell = \widetilde{S}_p(\widehat{\mathcal{T}}_\ell)$$



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Choice 2: degree reduction

$$\widehat{M}_\ell^2 = S_{p-2}(\widehat{\mathcal{T}}_\ell)$$

Indeed, it is true that

$$\dim(\widehat{M}_\ell^2) = \{\widehat{v} \in S_p(\widehat{\mathcal{T}}_{i_\ell})|_{\Gamma_\ell} \ : \ \widehat{v}|_{\partial\Gamma_\ell} = 0\}$$

- No need for degree reduction or other manipulation
- If stable, it will deliver a slightly suboptimal order : 1/2 suboptimal

Stability: numerical checks

Chapelle-Bathe test

Estimate on the inf-sup constant:

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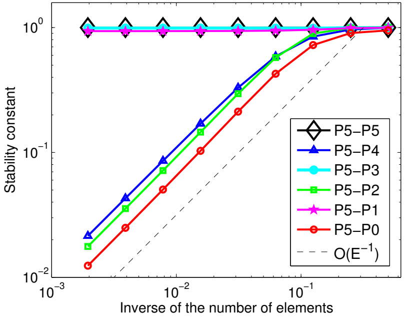
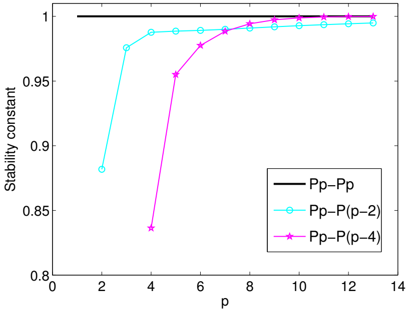
Chapelle-Bathe test

Estimate on the inf-sup constant:

Testing $\widehat{M}_\ell$ against $S_p(\widehat{\mathcal{T}}_{i_\ell})|_{\Gamma_\ell}$ without boundary conditions





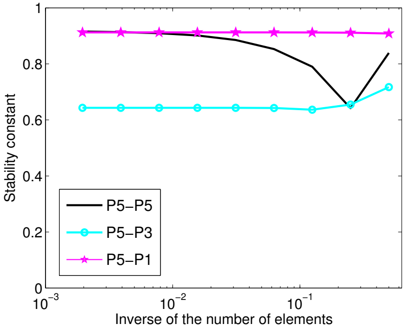
$p/p-2k$ couples are all stable! the pairings are also stable varying p !

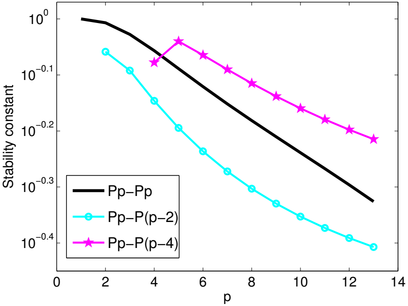
Stability: numerical checks

Chapelle-Bathe test

Estimate on the inf-sup constant:

Testing $\widehat{M}_\ell$ against $S_p(\widehat{\mathcal{T}}_{i_\ell})|_{\Gamma_\ell}$ with boundary conditions .... the right thing!





$p/p-2k$ couples are all stable!

exponential behavior in p !

Stability: Proof of the inf-sup condition

the $p/p-2$ case

We consider $\widehat{M}_\ell^2$ and can prove the following:

$$\inf_{\widehat{\mu}\in S_{p-2}} \sup_{\widehat{v}\in S_p\cap H_0^1} \frac{\int_{\widehat{\Gamma}} \widehat{\mu}\,\widehat{v}}{\|\widehat{v}\|_{L^2}\|\widehat{\mu}\|_{L^2}} \geq \alpha_0$$

+ quasi uniform meshes :

$$\inf_{\widehat{\mu}\in S_{p-2}} \sup_{\widehat{v}\in S_p\cap H_0^1} \frac{\int_{\widehat{\Gamma}} \widehat{\mu}\,\widehat{v}}{\|\widehat{v}\|_{H_{00}^{1/2}}\|\widehat{\mu}\|_{(H_{00}^{-1/2})'}} \geq \alpha_0$$

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Proof

In 2D:

- $S_p \cap H_0^1 \xrightarrow{\partial_x} S_{p-1} \cap L_0^2 \xrightarrow{\partial_x} S_{p-2}$ is exact
- choose $\widehat{v} \in S_p \cap H_0^1$ such that $\partial_{xx}^2 \widehat{v} = \widehat{\mu}$ and the work with Sobolev norms.

In 3D, basically the same applies...

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In 3D, basically the same applies...

It is stable! ... we need now to go to physical space

Stability in the physical space

the $p/p-2$ case

$$\inf_{\widehat{\mu}\in S_{p-2}} \sup_{\widehat{v}\in S_p\cap H^1_0} \frac{\int_{\widehat{\Gamma}} \widehat{\mu}\,\widehat{v}}{\|\widehat{v}\|_{L^2}\|\widehat{\mu}\|_{L^2}} \geq \alpha_0$$

⇓

$$\inf_{\mu\in M_\ell} \sup_{v\in V_{i_\ell}: v\in H^1_0(\Gamma_\ell)} \frac{\int_{\Gamma_\ell} \mu\, v}{\|v\|_{L^2}\|\mu\|_{L^2}} \geq \alpha_0$$

Stability in the physical space

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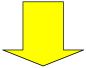

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$\int_{\Gamma_\ell} \mu\, v = \int_{\widehat{\Gamma}} \rho\, \widehat{\mu}\,\widehat{v} \quad \rho =$ weight, area change..

Stability in the physical space

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and by super-convergence results à la Wahlbin:

$$\Pi : L^2(\widehat{\Gamma}) \to \widehat{M}^2_\ell \quad \Rightarrow \quad \|\rho\widehat{\mu} - \Pi(\rho\widehat{\mu})\|_{L^2(\widehat{\Gamma})} \leq Ch\|\widehat{\mu}\|_{L^2(\widehat{\Gamma})}$$

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For h small enough the stability holds in physical space!

Back to our variational problem

Find $u_h \in V$, $\lambda_h \in M$ such that

$$\begin{aligned} a(u_h, v_h) + b(\lambda_h, v) &= R(v_h) \qquad && \forall v_h \in V \\ b(\mu_h, u_h) &= 0 && \forall \mu_h \in M \end{aligned}$$

Back to our variational problem

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$$a(u_h, v_h) + b(\lambda_h, v) = R(v_h) \qquad \forall v_h \in V$$
$$b(\mu_h, u_h) = 0 \qquad \forall \mu_h \in M$$

It is well-posed and verifies the following error estimate: if $u \in H^r(\Omega)$:

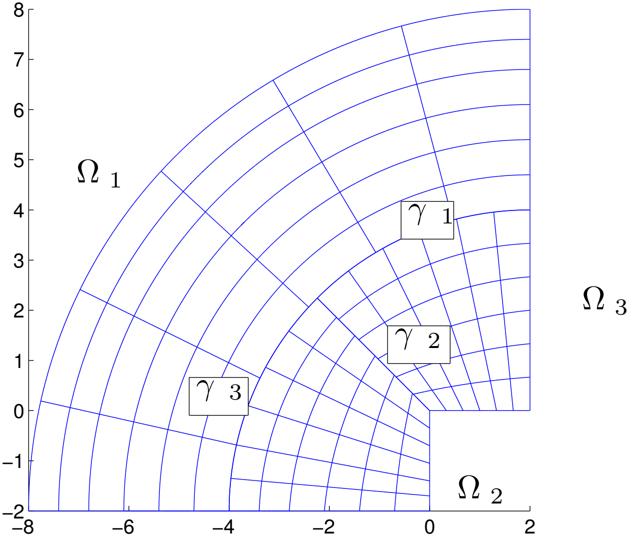
$$\|u - u_h\|_V \leq C \inf_{v_h \in V} \|u - v_h\|_V + \inf_{\mu_h \in M} \|\lambda - \mu_h\|_M \tag{1}$$
$$\leq Ch^t + Ch^s \qquad t = \min\{p, r-1\} \tag{2}$$

- $s = \min\{p + 1/2, r - 1\}$ for **Choice 1: same degree,**
- $s = \min\{p - 1/2, r - 1\}$ for **Choice 2: degree reduction**

Or, indeed:

$$\|u - u_h\|_V \leq C \inf_{v_h \in V \cap \mathrm{Ker}(B)} \|u - v_h\|_V \leq C \ldots$$

Numerical validation: problem 1



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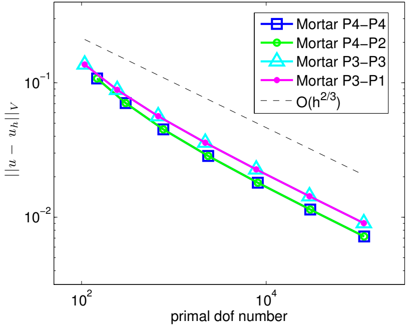
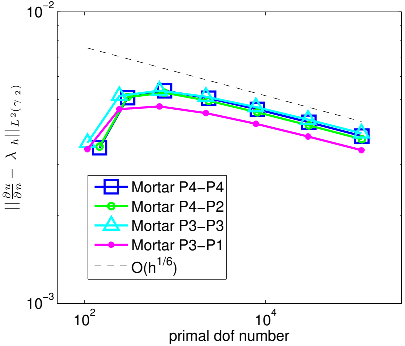
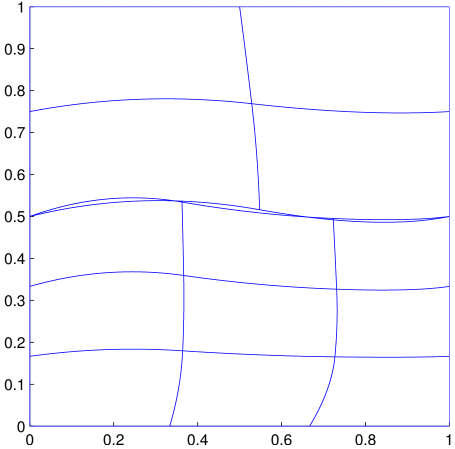


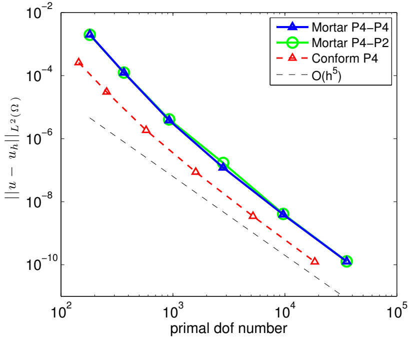


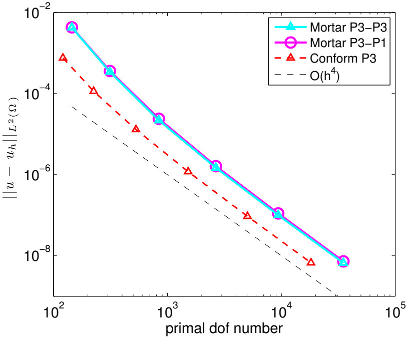
$$1/6 + 1/2 = 2/3$$

Numerical validation: problem 2



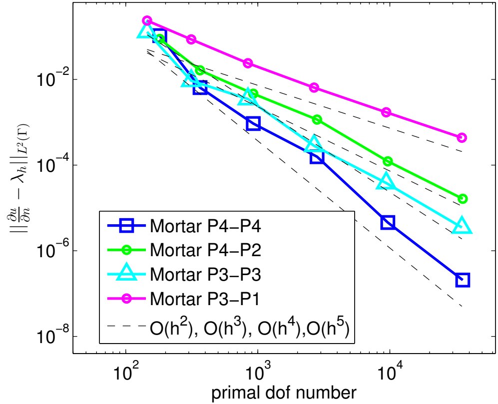
Numerical validation: problem 2





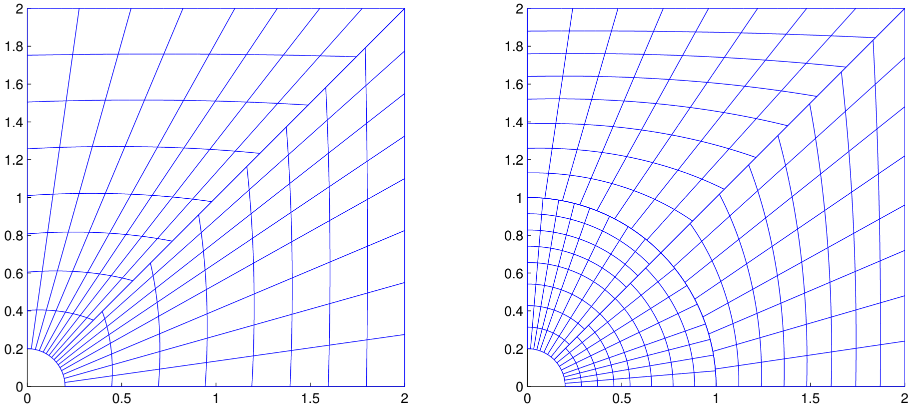
Multipliers' degree **does not** affect the order for the primal unknown!

Numerical validation: problem 2

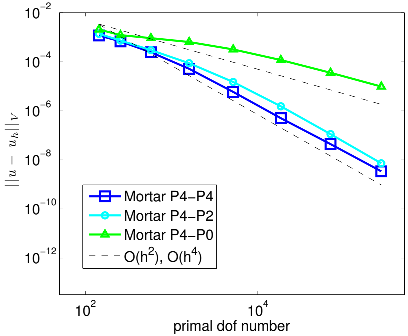


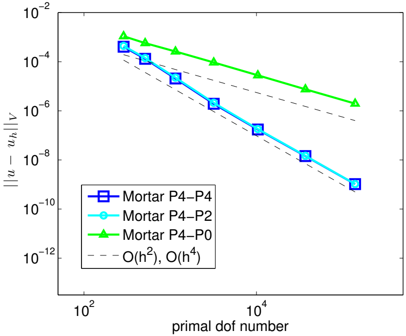
but it affects the convergence of the multiplier!

Numerical validation: elasticity

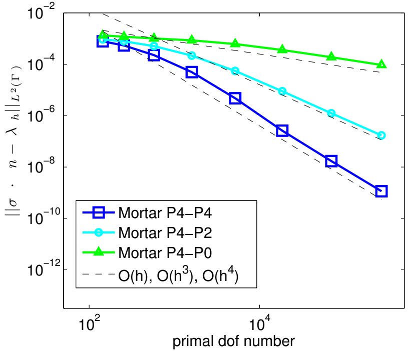


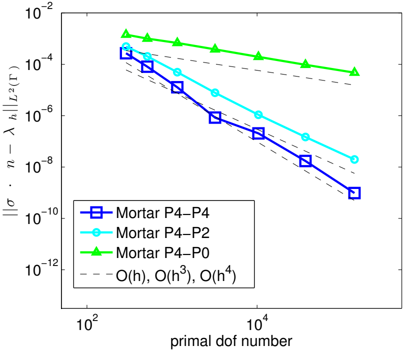
Numerical validation: elasticity





Numerical validation: elasticity





Final remarks

- This approach can be used to treat **contact** in a variationally consistent way
- For patch gluing: the geometric non-matching patch cases should be studied in details
- The question of exact / inexact integration for interface integrals remains open (robustness of splines thanks to regularity)

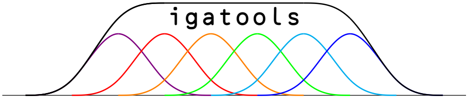
Final remarks

Surveys and Codes

- New **Acta Numerica** survey paper with several math results:

> L. Beirão Da Veiga, A. Buffa, G. Sangalli, R. Vázquez,
> *Mathematical analysis of variational isogeometric methods*

- We have two codes available to public :
  - GeoPDEs library is a GNU licensed software available here: `www.imati.cnr.it/geopdes`
  - IGATools is a C++ dimension independent library `http://code.google.com/p/igatools`



THANKS!