Finite Element Approximation To A Class of Interface Problems

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Outline

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Interface Problem

Interface Problem

$$\begin{aligned} -\Delta u^{\pm} &= f && \text{in } \Omega^{\pm} \\ u &= 0 && \text{on } \partial\Omega \\ [u] &= \alpha && \text{on } \Gamma \\ [\nabla u \cdot \boldsymbol{n}] &= \beta && \text{on } \Gamma. \end{aligned}$$

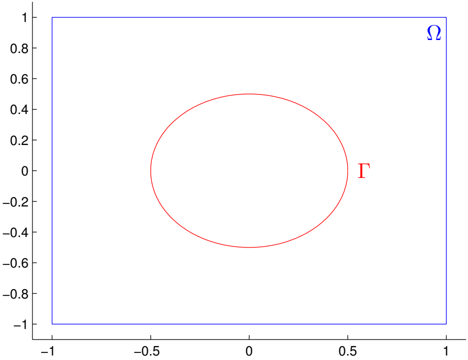
We denote $[u] = u^{+} - u^{-}$,
and

$$[\nabla u \cdot \boldsymbol{n}] = \nabla u^{-} \cdot \boldsymbol{n}^{-} + \nabla u^{+} \cdot \boldsymbol{n}^{+}$$

For simplicity we will assume that $\alpha \equiv 0$.

Illustration of interface

Illustration of Ω, Γ.



Equivalent Formulation

$$-\Delta u = f + F \qquad \text{in } \Omega \subset \mathbb{R}^2,$$
$$u = 0 \qquad \text{on } \partial\Omega$$

$$F(x) = \int_0^A \beta(s)\delta(x - X(s))ds \quad \forall x \in \Omega$$

where $X : [0, A) \longrightarrow \Gamma$ is the arch-length parametrization of the curve Γ (closed curve $X(0) = X(A)$), and δ is a two-dimensional Dirac function.

- This could be thought of as Peskin's Formulation.

Previous Work

Some Finite Difference methods

• The immersed boundary method of Peskin (1977). This method is only first order accurate near interface. Y. Mori proved error estimates in 2008.
• The immersed interface method by Li and LeVeque (1994). T. Beale and A. Layton proved second order estimates for this method in 2006.

Some Finite Element Methods

• Z. Li and T. Lin with collaborators have developed many finite element methods.
• X. He, T. Lin and Y. Li (2012). Very similar to our method.
• Y. Gong, B. Li and Z. Li (2007). Proved second order accuracy in L^2 based norms.

Outline

Variational Formulation for Interface Problem

Find $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx + \int_\Gamma \beta v ds$$

for all $v \in H_0^1(\Omega)$.

The natural method

Find $u_h \in V_h$ such that;

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f \, v \, dx + \int_\Gamma \beta v \, ds \quad \forall v \in V_h,$$

Here V_h is the space of piecewise linears.

• This is the method D. Boffi and L. Gastaldi have been analyzing and improving. Can be thought of as the finite element version of Peskin's method.

• Note that the left hand side does not change. Important for time dependent problem.

• Mesh does not have to conform to interface.

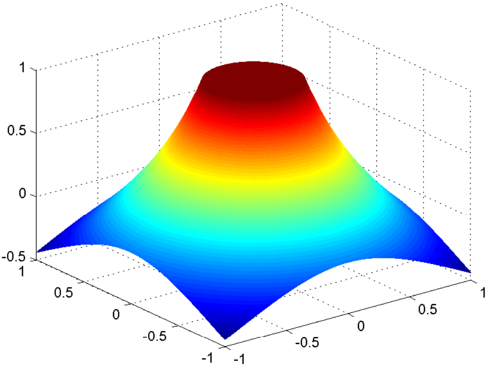
Numerical Example

Consider a exact solution of the interface problem for $x \in \Omega = [-1, 1]^2$

$$u(x) = \begin{cases} 1, & \text{if } r \leq R \\ 1 - \log(\frac{r}{R}), & \text{if } r > R \end{cases}$$

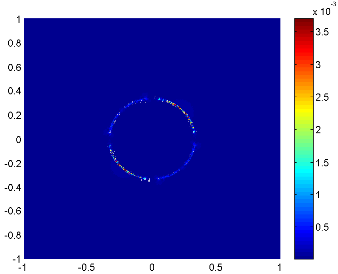
where $r = \|x\|_2$ and $R = 1/3$. Then, the data is given by $f^{\pm} = 0$, $\alpha = 0$ and $\beta = \frac{1}{R}$.

Numerical Example



Plot of the approximate solution on a non-uniform grid.

Numerical Example



Plot of the error at the nodes for the "Natural" method on a uniform grid.

- The method is not optimal near interface. However, away from interface it appears to be optimal.

Numerical Result

h	$\|e_h^N\|_{L^2}$	r	$\|\nabla e_h^N\|_{L^2}$	r	$\|e_h^N\|_{L^\infty}$	r	$\|\nabla e_h^N\|_{L^\infty}$	r
1.8e-1	1.02e-1		4.71e-1		1.63e-1		7.01e-1	
8.8e-2	1.57e-2	2.70	1.38e-1	1.78	4.09e-2	2.00	3.26e-1	1.10
4.4e-2	6.72e-3	1.22	1.30e-1	0.09	2.85e-2	0.52	5.48e-1	-0.75
2.2e-2	2.02e-3	1.74	7.88e-2	0.72	1.07e-2	1.42	5.87e-1	-0.10
1.1e-2	7.65e-4	1.40	6.16e-2	0.36	7.24e-3	0.56	6.24e-1	-0.09
5.5e-3	2.71e-4	1.50	4.27e-2	0.53	4.39e-3	0.72	6.24e-1	0.00
2.8e-3	9.09e-5	1.58	2.83e-2	0.59	2.04e-3	1.11	7.80e-1	-0.32
1.4e-3	3.53e-5	1.36	2.24e-2	0.34	1.38e-3	0.57	8.78e-1	-0.17

L^2 and L^∞ errors of the approximate solution of the natural method, on a non-uniform grid.

$$e_h^N := u_h^N - I_h u, \quad r(e, \|\cdot\|) := \frac{\log(\|e_{h_{l+1}}\|/\|e_{h_l}\|)}{\log(h_{l+1}/h_l)}.$$

Outline

Our method

Goal

Correct the natural method to render it nearly second order accurate at vertices.

Find $u_h \in V_h$ such that for all $v \in V_h$ the following holds

$$\int_\Omega \nabla u_h \cdot \nabla v \, dx = \int_\Omega f \, v dx + \int_\Gamma \beta v \, ds - \sum_{e \in \mathcal{E}_h^\Gamma} \frac{h_{e^-} h_{e^+}}{2} a_e \beta(x_e) [\nabla v \cdot n]|_e$$

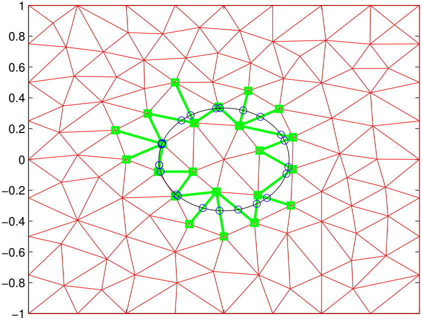
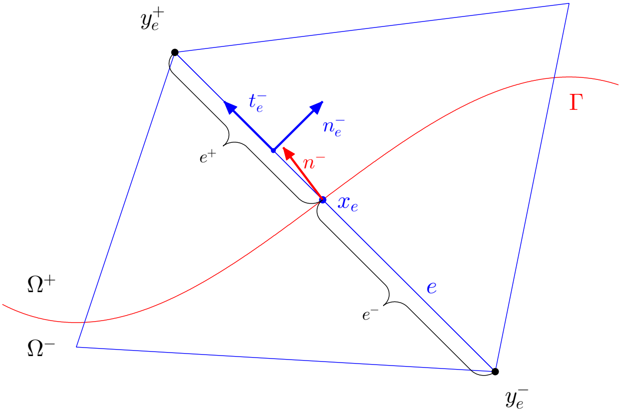


Illustration of definitions



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It holds

$$\int_\Omega \nabla(I_h u) \cdot \nabla v \, dx = \int_\Omega f \, v dx + \int_\Gamma \beta v \, ds \\ - \sum_{e \in \mathcal{E}_h^\Gamma} \frac{h_{e^-} h_{e^+}}{2} a_e \beta(x_e) [\nabla v \cdot n]|_e + F_u(\nabla v)$$

Where $F_u(\nabla v)$ is of higher order.

Summarizing

Therefore, we have defined our method such that

Lemma

Let $u_h \in V_h$ solution by our method, then it holds,

$$\int_\Omega \nabla(I_h u - u_h) \cdot \nabla v \, dx = F_u(\nabla v) \quad \textit{for all } v \in V_h,$$

where

$$|F_u(\nabla v)| \leq C_F h \|\nabla v\|_{L^1(\Omega)} \textit{ for all } v \in V_h.$$

and

$$C_F = C(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)})$$

Outline

Error analysis of Our method

Theorem

Then, there exists a constant C such that

$$\|\nabla(I_h u - u_h)\|_{L^\infty(\Omega)} \;\leq\; C C_F\, h$$

and

$$\|I_h u - u_h\|_{L^\infty(\Omega)} \;\leq\; C C_F\, h^2 \log(1/h)$$

where C is independent of h, the quasi-uniformity and shape regularity of the mesh.

and again

$$C_F \leq C(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)})$$

Error analysis for the Natural method

Theorem

Suppose that Ω is a rectangle and assume that u solves the interface problem with periodic boundary conditions . Let u_h be the approximation using the natural method. Let $z \in \Omega$ and let $d = dist(z, \Gamma) \geq \kappa h$ for a sufficiently large fixed constant κ. Then, we have

$$|\nabla(I_h u - u_h)(z)| \leq Ch(\log(1/h)\frac{h}{d^2} + 1)(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}).$$

In particular optimal estimates are obtained for points z that are $O(\sqrt{h\log(1/h)})$ away from Γ.

Outline

1st example

Consider a exact solution of the interface problem for $x \in \Omega = [-1, 1]^2$

$$u(x) = \begin{cases} 1, & \text{if } r \leq R \\ 1 - \log(\frac{r}{R}), & \text{if } r > R \end{cases}$$

where $r = \|x\|_2$ and $R = 1/3$. Then, the data is given by $f^{\pm} = 0$, $\alpha = 0$ and $\beta = \frac{1}{R}$.

1st example

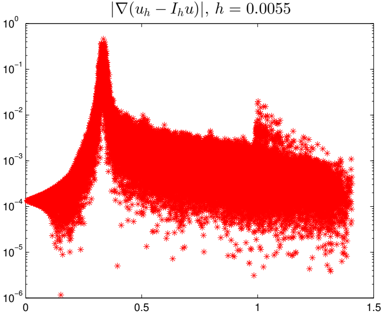
h	$\|e_h\|_{L^2}$	r	$\|\nabla e_h\|_{L^2}$	r	$\|e_h\|_{L^\infty}$	r	$\|\nabla e_h\|_{L^\infty}$	r
1.8e-1	1.39e-1		4.44e-1		2.53e-1		5.20e-1	
8.8e-2	3.09e-2	2.17	1.72e-1	1.37	6.40e-2	1.98	3.84e-1	0.44
4.4e-2	7.32e-3	2.08	5.75e-2	1.58	1.58e-2	2.02	1.79e-1	1.10
2.2e-2	1.81e-3	2.02	2.18e-2	1.40	4.19e-3	1.91	1.20e-1	0.58
1.1e-2	4.50e-4	2.01	8.57e-3	1.35	8.92e-4	2.23	6.45e-2	0.89
5.5e-3	1.12e-4	2.01	3.57e-3	1.26	2.37e-4	1.91	3.17e-2	1.02
2.8e-3	2.68e-5	2.06	1.55e-3	1.21	6.23e-5	1.93	1.71e-2	0.90
1.4e-3	6.89e-6	1.96	7.68e-4	1.01	1.68e-5	1.90	8.33e-3	1.03

L^2 and L^∞ errors of the approximate solution of our method , on a non-uniform grid.

$$e_h := u_h - I_h u, \quad r(e, \|\cdot\|) := \frac{\log(\|e_{h_{l+1}}\| / \|e_{h_l}\|)}{\log(h_{l+1}/h_l)}.$$

The result for the natural method

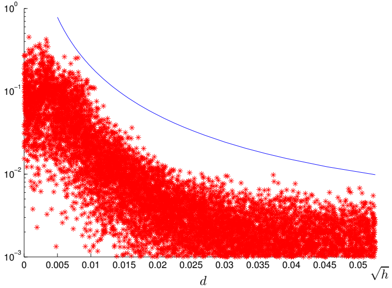
$$|\nabla(I_h u - u_h)(z)| \leq Ch\left(\log(1/h)\frac{h}{d^2} + 1\right)\left(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}\right).$$



Radial-Plot of the gradient of the error for the "Natural" method.

The result for the natural method

$$|\nabla(I_h u - u_h)(z)| \le Ch\left(\log(1/h)\frac{h}{d^2} + 1\right)\left(\|u\|_{C^2(\Omega^-)} + \|u\|_{C^2(\Omega^+)}\right).$$



Semi-log plot of gradient error for the natural method with $h = .0028$. $|\nabla e_h^N(d_T)|$ (red) for every triangle T and curve $2h + \log(1/h)(h/d)^2$ (blue).

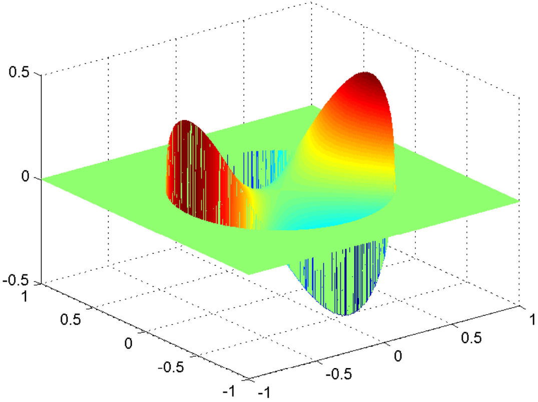
2nd example: Discontinuous u

Consider the exact solution

$$u(x_1, x_2) = \begin{cases} x_1^2 - x_2^2, & \text{if } r \le R \\ 0, & \text{if } r > R \end{cases}$$

Therefore, the data for the problem is given by $f^{\pm} = 0$, $\alpha(\theta) = -R^2(\cos^2(\theta/R) - \sin^2(\theta/R))$ and $\beta(\theta) = 2R\cos^2(\theta/R) - 2R\sin^2(\theta/R))$, for $\theta \in [0, 2\pi R]$, and $R = 2/3$.

2nd example



2nd example

h	$\|e_h\|_{L^2}$	r	$\|\nabla e_h\|_{L^2}$	r	$\|e_h\|_{L^\infty}$	r	$\|\nabla e_h\|_{L^\infty}$	r
1.8e-1	9.28e-3		3.27e-2		1.42e-2		4.23e-2	
8.8e-2	5.41e-3	0.78	3.50e-2	-0.10	8.23e-3	0.79	6.61e-2	-0.64
4.4e-2	1.19e-3	2.18	1.18e-2	1.56	2.19e-3	1.91	3.18e-2	1.06
2.2e-2	2.89e-4	2.05	5.06e-3	1.23	7.41e-4	1.56	2.25e-2	0.50
1.1e-2	7.51e-5	1.94	2.42e-3	1.06	1.64e-4	2.17	1.15e-2	0.97
5.5e-3	1.89e-5	1.99	1.18e-3	1.04	4.45e-5	1.88	5.57e-3	1.04
2.8e-3	4.71e-6	2.00	5.74e-4	1.03	1.20e-5	1.89	2.68e-3	1.06
1.4e-3	1.18e-6	2.00	2.86e-4	1.01	3.03e-6	1.98	1.35e-3	0.98

L^2 and L^∞ errors of the approximate solution of our method (EBC-FEI).

Outline

Summary

- New finite element method to solve the interface model problem
- Nearly optimal order of convergence in the maximum norm
- Error analysis for the natural method

Ongoing work

- Extend to fluid flow problems: Stokes, Navier-Stokes
- Higher order approximations
- 3d problems
- Discontinuous diffusion coefficients

THANKS.