A virtual element method for a Steklov eigenvalue problem

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The spectral problem.

Figure 1: H. MAYER AND R. KRECHETNIKOV, *Walking with coffee: Why does it spill?*. Phys. Rev. E, 85 (2012), 046117 (7 pp.).

Steklov Eigenvalue Problem:

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ. Let Γ_0 and Γ_1 be disjoint open subsets of Γ, with $|\Gamma_0| \neq 0$, such that $\Gamma = \bar{\Gamma}_0 \cup \bar{\Gamma}_1$. We consider the following spectral problem[a] [b]:

Find $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, such that

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \partial_n w = \begin{cases} \lambda w & \text{on } \Gamma_0, \\ 0 & \text{on } \Gamma_1, \end{cases} \end{cases}$$

where

- $\lambda = \frac{\omega^2}{g}$.
- w is the pressure of the fluid.

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[a] V.A. STEKLOV, *Sur les problèmes fondamentaux de la physique mathematique*, Annales sci. ENS, Sér. 3, 19, 1902, pp. 191–259 and pp. 455–490.

[b] N. KUZNETSOV, T. KULCZYCKI, M. KWAŚNICKI, A. NAZAROV, S. POBORCHI, I. POLTEROVICH, AND B. SIUDEJA, *The legacy of Vladimir Andreevich Steklov*, Notices Amer. Math. Soc., 61(1), 2014, pp. 9–22.

The spectral problem. (cont.)

Problem 1 *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\int_\Omega \nabla w \cdot \nabla v = \lambda \int_{\Gamma_0} wv \qquad \forall v \in H^1(\Omega).$$

We introduce the bilinear forms

$$a(w,v) := \int_\Omega \nabla w \cdot \nabla v \qquad \forall w, v \in H^1(\Omega),$$

$$b(w,v) := \int_{\Gamma_0} wv \qquad \forall w, v \in H^1(\Omega).$$

Problem 2 *Find* $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$, $w \neq 0$, *such that*

$$\hat{a}(w,v) = (\lambda + 1) b(w,v) \qquad \forall v \in H^1(\Omega),$$

where the bounded bilinear form is given by

$$\hat{a}(w,v) := a(w,v) + b(w,v) \qquad \forall w, v \in H^1(\Omega).$$

Spectral characterization.

We introduce the solution operator:

$$\begin{aligned} T : H^1(\Omega) &\longrightarrow H^1(\Omega), \\ f &\longmapsto Tf := u, \end{aligned}$$

where $u \in H^1(\Omega)$ is the solution of the source problem

$$\hat{a}(u,v) = b(f,v) \qquad \forall v \in H^1(\Omega).$$

Lemma 1 *There exists a constant $\alpha > 0$, depending on Ω, such that*

$$\hat{a}(v,v) \geq \alpha \|v\|_{1,\Omega}^2 \qquad \forall v \in H^1(\Omega).$$

The linear operator T is well defined and bounded. Moreover, $(\lambda, w) \in \mathbb{R} \times H^1(\Omega)$ solves **Problem 1** if and only if

$$Tw = \mu w, \quad \text{with } \mu := \frac{1}{1+\lambda} \neq 0, \quad \text{and } w \neq 0.$$

Spectral characterization. (cont.)

We have the following additional regularity result.

Lemma 2 *(i) For all $f \in H^1(\Omega)$ there exists $r \in (\frac{1}{2}, 1]$ such that the solution u of the source problem satisfies $u \in H^{1+r}(\Omega)$, and there exists $C > 0$ such that*

$$\|u\|_{1+r,\Omega} \leq C \|f\|_{1/2,\Gamma_0} \leq C \|f\|_{1,\Omega}.$$

(ii) If w is an eigenfunction of **Problem 2** *with eigenvalue λ, there exist $r_\Omega > \frac{1}{2}$ and $\tilde{C} > 0$ (depending on λ) such that for all $r \in (\frac{1}{2}, r_\Omega)$, the following estimate holds:*

$$\|w\|_{1+r,\Omega} \leq \tilde{C} \|w\|_{1,\Omega}.$$

Remark 1 *The constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. If Ω is convex, then $r_\Omega > 1$; otherwise, $r_\Omega := \frac{\pi}{\theta}$, where θ being the largest reentrant angle of Ω.*

Spectral characterization. (cont.)

Hence, because of the compact inclusion $H^{1+r}(\Omega) \hookrightarrow H^1(\Omega)$, T is a compact operator. Therefore, we have the following spectral characterization of T:

Theorem 1 *The spectrum of T decomposes as follows:*
$\mathrm{sp}(T) = \{0, 1\} \cup \{\mu_k\}_{k\in\mathbb{N}}$, *where:*

i) $\mu = 1$ *is an eigenvalue of T and its associated eigenspace is the space of constant functions in Ω;*

ii) $\mu = 0$ *is an eigenvalue of T and its associated eigenspace is* $H^1_{\Gamma_0}(\Omega) := \{q \in H^1(\Omega) : q = 0$ *on* $\Gamma_0\}$*;*

iii) $\{\mu_k\}_{k\in\mathbb{N}} \subset (0, 1)$ *is a sequence of finite-multiplicity eigenvalues of T which converges to 0 and the corresponding eigenspaces lie in $H^{1+r}(\Omega)$.*

Virtual Element Discretization [a]

Let $\{\mathcal{T}_h\}_h$ be a sequence of decompositions of Ω into polygons K and let $\mathcal{E}_h$ be the set of edges e of $\mathcal{T}_h$. Each edge $e \in \partial\mathrm{K}$ has a length h_e. Moreover, h_{K} denotes the diameter of the element K. Finally, h will also denote the maximum of the diameters of the elements, i.e., $h := \max_{\mathrm{K}\in\Omega} h_{\mathrm{K}}$.

We consider now a simple polygon K and we define for a fixed $k \geq 1$ (that will be our order of accuracy) the finite-dimensional space:

$$V^{\mathrm{K},k} := \{v \in H^1(\mathrm{K}) : v_{|_e} \in \mathbb{P}_k(e) \quad \forall e \in \partial\mathrm{K} \text{ and } \Delta v_{|_{\mathrm{K}}} \in \mathbb{P}_{k-2}(\mathrm{K})\},$$

where we denote $\mathbb{P}_{-1}(\mathrm{K}) := \{0\}$.

- the functions $v \in V^{\mathrm{K},k}$ are continuous and explicitly known on $\partial\mathrm{K}$.
- the functions $v \in V^{\mathrm{K},k}$ are virtually known inside the element K.
- there holds $\mathbb{P}_k(\mathrm{K}) \subseteq V^{\mathrm{K},k}$.

[a] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini and A. Russo, *A. Basic Principles of Virtual Element Methods*, Math. Models Methods Appl. Sci., 23(1), 2013, pp. 199–214.

The discrete problem. (cont.)

The dimension of the space $V^{\mathrm{K},k}$ is

$$dim(V^{\mathrm{K},k}) = N_e k + k(k-1)/2,$$

with N_e the number of edges of K.

Degrees of freedom for $V^{\mathrm{K},k}$:

- pointwise values for every vertex.
- for each edge e, $(k-1)$ pointwise values.
- volume moments:

$$\int_{\mathrm{K}} v p_{k-2} \qquad \forall p_{k-2} \in \mathbb{P}_{k-2}(\mathrm{K}).$$

The discrete problem. (cont.)

For every decomposition $\mathcal{T}_h$ of Ω into simple polygons K and a fixed $k \geq 1$, we define

$$V_h := \{v \in H^1(\Omega) : v|_{\mathrm{K}} \in V^{\mathrm{K},k}\}.$$

The total dofs are one per internal vertex, $k-1$ per internal edge and $k(k-1)/2$ per element.

The bilinear form $\hat{a}(\cdot,\cdot)$ can be split as

$$\hat{a}(u,v) = \sum_{\mathrm{K}\in\mathcal{T}_h} a^{\mathrm{K}}(u,v) + b(u,v) \qquad \forall u,v \in H^1(\Omega),$$

where $a^{\mathrm{K}}(\cdot,\cdot)$ is defined by

$$a^{\mathrm{K}}(u,v) := \int_{\mathrm{K}} \nabla u\cdot \nabla v \qquad \forall u,v \in H^1(\Omega).$$

The discrete problem. (cont.)

In order to construct the discrete scheme, we define the operator $\Pi_k^{\mathrm{K}} : V^{\mathrm{K},k} \to \mathbb{P}_k(\mathrm{K}) \subseteq V^{\mathrm{K},k}$ as the solution of

$$a^{\mathrm{K}}(\Pi_k^{\mathrm{K}} v, q) = a^{\mathrm{K}}(v, q) \qquad \forall q \in \mathbb{P}_k(\mathrm{K}),$$

$$\overline{\Pi_k^{\mathrm{K}} v} = \overline{v},$$

for all $v \in V^{\mathrm{K},k}$, where for any sufficiently regular function φ,

$$\bar{\varphi} := \frac{1}{n} \sum_{i=1}^{n} \varphi(\nu_i), \qquad \nu_i = \text{ vertices of } \mathrm{K}.$$

The discrete problem. (cont.)

Now, let $S^{\mathrm{K}}(u,v)$ be any symmetric positive definite bilinear form to be chosen to satisfy

$$c_0 a^{\mathrm{K}}(v,v) \leq S^{\mathrm{K}}(v,v) \leq c_1 a^{\mathrm{K}}(v,v) \qquad \forall v \in V^{\mathrm{K},k},$$

for some positive constants c_0, c_1 independent of K and h_{K}.

Then, the bilinear form

$$a_h(u_h,v_h) := \sum_{\mathrm{K}\in\mathcal{T}_h} a_h^{\mathrm{K}}(u_h,v_h) \qquad \forall u_h, v_h \in V_h,$$

where $a_h^{\mathrm{K}}(\cdot,\cdot)$ is the bilinear form on $V^{\mathrm{K},k} \times V^{\mathrm{K},k}$ defined by

$$a_h^{\mathrm{K}}(u,v) := a^{\mathrm{K}}(\Pi_k^{\mathrm{K}} u, \Pi_k^{\mathrm{K}} v) + S^{\mathrm{K}}(u - \Pi_k^{\mathrm{K}} u, v - \Pi_k^{\mathrm{K}} v) \qquad \forall u, v \in V^{\mathrm{K},k},$$

which is consistent and stable.

The discrete problem. (cont.)

More precisely:

- k-Consistency:

$$a_h^{\mathrm{K}}(p, v_h) = a^{\mathrm{K}}(p, v_h) \qquad \forall p \in \mathbb{P}_k(\mathrm{K}) \qquad \forall v_h \in V^{\mathrm{K},k}.$$

- Stability: There exist two positive constants α_* and α^*, independent of h_{K} and K, such that:

$$\alpha_* a^{\mathrm{K}}(v_h, v_h) \leq a_h^{\mathrm{K}}(v_h, v_h) \leq \alpha^* a^{\mathrm{K}}(v_h, v_h) \qquad \forall v_h \in V^{\mathrm{K},k}.$$

The discrete problem. (cont.)

The discrete virtual element formulation asociated to the spectral **Problem 1** reads:

Problem 3 *Find* $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, *such that*

$$a_h(w_h, v_h) = \lambda_h b(w_h, v_h) \qquad \forall v_h \in V_h.$$

We use again a shift argument to rewrite this discrete eigenvalue problem as follows:

Problem 4 *Find* $(\lambda_h, w_h) \in \mathbb{R} \times V_h$, $w_h \neq 0$, *such that*

$$\hat{a}_h(w_h, v_h) = (\lambda_h + 1) b(w_h, v_h) \qquad \forall v_h \in V_h,$$

where

$$\hat{a}_h(w_h, v_h) := a_h(w_h, v_h) + b(w_h, v_h) \qquad \forall w_h, v_h \in V_h.$$

We observe that from the stability condition and the trace theorem, the bilinear form $\hat{a}_h(\cdot, \cdot)$ is continuous, and uniformly elliptic.

The discrete problem. (cont.)

The discrete version of the operator T is then given by

$$\begin{aligned} T_h : H^1(\Omega) &\longrightarrow H^1(\Omega), \\ f &\longmapsto T_h f := u_h, \end{aligned}$$

where $u_h \in V_h$ is the solution of the discrete source problem,

$$\hat{a}_h(u_h, v_h) = b(f, v_h) \qquad \forall v_h \in V_h.$$

As in the continuos case, $(\lambda_h, w_h) \in \mathbb{R} \times V_h$ solves **Problem 3** if and only if

$$T_h w_h = \mu_h w_h, \quad \text{with } \mu_h := \frac{1}{1+\lambda_h} \neq 0 \quad \text{and } w_h \neq 0.$$

The discrete problem. (cont.)

As a consequence, the following spectral characterization holds true.

Theorem 2 *The spectrum of* T_h *consists of* $M = \dim(V_h)$ *eigenvalues, repeated accordingly to their respective multiplicities. The spectrum decomposes as follows:* $\mathrm{sp}(T_h) = \{0, 1\} \cup \{\mu_{hk}\}_{k=1}^{J}$, *where:*

i) *the eigenspace associated with* $\mu_h = 1$ *is the space of constant functions in* Ω*;*

ii) *the eigenspace associated with* $\mu_h = 0$ *is* $K_h := V_h \cap H^1_{\Gamma_0}(\Omega)$*;*

iii) $\mu_{hk} \subset (0, 1), k = 1, \ldots, J := M - \dim(K_h) - 1$, *are eigenvalues, repeated accordingly to their respective multiplicities.*

Spectral approximation.

To prove that T_h provides a correct spectral approximation of T, we will resort to the classical theory for compact operators[a].

Lemma 3 *There exists $C > 0$ such that, for all $f \in H^1(\Omega)$, if $u = Tf$ and $u_h = T_h f$, then*

$$\|(T - T_h)\, f\|_{1,\Omega} = \|u - u_h\|_{1,\Omega} \leq C \left(\|u - u_I\|_{1,\Omega} + |u - u_\pi|_{1,h} \right),$$

for all $u_I \in V_h$ and for all $u_\pi \in \mathrm{L}^2(\Omega)$ such that $u_\pi|_{\mathrm{K}} \in \mathbb{P}_k(\mathrm{K})\ \forall \mathrm{K} \in \mathcal{T}_h$.

[a] I. Babuška and J. Osborn, *Eigenvalue problems*, in *Handbook of Numerical Analysis*, Vol. II, P.G. Ciarlet and J.L. Lions, eds., North-Holland, Amsterdam, 1991, pp. 641–787.

Spectral approximation. (cont.)

Now, if the sequence of meshes $\mathcal{T}_h$ satisfy the following assumptions:

A0.1 There exists $\gamma > 0$ such that, for all h, each polygon K in $\mathcal{T}_h$ is star-shaped with respect to a ball of radius $\geq \gamma h_{\mathrm{K}}$.

A0.2 There exists $\delta > 0$ such that for all h and for each polygon K in $\mathcal{T}_h$, the distance between any two vertices of K is $\geq \delta h_{\mathrm{K}}$.

Spectral approximation. (cont.)

As a consequence we have the following results [a].

Proposition 1 *Assume that assumption* **A0.1** *is satisfied. Then, there exists a constant* C*, depending only on* k *and* γ*, such that for every* s *with* $1 \leq s \leq k+1$ *and for every* $u \in H^s(\mathrm{K})$ *there exists* $u_\pi \in \mathbb{P}_k(\mathrm{K})$ *such that*

$$\|u - u_\pi\|_{0,\mathrm{K}} + h_\mathrm{K}|u - u_\pi|_{1,\mathrm{K}} \leq Ch_\mathrm{K}^s|u|_{s,\mathrm{K}}.$$

Proposition 2 *Assume that assumptions* **A0.1** *and* **A0.2** *are satisfied. Then, there exists a constant* $C > 0$*, depending only on* k*,* δ *and* γ*, such that for every* s *with* $1 < s \leq k+1$*, for every* h*, for all* $\mathrm{K} \in \mathcal{T}_h$ *and for every* $u \in H^s(\mathrm{K})$ *there exists* $u_I \in V^{\mathrm{K},k}$ *such that*

$$\|u - u_I\|_{0,\mathrm{K}} + h_\mathrm{K}|u - u_I|_{1,\mathrm{K}} \leq Ch_\mathrm{K}^s|u|_{s,\mathrm{K}}.$$

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[a] S. C. Brenner and R. L. Scott, *The Mathematical Theory of Finite Element Methods*, Texts in Applied Mathematics, 15. Springer, New York, 2008.

Spectral approximation. (cont.)

The following theorem yields the convergence in norm of T_h to T as $h \to 0$.

Theorem 3 *There exist* $C > 0$ *and* $r \in (\frac{1}{2}, 1]$ *such that for all* $f \in H^1(\Omega)$,

$$\|(T - T_h)\, f\|_{1,\Omega} \leq C h^r \|f\|_{1,\Omega}.$$

Spectral approximation. (cont.)

Let (λ_h, w_h) be a solution of **Problem 3** with $\|w_h\|_{1,\Omega} = 1$. It can be proved that, there exists a solution (λ, w) of **Problem 1** with $\|w\|_{1,\Omega} = 1$. Moreover, the following error estimates hold true:

Theorem 4 *There exists $C > 0$ such that for all $r \in (\frac{1}{2}, r_\Omega)$*

$$\|w - w_h\|_{1,\Omega} \leq Ch^{\min\{r,k\}},$$

$$|\lambda - \lambda_h| \leq Ch^{2\min\{r,k\}},$$

$$\|w - w_h\|_{0,\Gamma_0} \leq Ch^{r_1/2+\min\{r,k\}},$$

where as before, the constant $r_\Omega > \frac{1}{2}$ is the Sobolev exponent for the Laplace problem with Neumann boundary conditions. If Ω is convex, then $r_\Omega > 1$; otherwise, $r_\Omega := \frac{\pi}{\theta}$, where θ being the largest reentrant angle of Ω, and $r_1 \in (\frac{1}{2}, 1]$.

Numerical tests.

VEM implementation [a]

- $\Omega := (0,1)^2$.
- We take Γ_0 as the top boundary.
- We take $k = 1$.

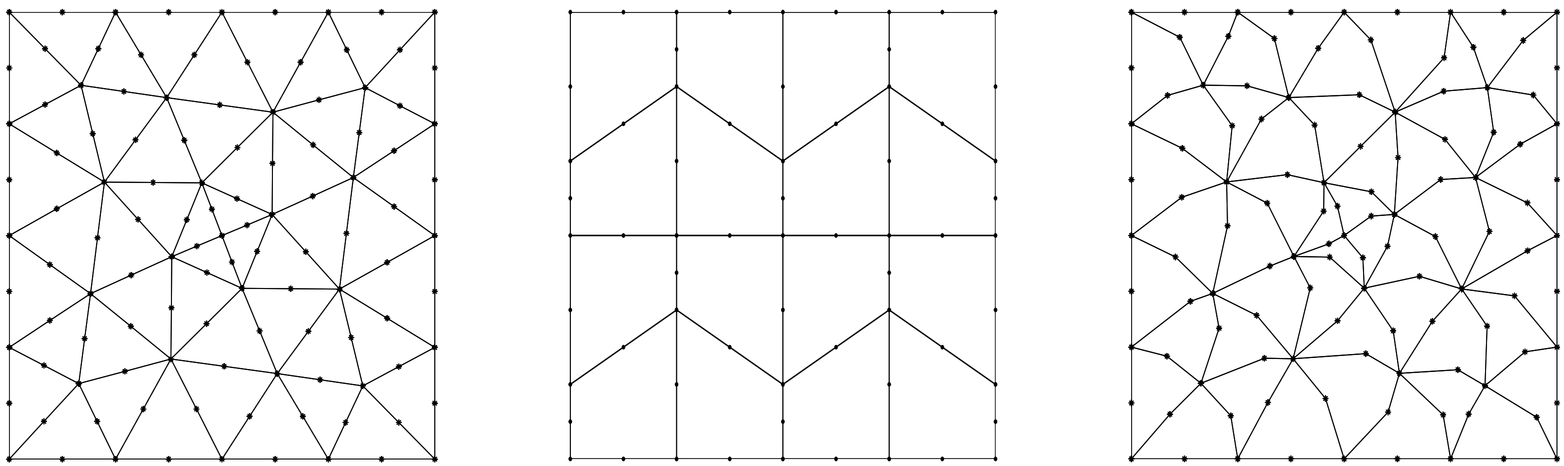
The analytical solution of this particular problem is given by:

$$\omega_n = \sqrt{n\pi \tanh(n\pi)},$$

$$w(x,y) = \cos(n\pi x)\sinh(n\pi y).$$

[a] L. Beirão da Veiga, F. Brezzi, L. D. Marini and A. Russo, *The Hitchhiker's Guide to the Virtual Element Method*, Math. Models Methods Appl. Sci., 24(8), 2014, pp. 1541–1573.

Numerical tests. (cont.)



- $\mathcal{T}_h^1$: Triangular mesh, considering the middle point of each edge as a new degree of freedom.
- $\mathcal{T}_h^2$: Trapezoidal meshes which consist of partitions of the domain into $N \times N$ congruent trapezoids, all similar to the trapezoid with vertexes $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{2}{3})$, and $(0,\frac{1}{3})$.
- $\mathcal{T}_h^3$: Meshes built from $\mathcal{T}_h^1$ with the edge midpoint moved randomly; note that these meshes contain non-convex elements.

Numerical tests. (cont.)

Table 1: Computed lowest sloshing frequencies $\omega_{hi}, i = 1, 2, 3$.

$\mathcal{T}_h$	ω_{hi}	$N = 16$	$N = 32$	$N = 64$	$N = 128$	Order	Extrap.	ω_i
	ω_{h1}	1.7716	1.7697	1.7693	1.7692	2.0400	1.7691	1.7691
$\mathcal{T}_h^1$	ω_{h2}	2.5211	2.5101	2.5074	2.5068	2.0700	2.5066	2.5066
	ω_{h3}	3.1114	3.0796	3.0723	3.0705	2.1100	3.0700	3.0700
	ω_{h1}	1.7897	1.7744	1.7705	1.7695	1.9500	1.7691	1.7691
$\mathcal{T}_h^2$	ω_{h2}	2.6133	2.5361	2.5142	2.5085	1.8400	2.5060	2.5066
	ω_{h3}	3.3267	3.1477	3.0906	3.0752	1.8900	3.0667	3.0700
	ω_{h1}	1.7721	1.7698	1.7692	1.7691	2.1600	1.7691	1.7691
$\mathcal{T}_h^3$	ω_{h2}	2.5242	2.5108	2.5075	2.5068	2.0600	2.5065	2.5066
	ω_{h3}	3.1203	3.0819	3.0727	3.0706	2.0800	3.0699	3.0700

Numerical tests. (cont.)

Table 2: Errors of the sloshing mode $\|w - w_h\|_{0,\Gamma_0}$ for the lowest sloshing frequency computed on meshes $\mathcal{T}_h^1$.

$1/h$	$\|w - w_h\|_{0,\Gamma_0}$	Order
16	4.6159e-3	-
32	1.1022e-3	2.07
64	2.9076e-4	1.92
128	7.0619e-5	2.04
256	1.8353e-5	1.94

Numerical tests. (cont.)

Figure 2 shows the first (left), second (middle) and third (right) sloshing modes of the fluid on the top.

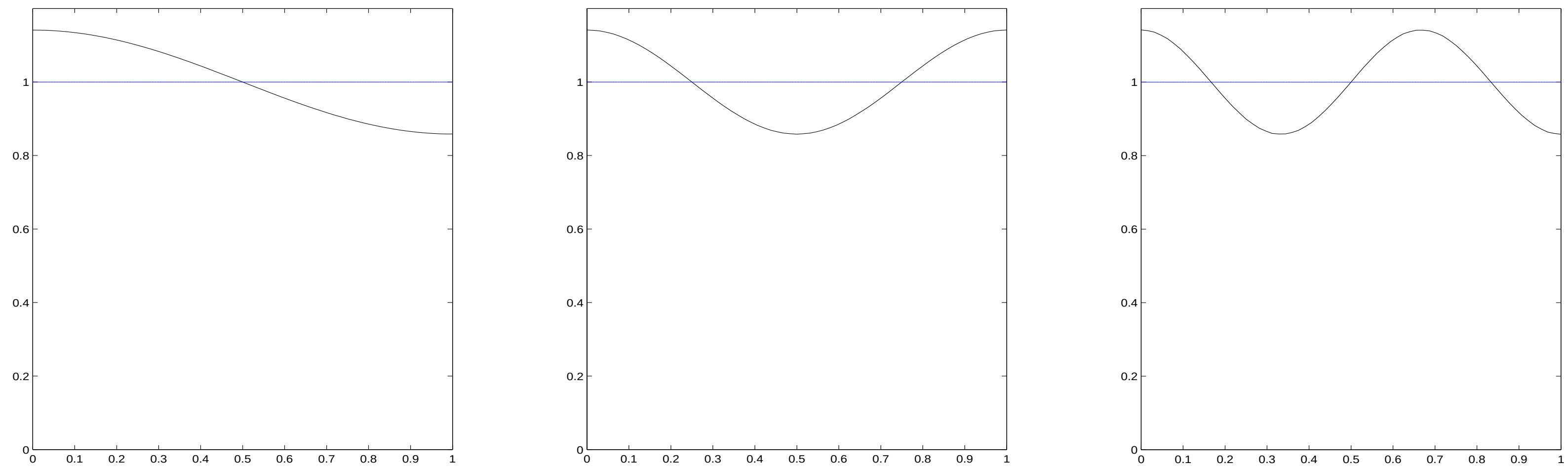


Figure 2: Vibration modes: u_{h1} (left), u_{h2} (middle) and u_{h3} (right) for $h = 1/256$.

Many thanks for your attention.