Topological clones

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Outline

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I: Algebras, function clones, abstract clones, Birkhoff’s theorem

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II: Topological clones, Topological Birkhoff

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III: Reconstruction

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VII: Topological clones revisited

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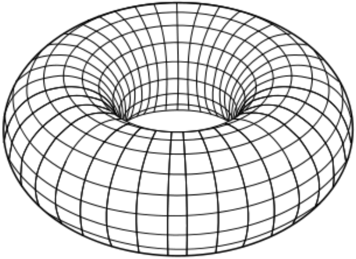
VIII: Discussion & Open Problems



I: Abstract clones

Algebras, function clones

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- closed under composition: $f(g_1(x_1, \ldots, x_m), \ldots, g_n(x_1, \ldots, x_m))$;
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Here: algebras up to "clone equivalence".

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Write $\mathcal{C} \to \mathcal{D}$ if there exists a clone homomorphism from $\mathcal{C}$ into $\mathcal{D}$.

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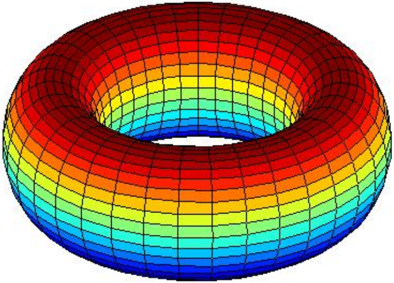
What about $\mathrm{HSP}^{\mathrm{fin}}$ of infinite function clones?

Analogy with groups and monoids

Permutation group	Abstract group
Transformation monoid	Abstract monoid
Function clone	Abstract clone



II: Topological clones

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Remark: For finite function clones: topology discrete.

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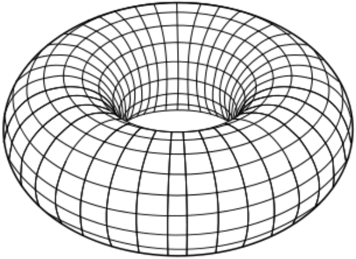

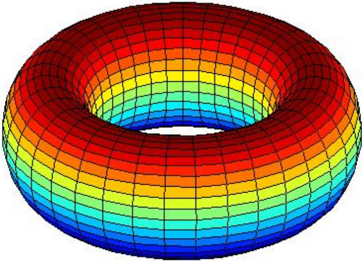

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- Christian Pech

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Two polymorphism clones of countable ω-categorical structures which are isomorphic, but not topologically.

(Bodirsky + Evans + Kompatscher + MP 2015)

IV: pp interpretations

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Observe: Pol(Γ) $\supseteq$ End(Γ) $\supseteq$ Aut(Γ).

Closed polymorphism clones

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$$\text{What does } \mathrm{Pol}(\Delta) \in \mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma)) \text{ imply for } \Gamma, \Delta?$$

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- Δ *has a pp interpretation in* Γ*:*

  it is a pp-definable homomorphic image
  of a pp-definable subuniverse
  of a finite power
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Primitive positive (pp) interpretations

A formula is primitive positive (pp) iff it is of the form

$$\exists x_1 \cdots \exists x_n \, \psi_1 \wedge \cdots \wedge \psi_m,$$

where ψ_i are atomic.

Theorem (Bulatov + Jeavons + Krokhin 2000; Bodirsky + MP 2011)

Let Γ, Δ be countable ω-categorical or finite relational structures. TFAE:

- $\mathrm{Pol}(\Delta)$ *contains a clone in* $\mathrm{HSP}^{\mathrm{fin}}(\mathrm{Pol}(\Gamma))$*;*
- Δ *has a pp interpretation in* Γ*:*

  it is a pp-definable homomorphic image
  of a pp-definable subuniverse
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  (There is a partial mapping from some Γ^n *onto* Δ *such that the preimage of every relation of* Δ *is pp-definable in* Γ*.)*

pp interpretations and topological clones

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Theorem (Bodirsky + MP 2011)

Let Γ *be countable* ω*-categorical or finite, and* Δ *be finite. TFAE:*

- $\mathrm{Pol}(\Gamma) \to \mathrm{Pol}(\Delta)$ *continuously;*
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Let Γ *be countable* ω*-categorical or finite. TFAE:*

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- *All finite structures have a pp interpretation in* Γ*.*

C			4		3		2	8			9				B
7						A				6			4		
	E		8	D				F		5	2		C	7	
			0		7				B		D		6		E
4				9							E		1		
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	C		B		6		F	A	2		5			0	4
A		2			5	D	0			C	8	3	B		1
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5			3		8				1		0	9	F		
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		C		F		1						B		E	
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		4		A	D		7		E		C	2			5

V: Constraint Satisfaction Problems

Constraint Satisfaction Problems (CSPs)

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Irrelevant whether Γ is finite or infinite. But language finite.

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Input: A finite system of equations using $=, +, \cdot, 1$

Question: Is there a solution in $\mathbb{Z}$?

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Corollary

Let Γ be finite or countable ω-categorical.

If Pol(Γ) $\to$ **1** continuously, then CSP(Γ) is NP-hard.

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In a sense unique.

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What does this mean for Pol(Γ)?

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II: Topological clones, Topological Birkhoff

III: Reconstruction

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Topological clones

Part II

Michael Pinsker

Technische Universität Wien / Univerzita Karlova v Praze

Funded by Austrian Science Fund (FWF) grant P27600

LMS-EPSRC Durham Symposium

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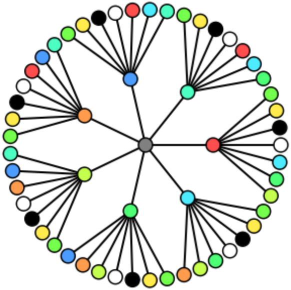

V: Projective clone homomorphisms

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Open problem

Is there a function clone with a projective clone homomorphism, but not a continuous one?

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So the Betweenness problem is NP-hard.

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Continuous iff U is principal.

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$\mathrm{Pol}(\Gamma) \not\twoheadrightarrow \mathbf{1}$.

But $\mathrm{Pol}(\Gamma, 0) \to \mathbf{1}$ continuously.

VII: Topological clones revisited

Homomorphic equivalence

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Γ, Δ homomorphically equivalent if $\Gamma \to \Delta$ and $\Delta \to \Gamma$.

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How does $\mathrm{Pol}(\mathfrak{C}(\Gamma))$ relate to $\mathrm{Pol}(\Gamma)$?

Double shrinks

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Proposition

Let Γ, Δ be structures, where Γ is ω-categorical. TFAE:

- Δ is homomorphically equivalent to a pp definable structure of Γ
- $\mathrm{Pol}(\Delta)$ contains a double shrink of $\mathrm{Pol}(\Gamma)$.

D, H, S, and weak homomorphisms

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If there exists such a function, we write $\mathcal{C} \rightsquigarrow \mathcal{D}$.

Birkhoff's theorem for D and P

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Theorem (Barto + MP 2015)

Let $\mathcal{C}, \mathcal{D}$ be function clones. TFAE:

- $\mathcal{D} \in \mathsf{D}\,\mathsf{P}(\mathcal{C})$;
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Let $\mathcal{C}, \mathcal{D}$ be function clones, $\mathcal{D}$ finite. TFAE:

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Meditation: What happened to $\mathcal{D}$ which is finitely generated?

The relational side

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Theorem (Barto + MP 2015)

Let Γ be finite or ω-categorical, let Δ be finite. TFAE:

- *Δ can be obtained from Γ by homomorphic equivalence, adding of constants to model-complete cores, and pp interpretations.*

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Let Γ be finite or ω-categorical, let Δ be finite. TFAE:

- Δ *can be obtained from* Γ *by*
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- $\mathrm{Pol}(\Gamma) \rightsquigarrow \mathrm{Pol}(\Delta)$ *uniformly contiuously.*

Infinite tractability conjecture, revisited

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Old Conjecture

Let Γ be definable in a countable finitely bounded homogeneous structure (implies ω-categorical). Then:

- there exists a finite tuple $\bar{c}$ such that $\mathrm{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \to \mathbf{1}$ continuously (and CSP(Γ) is NP-complete), or
- CSP(Γ) is polynomial-time solvable.

Infinite tractability conjecture, revisited

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Let Γ be definable in a countable finitely bounded homogeneous structure (implies ω-categorical). Then:

- there exists a finite tuple $\bar{c}$ such that $\mathrm{Pol}(\mathfrak{C}(\Gamma), \bar{c}) \to \mathbf{1}$ continuously (and CSP(Γ) is NP-complete), or
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Observation: Old $\implies$ New.

C			4		3		2	8			9				B
7						A				6			4		
	E		8	D				F		5	2		C	7	
			0		7				B		D		6		E
4				9							E		1		
	6		2							0		5			3
	0	B	1	4		2			9				E		
	9	5			A	B	C	6			7				
	C		B		6		F	A	2		5			0	4
A		2			5	D	0			C	8	3	B		1
		0	F	B								D		2	
5			3		8				1		0	9	F		
3	8			5		6	E	0		F				9	
		C		F		1						B		E	
0							8				6	7			D
		4		A	D		7		E		C	2			5

VIII: Discussion & Open Problems

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- Cannot expect weak homomorphism theorem with Δ infinite.

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- Useful?

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- If a closed function clone satisfies a linear equation, does it satisfy a special equation?

Reference

L. Barto, J. Opršal, and M. Pinsker

The wonderland of the double shrink

In preparation.

Wayne Ferrebee, *Torus with Spearman, Bagpipes and Barnacle*