Long chains of subsemigroups

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28th of July, 2015

Definitions 1

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In this talk:
Max:= Maximilien Gadouleau
Peter:= Peter Cameron
James:= James Mitchell

Definitions 2: Length of a group

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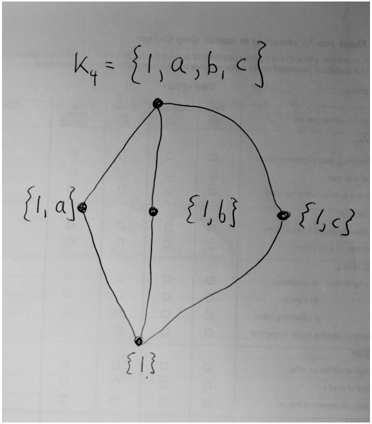

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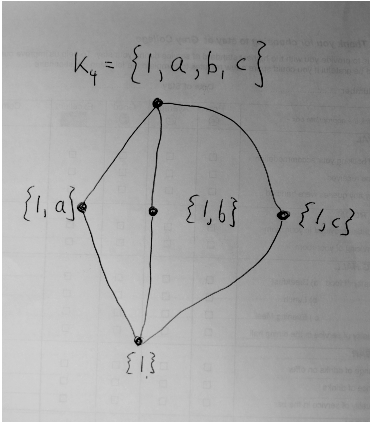

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If G has order n, then $l(G) \leq \Omega(n) \leq \log_2(n)$.
Equality holds, for example, for all soluble groups.

The symmetric group

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Theorem (Cameron-Solomon-Turull '89)

$$l(S_n) = \left\lceil \frac{3n}{2} \right\rceil - b(n) - 1$$

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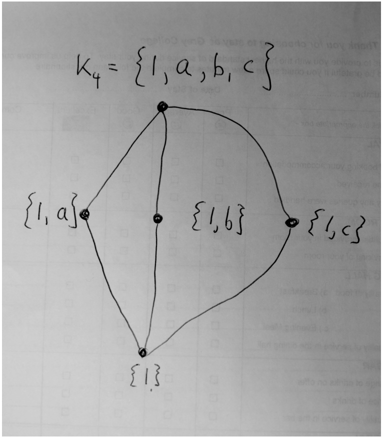

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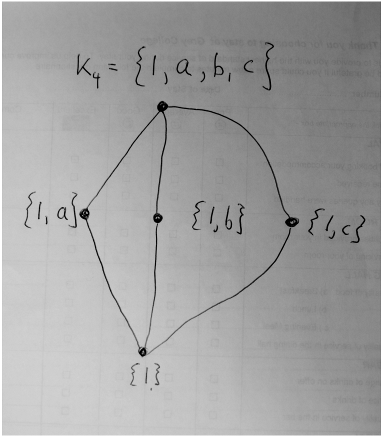

Forget that K_4 is a group.

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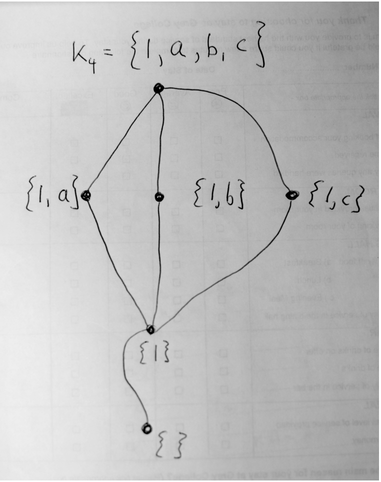

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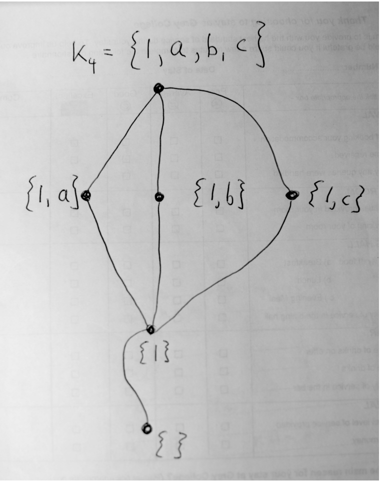

We get an extra subsemigroup: the empty semigroup!

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Semigroups don't have normal subgroups.
But Semigroups have ideals:
$I \leq S$ is an ideal $\iff$ $xi, ix \in I$ for every $s \in S, i \in I$.

Ideals

Proposition (cf. Ganyushkin-Livinsky '11)

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$S/I = S \setminus I \cup \{0\}$ with operation $*$.

$$s * t = \begin{cases} st \text{ if } st \in S \setminus I; \\ 0 \text{ otherwise}; \end{cases}$$

and $s0 = 0s = 00 = 0$.

Green's relations

If S is a semigroup and $x, y \in S$, then we write

- $x \mathscr{L} y$ if $S^1 x = S^1 y$
- $x \mathscr{R} y$ if $x S^1 = y S^1$
- $x \mathscr{J} y$ if $S^1 x S^1 = S^1 y S^1$
- $x \mathscr{H} y$ if $x \mathscr{L} y$ and $x \mathscr{R} y$

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These relations are equivalences called *Green's relations*, and their classes are *Green's classes*.

Principal factors

The *principal factor* J^* of a $\mathscr{J}$-class J is the set $J \cup \{0\}$ with multiplication

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Lemma

Let S be a finite regular semigroup and let $J_1, J_2, \ldots, J_m$ be the $\mathscr{J}$-classes of S. Then

$$l(S) = l(J_1^*) + l(J_2^*) + \cdots + l(J_m^*) - 1.$$

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Theorem

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Here are the first few values:

n	2	3	4	5	6	7	8
n^n	4	27	256	3 125	46 656	823 543	16 777 216
$a(n)$	0	0	7	110	1 921	37 795	835 290

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Similar lower bounds for the lengths of:

- order-preserving transformations O_n
- the general linear semigroup $GLS(n, q)$.

Inverse semigroups

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Theorem (cf. Ganyushkin and Livinsky (2011))

Let S be a finite inverse semigroup with $\mathscr{J}$-classes $J_1, \ldots, J_m$. If $n_i \in \mathbb{N}$ denotes the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J_i, and G_i is any maximal subgroup of S contained in J_i, then

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$$\begin{aligned} l(S) &= -1 + \sum_{i=1}^{m} l(J_i^*) \\ &= -1 + \sum_{i=1}^{m} n_i(l(G_i) + 1) + \frac{n_i(n_i - 1)}{2}|G_i| + (n_i - 1). \end{aligned}$$

The symmetric inverse monoid, part I

The *symmetric inverse monoid* I_n consists of all bijections between subsets of $X = \{1, \ldots, n\}$.

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If $f, g \in I_n$, then

- $f \mathscr{L} g$ if and only if $\mathrm{im}(f) = \mathrm{im}(g)$;
- $f \mathscr{R} g$ if and only if $\mathrm{dom}(f) = \mathrm{dom}(g)$;
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If J_i is the $\mathscr{J}$-class of I_n consisting of elements f with $|\,\mathrm{dom}(f)| = i$, then the number of $\mathscr{L}$- and $\mathscr{R}$-classes in J_i is $\binom{n}{i}$. Thus

$$l(I_n) \;=\; -1 + \sum_{i=1}^{n} \binom{n}{i} (l(S_i) + 1) + \frac{\binom{n}{i}(\binom{n}{i} - 1)}{2} |S_i| + \binom{n}{i} - 1.$$

The symmetric inverse monoid, part II

n	1	2	3	4	5	6	7	8
$\lvert I_n \rvert$	2	7	34	209	1 546	13327	130 922	1 441 729
$l(I_n)$	1	6	25	116	722	5 956	59 243	667 500

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The same limit holds for various other well-known inverse semigroups: the dual symmetric inverse monoid, the semigroup of partial order-preserving injective mappings, and so on.

Open problem

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Thank you for listening!