Synchronization Theory and Links to Combinatorics

Artur Schäfer

University of St. Andrews

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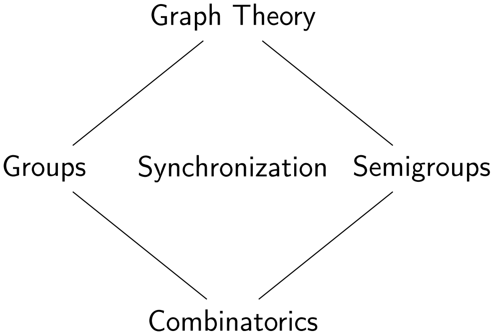
The Setting


Graph Theory
Groups
Synchronization
Semigroups
Combinatorics

Outline

1. Synchronization Theory
2. Hulls of Graphs
3. Endomorphisms and Combinatorics
4. Tilings and Semigroups

Synchronization Theory

Definition: Synchronization

- G **synchronizes** t, if the semigroup $\langle G, t \rangle$ has a map of rank 1 (size of its image).
- G is **synchronizing**, if G synchronizes all transformations t.

primitive $\Leftarrow$ almost-synchronizing $\Leftarrow$ synchronizing $\Leftarrow$ 2-transitive

The Synchronization Problem

What are the transformations (not) synchronized by G?

We know many examples of synchronizing groups are known.

Which ranks are synchronized by G?

Results

$n-1, n-2$ and 2, and $3, 4$ for non-uniform maps.

Recently (ABCRS) 2015: $n-3, n-4$, and $n-(1+\sqrt{n-1}/12)$ (for rank 3 groups)

How did we get the previous results? -> Use Connection to Graphs

Theorem (Cameron-2008)

G does not synchronize the map t, if and only if $\exists$ a **graph** X with

1. $G \leq \mathrm{Aut}(X)$,
2. $\omega(X) = \chi(X) = n$, and
3. t is a singular endomorphism of X.

The Programme:

Analyse synchronizing groups G
$\Leftrightarrow$ Find endomorphisms (of minimal rank n) of graphs .

Hulls of Graphs

The theorem uses the following graph construction:

Construction: Graph of a Semigroup S

S a semigroup on n points. Then, the graph $Gr(S)$ has vertices $\{1, .., n\}$, where

two vertices v and w are adjacent, if there is **no** map $f \in S$ with $vf = wf$.

Definition (Hull)

Let X be a graph with endomorphism monoid $S = \mathrm{End}(X)$. Then, the **hull** of X is

$$Hull(X) = Gr(S).$$

Properties of $Gr(S)$

Let $\Gamma = Gr(S)$, then

- $S \leq \mathrm{End}(\Gamma)$,
- Γ satisfies $\omega = \chi$,
- if S is synchronizing, then $Gr(S)$ is the null-graph,
- if S is a permutation group, then $Gr(S)$ is the complete graph.

Now, we go for the hull $Y = Hull(X)$ of a graph X.

- X is a (spanning) subgraph of Y.,
- $\mathrm{Aut}(X) \leq \mathrm{Aut}(Y)$,
- $\mathrm{End}(X) \leq \mathrm{End}(Y)$,
- $Hull(X) = Hull(Y)$.

What makes hulls so important?

We are going to ask 2 question:

1. Which graph is a hull? (satisfies $X = Hull(X)$)
2. What are the (minimal) generators of $Gr(S)$?

Graphs which are Hulls

Approach: Find graphs with endomorphisms and check.

Theorem

If X is a graph with non-trivial hull whose automorphism group G has permutation rank 3, then X is a hull.

Further Hulls:

- Multi-partite graphs + Complement
- Hamming graphs + Complement

Non-Hulls:

- Paths, even cycles,
- $C_n \boxdot C_n$, for C_n an odd cycle $n \geq 5$

Generators of $Gr(S)$: Part I

Question: Do we really need all elements of S to obtain $Gr(S)$?

Lemma

- Kernel class representatives in S (R-Class Reps) generate $Gr(S)$.
- The elements of minimal rank in S (its minimal ideal) generate $Gr(S)$.

Corollary

1. The idempotents of S generate $Gr(S)$.
2. The generating set can be chosen to form a left-zero semigroup.

Generators of $Gr(S)$: Part II Examples

Monogenic Semigroups

$S = \langle a \rangle$ with index m and period r, then $\{a^m\}$ generates $Gr(S)$.

Bands (every element is an idempotent)

Generators of the minimal ideal generate $Gr(S)$.

Left-(Right)Zero Semigroups

The generators of S generate $Gr(S)$.

Minimal generating Sets

Lemma: Minimal sets for $L_2(n)$

- If n is a prime power, then the minimal generating set is given by a complete set of $n-1$ MOLS.
- If not, then the minimal generating set contains at most $n(n-1)$ elements.

Lemma

For $\overline{L_2(n)}$ a minimal generating has size 2.

Endomorphisms and Combinatorics

Consider hypercuboids: $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}$
In $H^R(n_1, ..., n_m)$ two vertices are adjacent, if their Hamming distance is in $\{1, ..., R\}$. $\rightarrow$ $H^1(n, ..., n)$ =Hamming graph.

Lemma

The endomorphisms of minimal rank of $H^R(n_1, ..., n_m)$ are Latin hypercuboids of class R.

Example R=2

The two layers form a Latin hypercuboid

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \begin{pmatrix} 5 & 6 & 4 \\ 3 & 1 & 2 \end{pmatrix}$$

They don't exist for all parameters.!!!

Latin hypercuboids of class R have **not appeared in the literature** and have **not** been counted.

Mixed codes

Mixed codes = elements of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_m}$. (Brouwer et. al considered $n_i \in \{2, 3\}$ in $'90s$, others considered **perfect** mixed codes, but not much known, in general).

Definition (Mixed MDS-code)

A mixed MDS code is a mixed code C with minimum distance δ satisfying the generalized Singleton bound

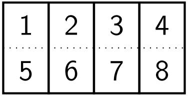
$$|C| \leq \prod_{i=1}^{m-\delta+1} n_{m-i+1} = n_\delta \cdots n_m.$$

Proposition

The Latin hypercuboids (of class R) are (almost) equivalent with and mixed MDS-codes.

Tilings and Semigroups

Idea: Tiling a 2×4 chess board with 2×1 tiles.



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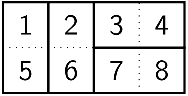

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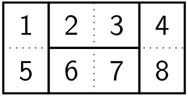

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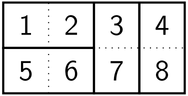

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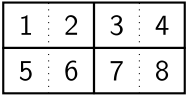

(5)

$\{1, 3, 6, 8\}$ and $\{2, 4, 5, 7\}$ are transversals of all tilings (partitions).
$\rightarrow$Let $f_1, \dots, f_{10}$ be the maps constructed from the partition-transversal combinations and $S = \langle f_1, \dots, f_{10} \rangle$

Tilings and Semigroups

Theorem

S satisfies the following

1. *S is idempotent generated, (and simple in this case)*
2. *For all $f_1, f_2 \in S$ it holds $\ker(f_1 f_2) = \ker(f_1)$ and $\operatorname{im}(f_1 f_2) = \operatorname{im}(f_2)$,*
3. *S is non-synchronizing.*

Consequences:

- New examples of non-synchronizing semigroups, and
- old examples seen in a new light $\overline{H^1(n, ..., n)}$.

Disjoint Decompositions

Def: S is decomposable, if $S = S_1 \uplus S_2 \uplus \cdots \uplus S_n$.

Definition

$S = \langle G, T \rangle \setminus G$, $T \subseteq T_n$ admits a **strong decomposition**, if for all $T' \subseteq T$ holds

$$\langle G, T' \rangle \setminus G = \biguplus_{a \in T'} \langle G, a \rangle \setminus G.$$

Theorem

Let S come from the tiling construction. If S is simple, then S admits a strong decomposition.

Question: Where does the group in S come from?

Problems

Problems:

- Find more families of hulls and their minimal generating sets.
- Count Latin hypercuboids.
- How good are mixed (MDS-)codes?
- Do non-synchronizing semigroups always admit some sort of decomposition?

Thank You for Your Attention!