"K3 Surface and Mathieu Moonshine"

T.E.,H.Ooguri and Y.Tachikawa

T.E. and K.Hikami

♣ Elliptic genus of K3

Elliptic genus in string theory is expressed as

$$Z_{elliptic}(z;\tau)=Tr_{\mathcal{H}_L\times\mathcal{H}_R}(-1)^{F_L+F_R}e^{4\pi izJ^3_{L,0}}q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}$$

and describes the topological invariants of the target manifold and counts the number of BPS states in the theory. Here L_0 denotes the zero mode of the Visasoro operators and F_L and F_R are left and right moving fermion numbers. In ellitpic genus the right moving sector is frozen to the supersymmetric ground states (BPS states) while in the left moving sector all the states in the Hilbert space $\mathcal{H}_L$ contribute.

SUSY algebra

$$\{\bar{G}_0^i, \bar{G}_0^{*\,j}\} = 2\delta^{ij}\bar{L}_0 - \frac{k}{2}\delta^{ij}, \ (i, j = 1, 2) \Longrightarrow \bar{L}_0 \geq \frac{k}{4}$$

BPS states

$$\bar{L}_0 = \frac{k}{4}$$

Elliptic genus of K3 surface is known: EOTY

$$Z_{K3}(z;\tau) = 8\left[\left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2\right]$$

$$Z_{K3}(z=0) = 24, \quad Z_{K3}(z=\frac{1}{2}) = 16 + O(q),$$

$$Z_{K3}(z=\frac{1+\tau}{2}) = 2q^{-\frac{1}{2}} + O(q^{\frac{1}{2}})$$

Elliptic genus of a complex D-dimensional manifold is a Jacobi form of weight=0 and index=D/2. When D=2, space of Jacobi form is one-dimensional and given by the above formula.

Jacobi form (weight k and index m)

$$\varphi(\tau, z + a\tau + b) = e^{-2\pi i m(a^2\tau + 2az)}\varphi(\tau, z)$$
$$\varphi(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}) = (c\tau + d)^k e^{\frac{2\pi i m c z^2}{c\tau + d}}\varphi(\tau, z)$$

String theory on K3 has an N=4 superconformal symmetry and its states fall into representations of N=4 superconformal algebra (SCA). N=4 SCA contains an affine $SU(2)_k$ symmetry and has a central charge $c = 6k$. $k = n$ case describes complex-$2n$ dimensional hyperKähler manifolds.

We would like to study the decomposition of the elliptic genus in terms of irreducible characters of N=4 SCA. In N=4 SCA, hightest-weight states $|h, \ell\rangle$ are charactered by

$$L_0|h, \ell\rangle = h|h, \ell\rangle, \qquad J_0^3|h, \ell\rangle = \ell|h, \ell\rangle$$

and the theory possesses two different type of representations, BPS and non-BPS representations. In the case of $k=1$ there are representations (in Ramond sector)

BPS rep.	$h=\frac{1}{4};$	$\ell=0,\ \frac{1}{2}$
non-BPS rep.	$h>\frac{1}{4};$	$\ell=\frac{1}{2}$

Character of a representation is given by

$$Tr_{\mathcal{R}}(-1)^F q^{L_0} e^{4\pi i z J_0^3}$$

Its index is given by the value at $z=0$, $Tr_{\mathcal{R}}(-1)^F q^{L_0}$.

BPS representations have a non-vanishing index

$$\text{index (BPS, } \ell=0) = 1$$
$$\text{index (BPS, } \ell=\frac{1}{2}) = -2$$

Character function of $\ell = 0$ BPS representation has the form

$$ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) = \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}\mu(z;\tau)$$

where

$$\mu(z;\tau) = \frac{-ie^{\pi i z}}{\theta_1(z;\tau)}\sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)}e^{2\pi i n z}}{1-q^n e^{2\pi i z}}$$

On the other hand the character of non-BPS representations are given by

$$ch^{\tilde{R}}_{h>\frac{1}{4},\ell=\frac{1}{2}} = q^{h-\frac{3}{8}}\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

These have vanishing index

$$\text{index (non-BPS rep)} = 0$$

At the unitarity bound non-BPS representation splits into a sum of BPS representations

$$\lim_{h \to \frac{1}{4}} q^{h-\frac{3}{8}} \frac{\theta_1^2}{\eta^3} = ch^{\tilde{R}}_{h=\frac{1}{4},\ell=\frac{1}{2}} + 2ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}$$

Function $\mu(z;\tau)$ is a typical example of the so-called Mock theta functions (Lerch sum or Appell function). Mock theta functions look like theta functions but they have anomalous modular transformation laws and are difficult to handle. Recently there were developments in understanding the nature of Mock theta functions due to Zwegers. He has introduced a method of regularization which is similar to the ones used in physics and improved the modular property of mock theta functions so that they transform as analytic Jacobi forms.

It is possible to derive the following idenities

$$
\begin{aligned}
ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) &= \left(\frac{\theta_2(z;\tau)}{\theta_2(0;\tau)}\right)^2 + \mu_2(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\
&= \left(\frac{\theta_3(z;\tau)}{\theta_3(0;\tau)}\right)^2 + \mu_3(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3} \\
&= \left(\frac{\theta_4(z;\tau)}{\theta_4(0;\tau)}\right)^2 + \mu_4(\tau)\frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}
\end{aligned}
$$

where

$$\mu_2(\tau)=\mu(z=\frac{1}{2};\tau),\ \mu_3(\tau)=\mu(z=\frac{1+\tau}{2};\tau),\ \mu_4(\tau)=\mu(z=\frac{\tau}{2};\tau)$$

$$\mu(z;\tau) = \frac{-ie^{\pi i z}}{\theta_1(z;\tau)} \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}$$

Then we can rewrite the ellitpic genus as

$$Z_{K3} = 24 ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z;\tau) - 8 \sum_{i=2}^{4} \mu_i(\tau) \frac{\theta_1(z;\tau)^2}{\eta(\tau)^3}$$

Using q-expansion of functions μ_i we find

$$8\left(\mu_2(\tau) + \mu_3(\tau) + \mu_4(\tau)\right) = -2 \sum_{n=0} A(n) q^{n-\frac{1}{8}}$$

$$Z_{K3} = 24 ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0}(z,\tau) + 2\sum_{n\geq 0} A(n) ch^{\tilde{R}}_{h=\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

At smaller values of n, Fourier coefficients $A(n)$ may be obtained by direct expansion. We find, $A(0) = -1$

n	1	2	3	4	5	6	7	8	$\cdots$
$A(n)$	45	231	770	2277	5796	13915	30843	65550	$\ldots$

Surprize: Dimensions of some irreducible reps. of Mathieu group M_{24} appear

dimensions : $\{\ 45 \quad 231 \quad 770 \quad 990 \quad 1771 \quad 2024 \quad 2277 \quad 3312 \quad 3520 \quad 5313 \quad 5544 \quad 5796 \quad 10395 \quad \cdots \}$

$$A(6) = 13915 = 3520 + 10395,$$

$$A(7) = 30843 = 10395 + 5796 + 5544 + 5313 + 2024 + 1771$$

Mathieu moonshine? T.E.-Ooguri-Tachikawa

cf. Monsterous moonshine:

$$J(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \cdots$$

q-expansion coeffcients of J-function are decomposed into a sum of irred. reps. of the monster group.

$$196884 = 1 + 196883, \;\; 21493760 = 1 + 196883 + 21296876$$

Mukai: enumeration of eleven K3 surfaces with finite non-Abelian automorphism group. All these groups are sugbgroups of M_{23}.

Fantasy: Is it possible that these automorphism groups at isoletd points in K3 moduli space are enhanced to M_{24} over the whole moduli space when one cos-nider the elliptic genus?

On the other hand, using the method of Rademacher expansion adapted to the case of Mock theta functions (Bringmann-Ono) we can determine the asymptotic behavior of coefficients $A(n)$ as

$$A(n) \approx \frac{2}{\sqrt{8n-1}} e^{2\pi\sqrt{\frac{1}{2}(n-\frac{1}{8})}}$$

T.E.-Hikami

♣ Twisted Elliptic Genus

Dimension of the representation equals the trace of the identity element: we may identify

$$A(n) = Tr_{V_n}\ 1$$

$$V_1 = 45 + 45^*,\ V_2 = 231 + 231^*,\ V_3 = 770 + 770^*, \cdots$$

We consider the trace of other group elements in M_{24}

$$A_g(n) = Tr_{V_n}\ g, \qquad g \in M_{24}$$

$Tr\ g$ depends only on the conjugacy class of g. There exists 26 conjugacy classes $\{g\}$ in M_{24} and also 26 irreducible representations $\{R\}$. We have the character table given by

$$\chi_R{}^g = Tr_R\ g$$

1A	2A	3A	5A	4B	7A	7B	8A	6A	11A	15A	15B	14A	14B	23A	23B	12B	6B	4C	3B
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
23	7	5	3	3	2	2	1	1	1	0	0	0	0	0	0	-1	-1	-1	-1
252	28	9	2	4	0	0	0	1	-1	-1	-1	0	0	-1	-1	0	0	0	0
253	13	10	3	1	1	1	-1	-2	0	0	0	-1	-1	0	0	1	1	1	1
1771	-21	16	1	-5	0	0	-1	0	0	1	1	0	0	0	0	-1	-1	-1	7
3520	64	10	0	0	-1	-1	0	-2	0	0	0	1	1	1	1	0	0	0	-8
45	-3	0	0	1	e_7^+	e_7^-	-1	0	1	0	0	$-e_7^+$	$-e_7^-$	-1	-1	1	-1	1	3
$\overline{45}$	-3	0	0	1	e_7^-	e_7^+	-1	0	1	0	0	$-e_7^-$	$-e_7^+$	-1	-1	1	-1	1	3
990	-18	0	0	2	e_7^+	e_7^-	0	0	0	0	0	e_7^+	e_7^-	1	1	1	-1	-2	3
$\overline{990}$	-18	0	0	2	e_7^-	e_7^+	0	0	0	0	0	e_7^-	e_7^+	1	1	1	-1	-2	3
1035	-21	0	0	3	$2\ e_7^+$	$2\ e_7^-$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3
$\overline{1035}$	-21	0	0	3	$2\ e_7^-$	$2\ e_7^+$	-1	0	1	0	0	0	0	0	0	-1	1	-1	-3
$1035'$	27	0	0	-1	-1	-1	1	0	1	0	0	-1	-1	0	0	0	2	3	6
231	7	-3	1	-1	0	0	-1	1	0	e_{15}^+	e_{15}^-	0	0	1	1	0	0	3	0
$\overline{231}$	7	-3	1	-1	0	0	-1	1	0	e_{15}^-	e_{15}^+	0	0	1	1	0	0	3	0
770	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^+	e_{23}^-	1	1	-2	-7
$\overline{770}$	-14	5	0	-2	0	0	0	1	0	0	0	0	0	e_{23}^-	e_{23}^+	1	1	-2	-7
483	35	6	-2	3	0	0	-1	2	-1	1	1	0	0	0	0	0	0	3	0
1265	49	5	0	1	-2	-2	1	1	0	0	0	0	0	0	0	0	0	-3	8
2024	8	-1	-1	0	1	1	0	-1	0	-1	-1	1	1	0	0	0	0	0	8
2277	21	0	-3	1	2	2	-1	0	0	0	0	0	0	0	0	0	2	-3	6
3312	48	0	-3	0	1	1	0	0	1	0	0	-1	-1	0	0	0	-2	0	-6
5313	49	-15	3	-3	0	0	-1	1	0	0	0	0	0	0	0	0	0	-3	0
5796	-28	-9	1	4	0	0	0	-1	-1	1	1	0	0	0	0	0	0	0	0
5544	-56	9	-1	0	0	0	0	1	0	-1	-1	0	0	1	1	0	0	0	0
10395	-21	0	0	-1	0	0	1	0	0	0	0	0	0	-1	-1	0	0	3	0

Here we have used $e_p^\pm = \frac{1}{2}\left(\pm\sqrt{-p} - 1\right)$.

There are two types of conjugacy classes in M_{24}, type I and type II.

Conjugacy class of type I fixes at least one element out of 24 and thus they arise from the conjugacy classes of M_{23}.

On the other hand conjugacy class of type II does not have a fixed point and is intrinsically M_{24}.

For each conjugacy class we want to construct a twisted genus (analogue of Thompson series in monstrous moonshine)

$$A_g = \sum_{n=1}^{\infty} Tr_{V_n}\, g \times q^n$$

For instance,

$$A_{2A} = -6q + 14q^2 - 28q^3 + 42q^4 - 56q^5 + 86q^6 + \cdots$$

and has the right modular property ($Z_{2A} \in \Gamma_0(2)$).

Twisted genus is decomposed into massless and massive parts

$$Z_g(\tau, z) = \chi_g \, ch^{\tilde{R}}_{h=\frac{1}{4},\ell=0} + \sum_{n\geq 0} A_g(n) ch^{\tilde{R}}_{\frac{1}{4}+n,\ell=\frac{1}{2}}(z,\tau)$$

Here χ_g is the Euler number assigned to the class g

g	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	others
χ_g	24	8	6	4	4	3	2	2	2	1	1	1	0

χ_g vanishes for type II classes. We note that χ_g can be written as $\chi_g = \chi_1^g + \chi_{23}^g$ which is equal to the number of fixed points of the permutation rep. of g.

conjugacy class	cycle shape
1A	1^{24}
2A	$1^8 \cdot 2^8$
3A	$1^6 \cdot 3^6$
5A	$1^4 \cdot 5^4$
4B	$1^4 \cdot 2^2 \cdot 4^4$
7A	$1^3 \cdot 7^3$
7B	$1^3 \cdot 7^3$
8A	$1^2 \cdot 2^1 \cdot 4^1 \cdot 8^2$
6A	$1^2 \cdot 2^2 \cdot 3^2 \cdot 6^2$
11A	$1^2 \cdot 11^2$
15A	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$
15B	$1^1 \cdot 3^1 \cdot 5^1 \cdot 15^1$
14A	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$
14B	$1^1 \cdot 2^1 \cdot 7^1 \cdot 14^1$
23A	$1^1 \cdot 23^1$
23B	$1^1 \cdot 23^1$
12B	12^2
6B	6^4
4C	4^6
3B	3^8
2B	2^{12}
10A	$2^2 \cdot 10^2$
21A	$3^1 \cdot 21^1$
21B	$3^1 \cdot 21^1$
4A	$2^4 \cdot 4^4$
12A	$2^1 \cdot 4^1 \cdot 6^1 \cdot 12^1$

Twisted genera for all conjugacy classes of M_{24} have been obtained by collective efforts by various authors. They reproduce correct lower-order expansion coefficients and are invariant under the Hecke subgroup $\Gamma_0(N)$

$$\Gamma_0(N) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; ad - bc = 1, \; c \equiv 0, \; mod\, N\}$$

N denotes the order of the element g.

M.Cheng, Gaberdiel,Hohenegger and Volpato,

T.E. and K.Hikami

From the study of K3 surface with $Z_p(p = 2, 3, \cdots)$ symmetry, for instance, twisted genera of classes

$pA(p=2,3,\cdots)$ are known

A.Sen

$$Z_{pA}(z;\tau) = \frac{2}{p+1}\phi_{0,1}(z;\tau) + \frac{2p}{p+1}\phi_2^{(p)}(\tau)\phi_{-2,1}(z;\tau)$$

where

$$\phi_{0,1}(z;\tau) = \frac{1}{2}Z_{K3}(z;\tau), \qquad \phi_{-2,1}(z;\tau) = -\frac{\theta_1(z;\tau)^2}{\eta(\tau)^6}$$

are the basis of Jacobi forms with index=1 and

$$\begin{aligned}\phi_2^{(p)}(\tau) &= \frac{24}{p-1} q\partial_q \log\left(\frac{\eta(p\tau)}{\eta(\tau)}\right), \\ &= \frac{24}{p-1}\sum_{k=1}\sigma_1(k)(q^k - pq^{pk})\end{aligned}$$

is an element of $\Gamma_0(p)$.

In the case of type II twisted genera are modular forms of $\Gamma_0(N)$ with a multiplier system (invariant up to a phase). They are given in terms of quotients of eta functions.

$$
\begin{aligned}
Z_{2B}(z;\tau) &= 2\frac{\eta(\tau)^8}{\eta(2\tau)^4}\phi_{-2,1}(z;\tau), \\
Z_{3B}(z;\tau) &= 2\frac{\eta(\tau)^6}{\eta(3\tau)^2}\phi_{-2,1}(z;\tau), \\
Z_{4A}(z;\tau) &= 2\frac{\eta(2\tau)^8}{\eta(4\tau)^4}\phi_{-2,1}(z;\tau), \\
Z_{4C}(z;\tau) &= 2\frac{\eta(\tau)^4\eta(2\tau)^2}{\eta(4\tau)^2}\phi_{-2,1}(z;\tau) \\
\cdots
\end{aligned}
$$

etc. The multiplier sytem has been studied in detail by Gaberdiel, Persson, Ronellenfitsch and Volpato .

Thus we have a complete list of the twisted genera for 26 conjugacy classes. Making use of them we can uniquely decompose the coefficients of K3 elliptic genus into irreducible representations of M_{24} at arbitrary level.

n	1A	2A	3A	5A	4B	7A	8A	6A	11A	15A	14A	23A	12B	6B	4C	3B
1	90	-6	0	0	2	-1	-2	0	2	0	1	-2	2	-2	2	6
2	462	14	-6	2	-2	0	-2	2	0	-1	0	2	0	0	6	0
3	1540	-28	10	0	-4	0	0	2	0	0	0	-1	2	2	-4	-14
4	4554	42	0	-6	2	4	-2	0	0	0	0	0	0	4	-6	12
5	11592	-56	-18	2	8	0	0	-2	-2	2	0	0	0	0	0	0
6	27830	86	20	0	-2	-2	2	-4	0	0	2	0	0	0	6	-16
7	61686	-138	0	6	-10	2	-2	0	-2	0	2	0	-2	-2	-2	30
8	131100	188	-30	0	4	-3	0	2	2	0	-1	0	0	0	-12	0
9	265650	-238	42	-10	10	0	-2	2	0	2	0	0	-2	6	10	-42
10	521136	336	0	6	-8	0	-4	0	0	0	0	2	-2	2	16	42
11	988770	-478	-60	0	-14	6	2	-4	2	0	-2	0	0	0	-6	0
12	1830248	616	62	8	8	0	0	-2	2	2	0	0	2	-6	-16	-70
13	3303630	-786	0	0	22	-6	2	0	0	0	-2	2	0	-4	6	84
14	5844762	1050	-90	-18	-6	0	2	6	0	0	0	2	0	0	18	0
15	10139734	-1386	118	4	-26	-4	-2	6	0	-2	0	0	2	2	-10	-110
16	17301060	1764	0	0	12	0	0	0	-4	0	0	0	2	6	-28	126
17	29051484	-2212	-156	14	28	0	-4	-4	0	-1	0	0	0	0	12	0
18	48106430	2814	170	0	-18	8	-2	-6	-2	0	0	-2	2	-6	38	-166
19	78599556	-3612	0	-24	-36	0	0	0	2	0	0	0	-2	-6	-20	210
20	126894174	4510	-228	14	14	-6	-2	4	0	2	2	0	0	0	-42	0
21	202537080	-5544	270	0	48	4	4	6	-2	0	0	0	-2	6	16	-282
22	319927608	6936	0	18	-16	-7	4	0	0	0	-1	0	0	4	48	300
23	500376870	-8666	-360	0	-58	0	-2	-8	4	0	0	2	0	0	-18	0
24	775492564	10612	400	-36	28	0	0	-8	0	0	0	0	0	-8	-60	-392
25	1191453912	-12936	0	12	64	12	-4	0	0	0	0	0	2	-10	32	462
26	1815754710	15862	-510	0	-34	0	-6	10	0	0	0	-1	0	0	78	0
27	2745870180	-19420	600	30	-76	-10	4	8	-2	0	-2	0	0	8	-36	-600
28	4122417420	23532	0	0	36	2	0	0	0	0	-2	0	0	12	-84	660
29	6146311620	-28348	-762	-50	100	-6	4	-10	-2	-2	2	0	0	0	36	0
30	9104078592	34272	828	22	-40	0	4	-12	4	-2	0	0	0	-8	96	-840
31	13401053820	-41412	0	0	-116	0	-4	0	0	0	0	-2	-2	-10	-44	966
32	19609321554	49618	-1062	34	50	18	2	10	-2	-2	2	0	0	0	-126	0
33	28530824630	-59178	1220	0	126	0	-6	12	0	0	0	2	-4	12	62	-1204
34	41286761478	70758	0	-72	-66	-10	-6	0	6	0	2	0	0	12	150	1332
35	59435554926	-84530	-1518	26	-154	6	2	-14	0	2	2	0	0	0	-66	0
36	85137361430	100310	1670	0	70	-12	-2	-10	0	0	0	0	-2	-18	-170	-1666

♣ Proof of Mathieu moonshine

Orthogonality relation of characters:

$$\sum_g n_g \, \chi_{R'}^g \, \bar{\chi}_R^g = |G|\delta_{RR'}$$

n_g is the number of elements in the conjugacy class g and $|G|$ denotes the order of the group. Let $c_R(n)$ be he multiplicity of representation R in the decom-postion of K3 elliptic genus at level n. We then have

$$\sum_R c_R(n)\chi_R^g = A_g(n)$$

Then using the orthogonality relation we find

$$\sum_g \frac{1}{|G|} n_g \bar{\chi}_R^g A_g(n) = c_R(n)$$

We have checked that the multiplicities $c_R(n)$ are all positive integers upto $n = 1000$ and this gives a very strong evidence for Mathieu moonshine conjecture.

n	1	23	252	253	1771	3520	$\dfrac{45}{45}$	$\dfrac{990}{990}$	$\dfrac{1035}{1035}$	$1035'$	$\dfrac{231}{231}$	$\dfrac{770}{770}$	483
1	0	0	0	0	0	0	1	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	1	0	0
3	0	0	0	0	0	0	0	0	0	0	0	1	0
4	0	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	2	0	0	0	0	0	0	0
7	0	0	0	0	2	0	0	0	0	0	0	0	0
8	0	0	0	0	0	2	0	1	1	0	0	0	0
9	0	0	0	0	2	4	0	0	2	2	0	2	2
10	0	0	0	2	4	8	0	2	2	2	2	0	2
11	0	0	0	0	8	12	0	4	4	6	0	4	0
12	0	2	2	4	12	30	0	8	8	4	2	6	4
13	0	0	4	2	26	44	2	14	14	18	2	10	6
14	0	0	4	6	38	86	0	24	24	22	8	16	14
15	0	0	12	8	78	144	2	40	44	46	8	38	18
16	0	2	18	22	122	252	2	72	72	68	18	50	36
17	0	2	30	26	212	410	8	116	124	130	25	94	54
18	0	6	50	58	342	704	6	194	202	192	50	148	100
19	0	4	80	72	582	1116	18	318	332	346	68	252	150
20	0	14	128	138	904	1836	20	516	536	520	126	390	254
21	2	20	214	200	1476	2902	40	814	860	872	182	652	396
22	2	32	328	346	2302	4616	55	1298	1348	1336	314	988	640
23	2	40	512	496	3638	7166	98	2020	2118	2144	460	1590	972
24	0	80	798	824	5584	11192	132	3140	3278	3236	744	2426	1544
25	8	108	1232	1208	8654	17084	234	4814	5038	5084	1106	3764	2336
26	6	174	1860	1904	13090	26148	322	7348	7670	7626	1742	5677	3602
27	12	252	2836	2802	19914	39436	514	11092	11618	11666	2560	8688	5394
28	16	398	4238	4310	29772	59330	742	16686	17418	17356	3922	12912	8160
29	26	560	6328	6286	44512	88280	1154	24840	25994	26078	5758	19380	12090
30	34	876	9368	9486	65776	131020	1642	36824	38480	38368	8642	28580	18008
31	58	1236	13802	13764	97060	192538	2500	54178	56660	56800	12582	42218	26384
32	76	1866	20166	20356	141714	282074	3564	79320	82884	82730	18576	61574	38738
33	122	2664	29396	29374	206524	410062	5286	115334	120644	120798	26830	89868	56226
34	166	3900	42474	42810	298508	593800	7542	166990	174510	174330	39066	129694	81546
35	248	5536	61184	61234	430134	854284	10988	240304	251292	251544	55956	187094	117138

Recently Gannon has proved by mathematical induction that the multiplicities are all positive integers for all n.

Summary

- There is a strong evidence for Mathieu moonshine phenomenon for K3 surface.
- It is beyond classical geometry and no fundamental explanations so far.
- Hilbert space of string theory compactified on K3

surfaces does not possess symmetry under M_{24}.

Gaberdiel-Hohenegger-Volpato, Taormina-Wendland.

M_{24} symmetry should be searched within the BPS or

topololgical sector of the theory ?

- Discovery of more moonshine phenomena;

Umbral moonshine, free fermions on 24 dim. lattice

Spin 7 manifold, N=2 extremal Jacobi form, ,,,

♠ Zweger's method

Function μ

$$\mu(z;\tau) = \frac{-ie^{i\pi z}}{\theta_1(z;\tau)} \sum_n (-1)^n \frac{q^{\frac{1}{2}n(n+1)} e^{2\pi i n z}}{1 - q^n e^{2\pi i z}}$$

has the S-tranformation

$$\mu(z;\tau) + \sqrt{\frac{i}{\tau}}\mu(\frac{z}{\tau}; -\frac{1}{\tau}) = \frac{1}{2}M(\tau)$$

$$M(\tau) = \int_{-\infty}^{+\infty} \frac{e^{\pi i \tau \alpha^2}}{\cosh \pi\alpha} d\alpha$$

$$\Uparrow$$

Modell integral

Zwegers prescription to cure the modular property is to first introduce a non-holomrophic partner of $\mu(z;\tau)$,

$$R(\tau) = \sum(-1)^n[sgn(n+\frac{1}{2}) - E((n+\frac{1}{2})\sqrt{2\tau_2})]$$
$$\times q^{-\frac{1}{2}(n+\frac{1}{2})^2}, \quad \tau = \tau_1 + i\tau_2$$

Here E denotes the error function. By construction $R(\tau)$ obeys a transformation law

$$R(\tau) + \sqrt{\frac{i}{\tau}}R(-\frac{1}{\tau}) = M(\tau)$$

Then we form a combination

$$\hat{\mu}(z;\tau) \equiv \mu(z;\tau) - \frac{1}{2}R(\tau)$$

The Mordell integral cancels out and $\hat{\mu}$ has a good modular property

$$\hat{\mu}(z;\tau) = -\sqrt{\frac{i}{\tau}}\hat{\mu}(\frac{z}{\tau};-\frac{1}{\tau})$$

and is in fact an analytic Jacobi form of weight=1/2 and index=0. Explicitly R is expressed by a contour integral

$$R(\tau) = -i\int_{-\bar{\tau}}^{i\infty} \frac{\eta(z)^3}{\sqrt{\frac{z+\tau}{i}}}dz$$

Completion at higher level

$$\widehat{f_u^{(k)}}(\tau, z) = f_u^{(k)}(\tau, z) - \frac{1}{2} \sum_{m \in \mathrm{Z}_{2k}} R_{m,k}(\tau, u) \Theta_{m,k}(\tau, 2z), \; (k \in \mathrm{Z} > 0)$$

$$f_u^{(k)}(\tau, z) = \sum_{n \in Z} \frac{q^{kn^2} y^{2kn}}{1 - y w^{-1} q^n}, \; (q = e^{2\pi i \tau}, \; y = e^{2\pi i z}, \; w = e^{2\pi i u})$$

$$R_{m,k}(\tau, u) = \sum_{\nu \in m + 2kZ} \left[\mathrm{sgn}(\nu + 0) - \mathrm{Erf}\left\{ \sqrt{\frac{\pi \tau_2}{k}} \left(\nu + 2k \frac{u_2}{\tau_2} \right) \right\} \right] w^{-\nu} q^{-\frac{\nu^2}{4k}}$$

$$(\tau_2 = \mathrm{Im}\tau, u_2 = \mathrm{Im} u)$$

Error function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

Integral representation

$$\mathrm{sgn}(\nu + 0) - \mathrm{Erf}(\nu) = \frac{1}{i\pi} \int_{\mathbb{R}-i0} dp \frac{e^{-(p^2+\nu^2)}}{p - i\nu}$$

Poincare series representation (we set $u = 0$)

T.E.-Sugawara

$$\widehat{f}^{(k)}(\tau, z) = \frac{i}{2\pi} \sum_{n,m \in \mathbb{Z}} q^{km^2} y^{2km} \frac{e^{-\frac{\pi k}{\tau_2}(z+m\tau+n)^2}}{z + m\tau + n}$$

$$
\begin{aligned}
& ch^{\tilde{R}}_{h=\frac{k}{4},\ell}(z;\tau) \\
&= (-1)^{k-\ell}\frac{\theta_1(\tau,z)^2}{i\eta(\tau)\theta_1(\tau,2z)}\sum_{a=-\ell}^{a=\ell+1}\sum_n \frac{(yq^n)^a}{1-yq^n}q^{(k+1)n^2}y^{2(k+1)n} \\
&= (-1)^{k-\ell}\frac{\theta_1(\tau,z)^2}{i\eta(\tau)\theta_1(\tau,2z)}\sum_{a=-\ell}^{\ell+1} q^{-\frac{a^2}{4(k+1)}} s^{(k+1)}_{\frac{a}{2(\ell+1)}\tau}\cdot f^{(k+1)}_{\frac{a}{2(k+1)}\tau}(\tau,z)
\end{aligned}
$$

$$
s^{(k)}_{\lambda}\cdot f_u(\tau,z) = q^{k\alpha^2}y^{2k\alpha}e^{2\pi i k\alpha\beta}f_u(\tau, z+\alpha\tau+\beta),
$$

$$
(\lambda = \alpha\tau+\beta, \alpha,\beta\in R)
$$

Completion

$$f \to \hat{f}$$

Identity

$$q^{-\frac{a^2}{4(k+1)}} s^{(k+1)}_{\frac{a}{2(\ell+1)}\tau} \cdot \hat{f}^{(k+1)}_{\frac{a}{2(k+1)}\tau}(\tau, z) = \hat{f}^{(k+1)}_0(\tau, z)$$

Thus

$$\begin{aligned} \hat{ch}^{\tilde{R}}_{h=\frac{k}{4},\ell}(\tau, z) &= (-1)^{k-\ell} \frac{2(\ell+1)\theta_1(\tau, z)^2}{i\eta(\tau)^3\theta_1(\tau, 2z)} \hat{f}^{(k+1)}_0(\tau, z) \\ &= (-1)^{\ell}(\ell+1)\hat{ch}^{\tilde{R}}_{h=\frac{k}{4},0}(\tau, z) \end{aligned}$$

Thus the completed characters of all level-k BPS rep-resenations become proportional to each other. What

does this mean for the representation theory of su-
perconformal algebra?