2015, LMS EPSRC Symposium

Umbral Moonshine and K3 CFTs

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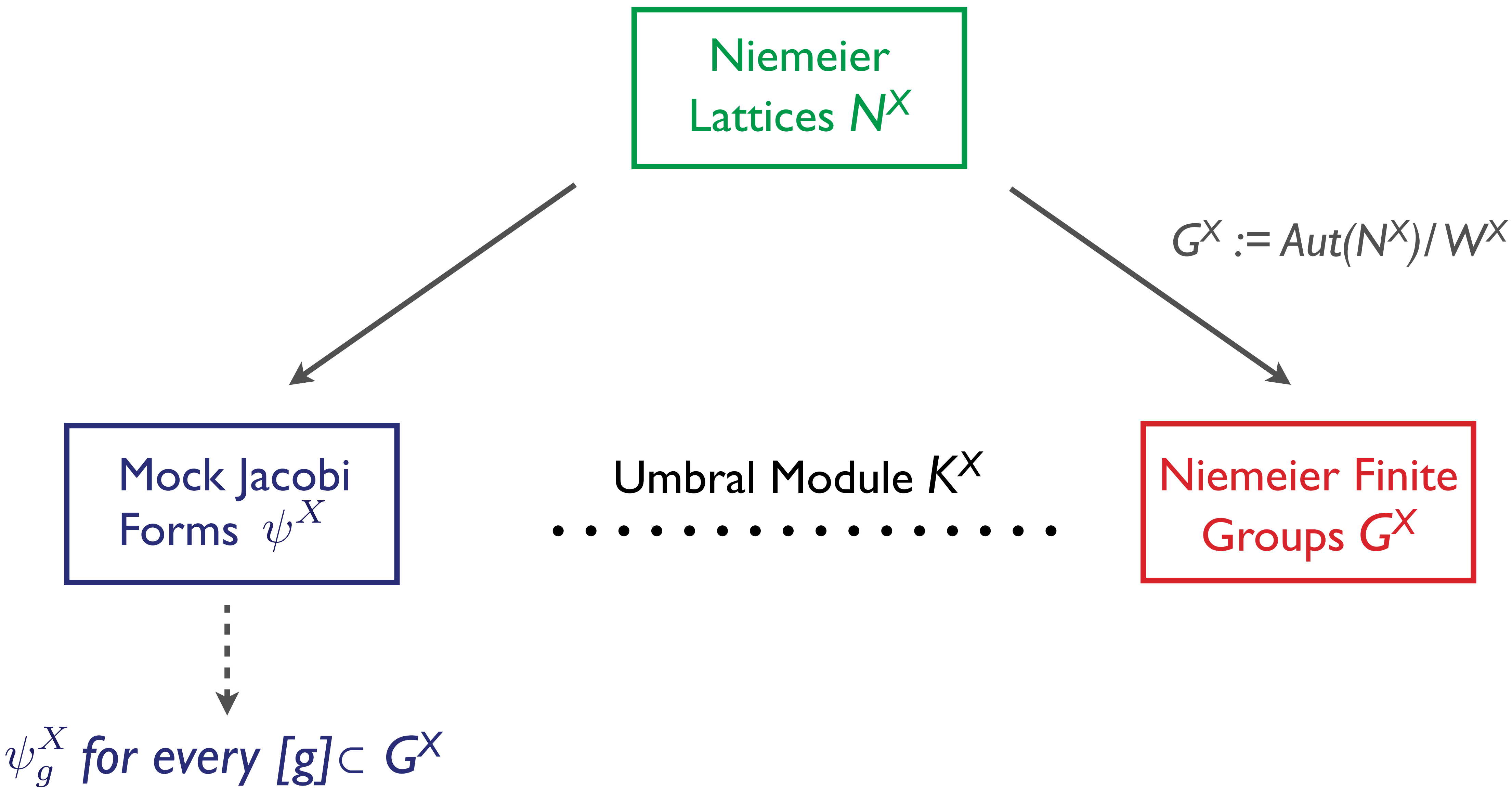
*: on leave from CNRS, France.

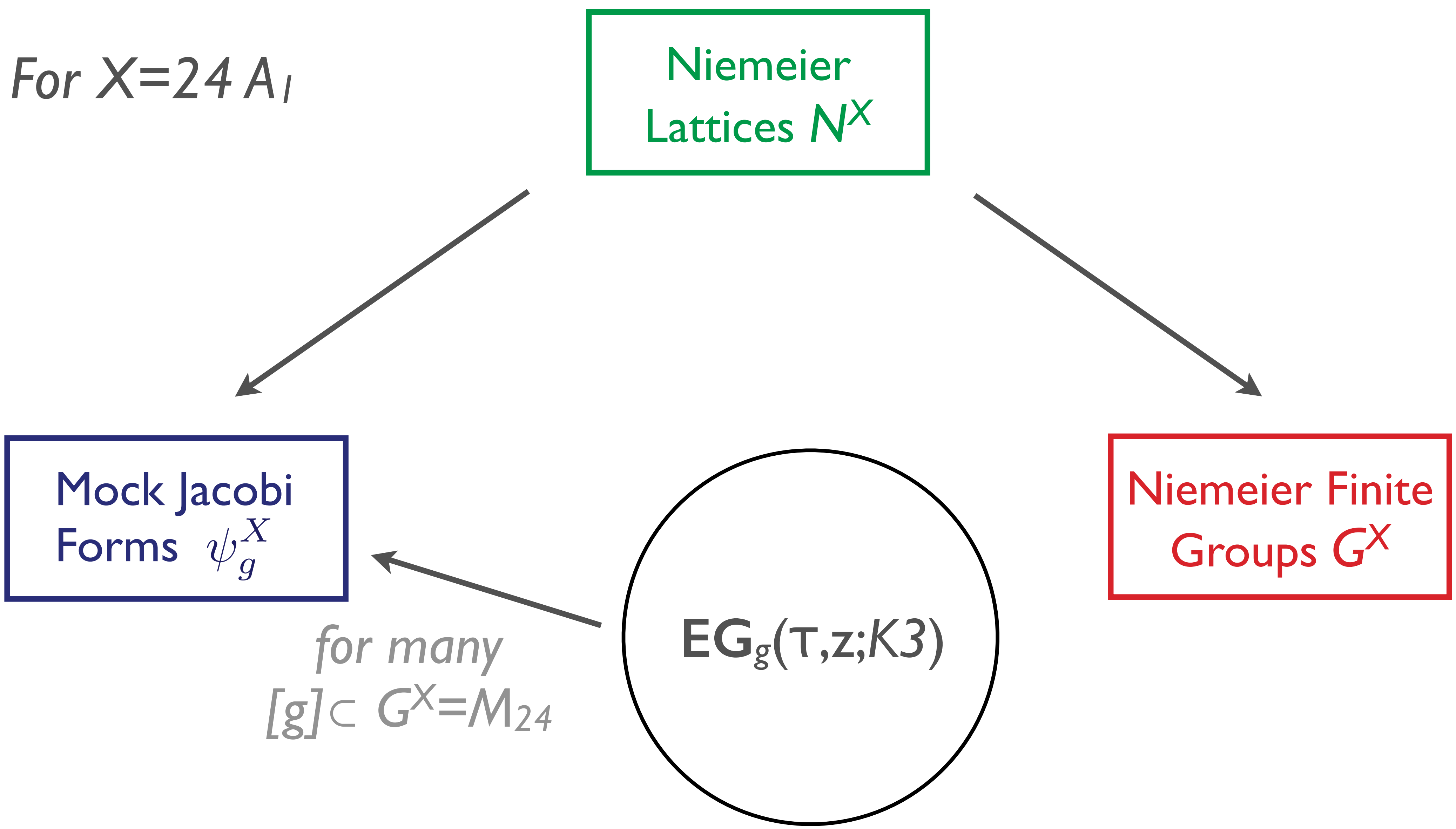
Motivation

On the one hand, there is Umbral Moonshine.



String Theory on *K3* and Mathieu Moonshine

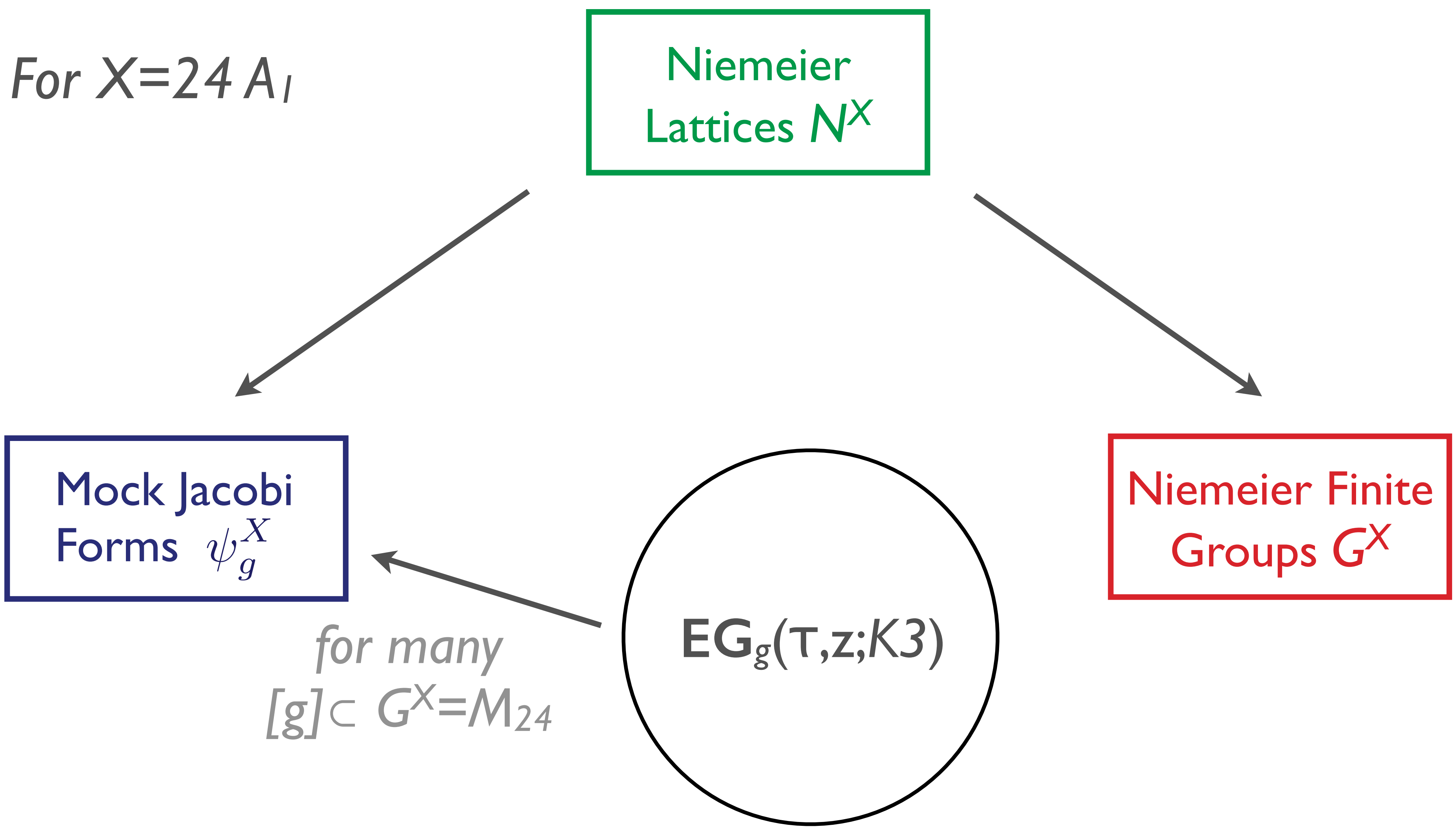
The M_{24} case of Umbral Moonshine is the Mathieu Moonshine, known to have an intricate relation to K3 CFTs.



[Eguchi-Ooguri-Tachikawa/MC/Gaberdiel-Hohenneger-Volpato/Eguchi-Hikami '10]

String Theory on *K3* and Umbral Moonshine?

Q: What about the other 22 cases ($X \neq 24\,A_1$) of umbral moonshine?
What is the physical and geometric context of umbral moonshine?



Outline

I. The Idea:
   Umbral Moonshine and *K3* CFT

II. Further exploration:
   Landau-Ginzburg Models and Umbral Moonshine

Mostly based on work (in progress) with

Sarah Harrison
(Harvard)

and many other fantastic collaborators!

Part I
The Idea:
Umbral Moonshine and K3 CFT

Mostly based on 1406.0619 w. S. Harrison

UM($24A_1$), EG(K3) and Representations of $N=4$ *SCA*

UM associates a unique optimal mock Jacobi form to the 23 N^X.

[Dabholkar–Murthy–Zagier'12 /MC–Duncan–Harvey '13/MC–Duncan to appear]

$$\psi^X(\tau,z) = \sum_{r\in\mathbb{Z}/2h} H_r^X(\tau)\theta_{h,r}(\tau,z)$$

$$\theta_{h,r}(\tau,z) = \sum_{k\in\mathbb{Z},\ k=r\ (2m)} q^{k^2/4h}y^k$$

$$= \text{the index } h \text{ theta function}$$

$$h = \mathrm{Cox(X)}$$

vector-valued mock theta functions of wt 1/2

UM($24A_1$), EG(K3) and Representations of $N=4$ SCA

UM associates a unique mock Jacobi form to the N^X, $X=24A_1$.

$$\psi^X = \sum_{r\in\mathbb{Z}/4} H_r^X \theta_{2,r}$$

$$\begin{aligned} H_1^X &= -H_{-1}^X \\ &= 2q^{-1/8}(-1 + 45\,q + 231\,q^2 + 770\,q^3 + 2277\,q^4 + 5796\,q^5 \ldots) \end{aligned}$$

UM($24A_1$), EG(K3) and Representations of N=4 SCA

Recall: 2d sigma model on K3 is a N=(4,4) SCFT.

$\Rightarrow$ The spectrum fall into irred. representations of the N=4 SCA.

[Eguchi–Ooguri–Taormina–Yang '88]

$$\mathrm{EG}(\tau, z; K3)$$

$$= \frac{i\theta_1^2(\tau,z)}{\eta^3(\tau)\theta_1(\tau,2z)}(\psi^X_{polar} + \psi^X)$$

[Zwegers, DMZ]

polar part cor. into a mero. Jac form

UM($24A_1$) mock Jacobi form

$$= \frac{\theta_1^2(\tau,z)}{\eta^3(\tau)}\Big(24\,\mu(\tau,z) + 2\,q^{-1/8}(-1 + 45\,q + 231\,q^2 + 770\,q^3 + \dots)\Big)$$

Appell-Lerch function

number of long N=4 multiplets

$$= 24 \times \text{short multiplet ch} + \text{long multiplet ch's}$$

It seems difficult to generalise such an **algebraic** relation between EG(K3) and UM($24A_1$) to the other cases.

Roots and UM Mock Mod. Forms

Niemeier Lattices N^X → Mock Jacobi Forms ψ_g^X

The lattice roots X (ADE) determines the modular properties in a way reminiscent to the ADE classification of N=2 minimal models of [Cappelli–Itzykson–Zuber '87].

Q: What is the **physical and geometric meaning** of this construction?

Q: What is CIZ classification doing here ??

ADE Classifications *N*=2 Minimal Models

- Based on representations of discrete series of *N=2 SCA* with

$$\hat{c} = \frac{c}{3} = 1 - \frac{2}{h},\ h = 2, 3, \ldots$$

where *h* is identified with the Coxeter number of the simply-laced root system *Y* in the ADE classification. **[Cappelli–Itzykson–Zuber '87]**

- The mysterious ADE classification of minimal models can be understood through the *ADE classification of du Val (or Kleinian) surface singularities.* **[E. Martinec, Vafa-Warner '89]**

They occur at the point (0,0,0) in the surface $W_Y^0=0$, with

$$\begin{aligned} W^0_{A_{m-1}} &= x_1^2 + x_2^2 + x_3^m \\ W^0_{D_{m/2+1}} &= x_1^2 + x_2^2 x_3 + x_3^{m/2} \\ W^0_{E_6} &= x_1^2 + x_2^3 + x_3^4 \\ W^0_{E_7} &= x_1^2 + x_2^3 + x_2 x_3^3 \\ W^0_{E_8} &= x_1^2 + x_2^3 + x_3^5. \end{aligned}$$

Landau–Ginzburg Models

Φ_i : chiral superfields

$$L = \int d^2z\, d^4\theta K(\bar{\Phi}_i, \Phi_i) + \int d^2z\, d^2\theta\, \underbrace{W(\Phi_i)}_{\text{superpotential}} + c.c.$$

$(z, \bar{z}; \theta^+, \theta^-, \bar{\theta}^+, \bar{\theta}^-)$: local coordinates on the super Riemann surface

- It is an *N*=2, *d*=2 supersymmetric quantum field theory.
- It does not have conformal symmetry, but "flow to the IR" to a 2d CFT.
- From the LG point of view, the EG computes the graded index of *Q*, one of the susy generators. The *Q-cohomology* forms representations of an *N*=2 superconformal algebra. The EG is invariant under the RG flow.
- Deformations of *W* are *Q-exact* and hence one can send *W*→*0* and compute the EG using a free field theory.
- For our purpose, it can be thought of as a gadget to compute EG(CFT) and study the symmetries easily.

[Witten 1994]

ADE Classification in terms of ADE Sing.

LG model with $W = W_Y{}^0$ $\longleftrightarrow$ The *Y* type *N*=2 minimal model "*min(Y)*" of CIZ

eg. LG model with $W(\Phi) = \Phi^m$ $\longleftrightarrow$ the A_{m-1} type minimal model

$\longleftrightarrow$ $\dfrac{SU(2)}{U(1)}$ WZW model at level *m*-2

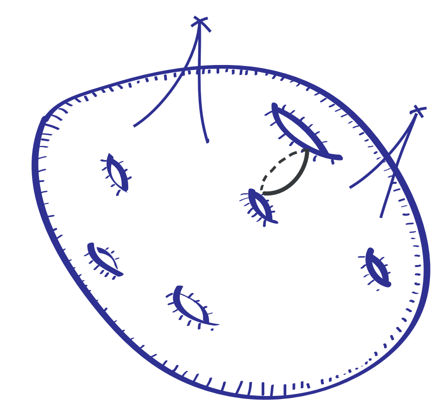
Its elliptic genus, $\mathrm{EG}^{Y}_{\mathrm{min}}$, can be computed either from the LG description via its chiral ring, or from the *min(Y)* description in terms of the Ramond *N*=2 characters at c=3(1-2/h).

[E. Martinec, Vafa-Warner, Schellekens–Waarner, Witten, Qiu, Di Francesco–Yankielowicz …..late 80']

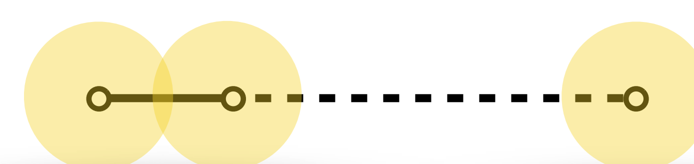
Roots, Singularities, and Curves

•*ADE classification of minimal models* $\longleftrightarrow$ *ADE singularities*

• They are singularities a K3 can develop. Locally they look like C^2/Γ.



• Their resolution gives rise to genus 0 curves with the corresponding ADE intersection matrix.



ADE root system $X \leftrightarrow$ **<u>ADE singularities</u>** X **?**

Elliptic Genus of $K3$ Singularities

A CFT description of the ADE singularities: [Ooguri–Vafa '95]

LG model with
$W = W_0^Y - \mu x^{-h}$ $\longleftrightarrow$ $\left(\min(Y) \times \dfrac{SL(2,\mathbb{R})}{U(1)} \text{ supercoset}\right) / (\mathbb{Z}/h)$

cf. The 5-brane setup in the work by Harvey–Murthy is basically given by $K3 \text{ CFT} \otimes T^Y$.

Their holomorphic part, denoted by **EG**(Y), are mock modular for all $Y=A_{m-1}, D_{1+m/2}, E_6, E_7, E_8$ due to the non-compactness of the target space.

$$\mathrm{EG}(Y;\tau,z) = \frac{1}{2h}\frac{i\theta_1(\tau,z)}{\eta^3(\tau)} \sum_{a,b\in\mathbb{Z}/h} (-1)^{a+b} q^{a^2/2} y^a \, \mathrm{EG}^Y_{\min}(\tau, z+a\tau+b)\mu_{h,0}(\tau, \tfrac{z+a\tau+b}{h})$$

[See also Troost, Eguchi–Sugawara '10, Eguchi–Sugawara–Taormina '08]

eg. $\mathrm{EG}(A_1;\tau,z) = \text{short multiplet ch of } N=4 \text{ SCA}$

Elliptic Genus of *K3* Singularities

What if we take

ADE root system *X* ↔ ADE singularities *X*

seriously. . . .

$$\mathrm{EG}(\tau, z; K3) = 24 \times \text{short multiplet ch} + \text{long multiplet ch's}$$

$$\| $$

$$\mathrm{EG}(24A_1; \tau, z)$$

For unions of simply-laced root systems, we define $\mathrm{EG}(\cup_i Y_i; \tau, z) := \sum_i \mathrm{EG}(Y_i; \tau, z)$

A Geometric Interpretation of the N=4 Decomposition

$$\mathrm{EG}(\tau, z; K3)$$

$$= \frac{i\theta_1^2(\tau,z)}{\eta^3(\tau)\theta_1(\tau,2z)}(\underline{\psi^X_{polar}} + \underline{\psi^X})$$ (modular)

$$= 24 \times \underline{\text{short multiplet ch}} + \underline{\text{long multiplet ch's}}$$ (algebraic)

$$= \underline{\mathrm{EG}(\tau, z; X)} + \underline{\text{contribution from the umbral function } \psi}^X$$ (geometric)

in the case X=$24A_1$

Q: Can we generalise this **geometric** relation between EG(K3) and UM($24A_1$) to the other cases?

A Geometric Decomposition for all UM(X)

It turns out that the following is true

$$\underbrace{\mathrm{EG}(\tau, z; K3) = \mathrm{EG}(\tau, z; X)}_{\text{contribution from the root system}} + \underbrace{\frac{\theta_1^2(\tau, z)}{\eta^6(\tau)} \tilde{\partial}\psi^X(\tau, 0)}_{\text{contribution from the umbral function}}$$

for all 23 Niemeier lattices with $X = 24A_1, 12A_2, \ldots, A_{11}D_7E_6, \ldots, 3E_8, D_{24}$.

where $\tilde{\partial} f(\tau, z) = \dfrac{1}{2\pi i} \dfrac{\partial}{\partial z} f(\tau, z)$.

Q: What could be the **physical and geometric meaning** of this construction?

Niemeier lattices provide a framework to study K3 geometry.

[Nikulin '11, '14, see also Taormina–Wendland '11]

Every K3 surface M can be associated with (marked by) at least one Niemeier lattice N^X (including $N^0 \simeq \Lambda_{Leech}$), and such that

- The genus 0 curves arising from resolving the *ADE* singularity have intersections given by a sub-diagram of X.
- The group of hyper-Kähler structure-preserving symmetries of M is a subgroup of G^X.

Moreover, it was conjectured (and proven for most Xs) that for each N^X there exists an M that can only be marked by N^X and not by other Niemeier lattices.

eg. $M_1 = T^4/Z_2$, $sing(M_1) = 16A_1 \subset X = 24A_1 \Rightarrow Sym(M_1) \subset G^X = M_{24}$.

$M_2 = T^4/Z_3$, $sing(M_2) = 9A_2 \subset X = 12A_2 \Rightarrow Sym(M_2) \subset G^X = 2.M_{12}$.

Checks: How do we twine?

$$\mathrm{EG}^X_g(\tau,z;K3) = \mathrm{EG}_g(\tau,z;X) + \frac{\theta_1^2(\tau,z)}{\eta^6(\tau)}\tilde{\partial}\psi^X_g\,(\tau,0)$$

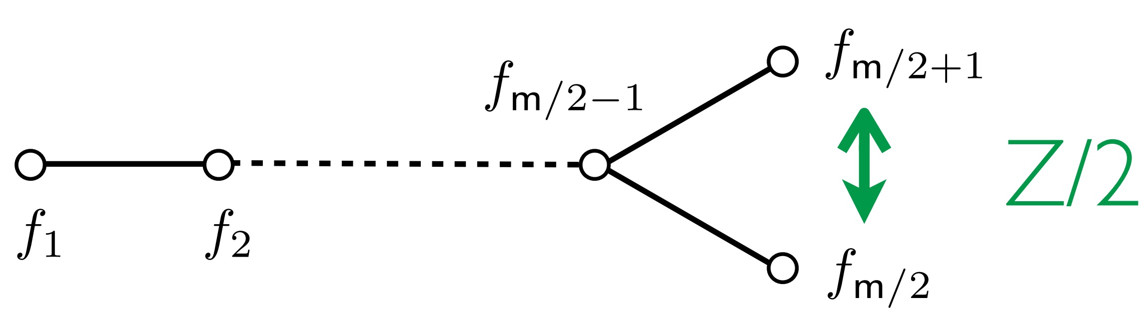
??

given by
UM(X) for all g∈G^X

Q: How to twine the root system contribution to EG(K3) for all g∈G^X?

Recall that G^X acts on the root system $X=\cup_i Y_i$ as permutation of the identical components Y_i (*eg. $24A_1$, $3E_8$ or $2A_9$ in $2A_9 \oplus D_8$*) and as diagram automorphisms of Y_i *(Z/2 for $A_{n>1}$, E_6 and $D_{>1}$ except for D_4 which have Dih_3 automorphism).*

$D_{1+m/2}$



Checks: How do we twine?

Recall: we have an explicit G^X action on the CFT on singularity X.

If $X = \cup_{i=1,\ldots,n} Y_i$,

$$g : (Y_1, \ldots, Y_n) \to (h_1(Y_{\pi(1)}), \ldots, h_n(Y_{\pi(n)})) \ , \ \ h_i \in \mathrm{Aut(Y_{\pi(i)})}$$

$$\Rightarrow \mathrm{EG}_g(\tau, z; X) = \sum_i \delta_{i,\pi(i)} \, \mathrm{EG}_{h_i}(\tau, z; Y_i)$$

As a result, we just need to compute the the EG of a simply-laced root system twined by its automorphisms.

This can be done using the Landau–Ginzburg description of the *ADE* minimal model and the identification between the chiral ring & the corresponding *ADE* weight space.

twine this part

$$\mathrm{EG}_h(Y; \tau, z) = \frac{1}{2h} \frac{i\theta_1(\tau, z)}{\eta^3(\tau)} \sum_{a,b \in \mathbb{Z}/h} (-1)^{a+b} q^{a^2/2} y^a \, \underline{\mathrm{EG}^Y_{\min}(\tau, z + a\tau + b)} \mu_{h,0}(\tau, \tfrac{z+a\tau+b}{h})$$

Twining: the Results

In the way we obtain, for each of the 23 X, the twinings

$$\mathrm{EG}_g^X(K3;\tau,z)\ \forall\ \ g\in G^X$$

- For the case $X=24A_1$, by construction it agrees with the result of Mathieu Moonshine.

[EOT, MC, Gaberdiel–Hohenegger–Volpato, Eguchi-Hikami '10-11]

- Sanity Checks: All <g> corresponding to geometric K3 sym. have the same result independent of from which UM(X) it arises, as required by the global Torelli theorem.

$$\mathrm{EG}_g^X(K3;\tau,z)=\mathrm{EG}_{g'}^{X'}(K3;\tau,z)\ \text{ if }\ \pi_g=\pi_{g'}\ \text{has at least 1 fixed point and 5 orbits.}$$

- For non-geometric type symmetries, no such theorem applies; twinings from different X can differ.

24 Ways to Twine EG($K3$)

N^X	UM(X)	$\mathbf{EG}_g^X(K3)$ all $g \in G^X$
$\Lambda_{Leech}=N^0$	$V^{s\natural}$	$\mathbf{EG}_g^0(K3)$ all $g \in G^0 \cong Co_0$ which preserves a 4-dim sublattice

[Duncan–Mack-Crane '15]

- Twinings for geometric type symmetries all agree.
- Twinings for non-geometric type depends on X.

eg. $\mathrm{EG}_g^{24A_1} = \mathrm{EG}_{g_1}^0 \neq \mathrm{EG}_{g'}^{12A_2} = \mathrm{EG}_{g_2}^0 \quad , \quad \pi_g = \pi_{g'} = \pi_{g_1} = \pi_{g_2} = 1^2 11^2$

$$\mathrm{EG}_g^{24A_1} \neq \mathrm{EG}_{g'}^{12A_2} = \mathrm{EG}_{g''}^0 \quad , \quad \pi_g = \pi_{g'} = \pi_{g''} = 3^8$$

$$\mathrm{EG}_g^{24A_1} = \mathrm{EG}_{g'}^{6A_4} \neq \mathrm{EG}_{g''}^0 \quad , \quad \pi_g = \pi_{g'} = \pi_{g''} = 6^4$$

Q: Which one(s) of the twinings is relevant for a given $K3$ CFT?

More on the Physical Interpretation

[in discussion w. *M. Gaberdiel, S. Harrison, R. Volpato*]

The CFT counterpart of the result by Nikulin, generalising the classification of [Gaberdiel–Hohenegger–Volpato '11] to include singular CFTs, is expected to hold.

Recall that a K3 CFT is specified by a 4-dimensional positive definite subspace $\Pi \subset \Gamma^{4,20} \otimes_{\mathbb{Z}} \mathbb{R}$. [Aspinwall–Morrison '94, Nahm–Wendland '99]

Let Λ_Π be the orthogonal complement of Π in $\Gamma^{4,20}$.
Then we can show that Λ_Π, whether or not it contains a root vector, can always be embedded into one of the 24 even, self-dual negative definite lattices N^X.

From this one concludes that the N=(4,4) preserving symmetries of the corresponding theory has symmetries $G \subset G^X$.

In a theory whose Λ_Π can only be embedded in N^X, it's natural to conjecture that the corresponding twinings coincide with those arising from UM(X).

Part II
Further Exploration:
Symmetries of the Landau–Ginzburg Models

Based on work in progress w. F. Ferrari, S. Harrison, N. Paquette

LG/CY Correspondence

Sigma models on CY that are hypersurfaces (W=0) in the weighted projected space

LG orbifold with the superpotential W

corresponding to the residual *Z/n* symmetries of the U(1) acting on the weighted projective space

This correspondence is well-tested and has played a very important role in the study of CY manifolds, in particular mirror symmetries.

[cf. Chiodo–Ruan '10, Libgober '14]

LG/CY Correspondence

eg1. Quintic 3-folds

the 3-fold: $W(z_1, \ldots, z_5) = \sum_{i=1}^{5} z_i^5 = 0$

the orbifold group: W invariant under *Z/5* generated by $z_i \mapsto e^{2\pi i/5} z_i$

⇒ the LG theory: the theory with chiral superfields $\Phi_1, \ldots, \Phi_5$ and superpotential $W = W(\Phi_1, \ldots, \Phi_5) = \sum_{i=1}^{5} \Phi_i^5$ orbifolded by *Z/5*

LG — minimal model

the CFT: $(3)^5 := (A_4)^{\otimes 5}/(Z/5)$ [Gepner '87, '88]

eg2. the non-compact CY: the E_8 ALE space

$W_{E_8} = -\mu\Phi_0^{-30} + \Phi_1^2 + \Phi_2^3 + \Phi_3^5$

the orbifold group *Z/30*: $\Phi \mapsto e^{2\pi i q_i/30}, \quad (q_0, \ldots, q_3) = (-1, 15, 10, 6)$

the CFT: $\left(\mathrm{min}(E_8) \times \frac{SL(2,\mathbb{R})}{U(1)} \text{ supercoset} \right) / (\mathbb{Z}/30)$

Why Studying the Landau–Ginzburg Orbifolds?

- Apart from torus orbifold, these (including the corresponding Gepner models) are the only computable models. Moreover, here we have access to the less "exceptional" points in the moduli space with more "generic" symmetries.

- The LG models with the same field content but different superpotentials W have the same free field representation for their Q-cohomology: a promising setup to "collect" symmetries at different point of the moduli space corresponding to different superpotentials.

- They can provide circumstantial tests to the conjecture: if a (necessarily singular) K3 sigma model has orthogonal lattice embeddable in a given N^X only, then its twining should coincide with the UM(X) twining.

- For our purpose, it can be thought of as a gadget to compute EG(CY) and study the symmetries easily.

Our Main Examples

chiral superfields: $\Phi_1, \Phi_2, \ldots, \Phi_6$

orbifolded by *Z/3*: $\Phi_i \mapsto e^{2\pi i/3}\Phi_i$

EG: independent of the superpotentials and can be computed using free fields

$$\mathrm{EG}(\tau, z; K3) = \frac{1}{3}\sum_{a,b\in\mathbb{Z}/3} q^{a^2} y^{2a} \prod_{i=1}^{6} \frac{\theta_1(\tau, \frac{-2}{3} z_{a,b})}{\theta_1(\tau, \frac{1}{3} z_{a,b})}\ , \quad z_{a,b} = z + a\tau + b$$

Typical symmetries of the theory are given by permutations and phase multiplications of $\mathbf{\Phi}_i$. The corresponding twining genus are given by

$$\mathrm{EG}_g(\tau, z; K3) = \frac{1}{3}\sum_{a,b\in\mathbb{Z}/3} q^{a^2} y^{2a} \prod_{i=1}^{6} \frac{\theta_1(\tau, \frac{-2}{3} z_{i,a,b})}{\theta_1(\tau, \frac{1}{3} z_{i,a,b})}\ , \quad z_{i,a,b} = z + a\tau + b + \alpha_i$$

Superpotentials and their Symmetries

UM($12A_2$)
distinct ord. 11

$$L_2(11)|W_1(\underline{\Phi}) = \Phi_0^3 + \Phi_1^2\Phi_5 + \Phi_2^2\Phi_4 + \Phi_3^2\Phi_2 + \Phi_4^2\Phi_1 + \Phi_5^2\Phi_3$$

UM($12A_2$)

$$M_{10}|W_2(\underline{\Phi}) = \sum_{i=0}^{5} \Phi_i^3 + \lambda.\sigma_3(\Phi_0, \ldots, \Phi_5)$$

[see Höhn–Mason '14]

exceptional

$$3^4 : A_6|W_3(\underline{\Phi}) = \sum_{i=0}^{5} \Phi_i^3 \rightarrow$$

equivalent to $(1)^6$ Gepner model,
conjectured to be equivalent to a T^4/Z_3 model
[Gaberdiel–Hohenegger–Volpato '11]

exceptional

$$3^{1+4} : 2.2^2|W_4(\underline{\Phi}) = \sum_{i=0}^{5} \Phi_i^3 + 3(i - 2e^{\pi i/6} - 1)(\Phi_0\Phi_1\Phi_2 + \Phi_3\Phi_4\Phi_5)$$

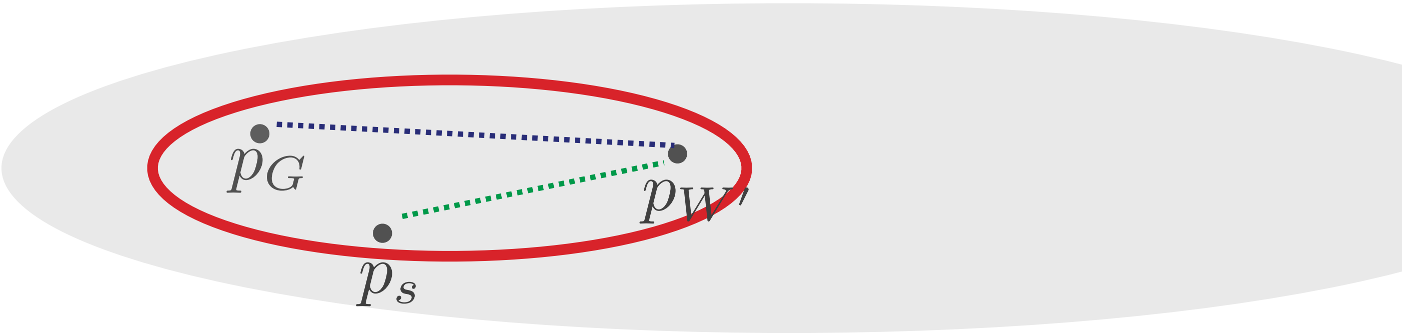
Summary of the Cubic LG Symmetries

- From these symmetries one obtains all the $M_{11} \subset 2.M_{12} \simeq G(12A_2)$ twinings of UM$(12A_2)$, including the one (ord. 11) distinct from M_{24}-twinings.

- If we just twine the free field theory without requiring them to be symmetries of a concrete superpotential, we obtain more $2.M_{12} \simeq G(12A_2)$ twinings.

- This shows that LG theories is a promising setup to study the symmetries of K3 sigma models.

- Moreover, the close connection to UM$(12A_2)$ provides circumstantial evidence for the relation between ADE theories and $K3$ sigma models: twinings other than M_{24} twinings (even with the same action on 24-dim rep) do appear where expected.



Comments on the Relation to $K3^{[2]}$

- Note that our cubic superpotential defines a cubic 4-fold. Such cubic 4-folds have a very close relation to K3's. [see Kuznetsov, '10 Huybrechts '15]

- In particular, through the (Fano scheme of lines of) cubic 4-folds one can study the automorphisms of $X^{[2]}:=Hilb(X^{[2]})$ where X is a K3. [Beauville–Donagi …]

- The classification of symplectic automorphisms of hyper-Kähler manifolds that are deformation equivalent to $X^{[2]}$: *it must be contained in either one of the 13 subgroups of M_{23}, or subgroups of the 2 S-lattice subgroups of Co_0.*

[see Höhn–Mason '14]

2 of them realised by W_1, W_2
their twinings given by UM($12A_2$)

realised by W_3, W_4,
their twinings given by $V^{s\natural}$

Geometric sym of $X^{[2]}$ are realised as stringy sym. of K3 CFT?
Natural to expect:
1: Twinings are related via the "2nd quantisation/Borcherds lift" formula.
2: Similarly for $X^{[n]}$.

Thank you!