Umbral Moonshine

New Moonshines, Mock Modular Forms and String Theory
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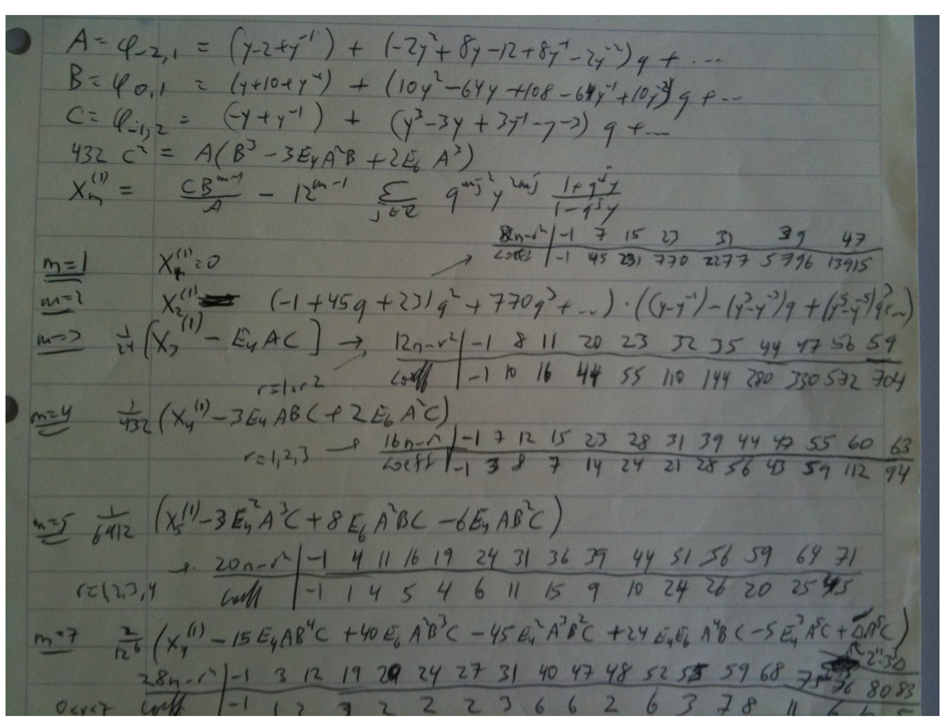
based primarily on two papers with
Miranda Cheng and John Duncan.

This workshop is in part about new moonshines. What are these new moonshines? What are their properties, what kind of taste and aroma do they have? How were they brewed?

HISTORY

Umbral Moonshine was inspired by trying to understand and generalize the observations of Eguchi, Ooguri and Tachikawa on M24 and the elliptic genus of K3 and by results of Dabholkar, Murthy and Zagier on Black hole counting in string theory and mock modular forms.

"Don's piece of paper"



Primary references:

M.Cheng, J. Duncan and JH, *Umbral Moonshine*, Commun. Num. Theor. Phys. **08** (2014) 101.

M. Cheng, J. Duncan and JH, *Umbral Moonshine and the Niemeier Lattices*, Research in the Mathematical Sciences, 2014 ,**1**:3.

OUTLINE

1. Umbral Moonshine

A. Umbral Groups and Niemeier Lattices

B. Genus zero subgroups of SL(2,R)

C. Shadows and ADE

D. The Umbral mock modular forms

E. Discriminant property

2. Comments about relations between different types of moonshine

Umbral Moonshine

The objects $\mathbf{X}$ which classify Umbral Moonshine are rank 24 root systems with A,D,E components, all with equal Coxeter number. There are 23 of these:

$$A_1^{24},\ A_2^{12},\ A_3^8,\ A_4^6,\ A_5^4D_4,\ A_6^4,\ A_7^2D_5^2,\ A_8^3,\ A_9^2D_6,\ A_{11}D_7E_6,\ A_{12}^2,\ A_{15}D_9,\ A_{17}E_7,\ A_{24},$$ A-type

$$D_4^6,\ D_6^4,\ D_8^3,\ D_{10}E_7^2,\ D_{12}^2,\ D_{16}E_8,\ D_{24},$$ D-type

$$E_6^4,\ E_8^3.$$ E-type

Given an $\mathbf{X}$ there are classical constructions of

$L^{\mathbf{X}}$ The Niemeier lattice constructed from $\mathbf{X}$ and glue.

$G^{\mathbf{X}} = Aut(L^{\mathbf{X}})/Weyl(\mathbf{X})$ The Umbral groups

Umbral Groups (see SPLAG and Wilson's talk)

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
G^X	M_{24}	$2.M_{12}$	$2.AGL_3(2)$	$GL_2(5)/2$	$GL_2(3)$	$SL_2(3)$	Dih_4
$\bar{G}^X$	M_{24}	M_{12}	$AGL_3(2)$	$PGL_2(5)$	$PGL_2(3)$	$PSL_2(3)$	2^2

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
G^X	Dih_6	4	2	4	2	2	2
$\bar{G}^X$	Sym_3	2	1	2	1	1	1

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
G^X	$3.Sym_6$	Sym_4	Sym_3	2	2	1	1
$\bar{G}^X$	Sym_6	Sym_4	Sym_3	2	2	1	1

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
G^X	$GL_2(3)$	Sym_3
$\bar{G}^X$	$PGL_2(3)$	Sym_3

A new element, discovered in the context of Umbral Moonshine is that a genus zero subgroup of $SL(2, \mathbb{R})$ and its hauptmodul can be attached to each $\mathbf{X}$.

A-type: The Coxeter numbers appearing are 2,3,4,5,6,7,8,9,10,12,13,16,18,25 and these are precisely the n for which $\Gamma_0(n)$ is genus zero.

D-type: The Coxeter numbers appearing are 6,10,14,18,22,30,46 and these are precisely the n for which $\Gamma_0(2n) + n$ are genus zero.

E-type: The Coxeter numbers are 12, 30 and $\Gamma_0(12) + 4, \ \Gamma_0(30) + 6, 10, 15$ are genus zero.

The notation is that of Conway-Norton in which

$$\Gamma_0(N) + e, f, g, \cdots$$

indicates the group obtained from $\Gamma_0(N)$ by adjoining Atkin-Lehner involutions $w_e, w_f, w_g, \cdots$ determined by exact divisors $e, f, g, \cdots$ of N, $\Gamma_0(N)$ is the congruence subgroup

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}), \quad c = 0 \bmod N$$

The hauptmoduls or principal moduli are constructed from the Coxeter frame shapes. For each A,D,E component of $\mathbf{X}$ consider the Coxeter element

$$w = r_1 r_2 \cdots r_r$$

which is the product of reflections in the simple roots. It has order the Coxeter number m. Let its eigenvalues (counting multiplicity) be

$$e^{2\pi i u_1/m}, \cdots, e^{2\pi i u_r/m}$$

and encode these eigenvalues in the frame shape

$\prod_i n_i^{k_i}$ such that the polynomial $\prod_{\substack{i\\ r}} (x^{n_i} - 1)^{k_i}$ coincides with the characteristic polynomial $\prod_{i=1}^{r} (x - e^{2\pi u_i/m})$ of w

Define the Coxeter frame shape of each $\mathbf{X}$ to be the product of these frame shapes for each component and define a corresponding eta product by

$$\pi = \prod_i n_i^{k_i} \to \eta_\pi(\tau) = \prod_i \eta(n_i \tau)^{k_i}$$

Then the hauptmodul $T^{\mathbf{X}}$ with normalization

$$T^{\mathbf{X}} = q^{-1} - \#\text{irreducible comp of X} + O(q)$$

is given by

$$T^{\mathbf{X}} = \frac{1}{\eta_{\pi^{\mathbf{X}}}}$$

The proof of this correspondence is by inspection. The connection with genus zero is suggestive of Monstrous Moonshine and can be used to label instances of Umbral Moonshine by the corresponding conjugacy classes of the Monster (including one ghost element).

Table 2: Niemeier Root Systems and Principal Moduli

X	A_1^{24}	A_2^{12}	A_3^8	A_4^6	$A_5^4D_4$	A_6^4	$A_7^2D_5^2$
ℓ	2	3	4	5	6	7	8
π^X	$\frac{2^{24}}{1^{24}}$	$\frac{3^{12}}{1^{12}}$	$\frac{4^8}{1^8}$	$\frac{5^6}{1^6}$	$\frac{2^16^5}{1^53^1}$	$\frac{7^4}{1^4}$	$\frac{2^28^4}{1^44^2}$
Γ^X	$2B$	$3B$	$4C$	$5B$	$6E$	$7B$	$8E$

X	A_8^3	$A_9^2D_6$	$A_{11}D_7E_6$	A_{12}^2	$A_{15}D_9$	$A_{17}E_7$	A_{24}
ℓ	9	10	12	13	16	18	25
π^X	$\frac{9^3}{1^3}$	$\frac{2^110^3}{1^35^1}$	$\frac{2^23^112^3}{1^34^16^2}$	$\frac{13^2}{1^2}$	$\frac{2^116^2}{1^28^1}$	$\frac{2^13^118^2}{1^26^19^1}$	$\frac{25^1}{1^1}$
Γ^X	$9B$	$10E$	$12I$	$13B$	$16B$	$18D$	$(25Z)$

X	D_4^6	D_6^4	D_8^3	$D_{10}E_7^2$	D_{12}^2	$D_{16}E_8$	D_{24}
ℓ	6+3	10+5	14+7	18+9	22+11	30+15	46+23
π^X	$\frac{2^66^6}{1^63^6}$	$\frac{2^410^4}{1^45^4}$	$\frac{2^314^3}{1^37^3}$	$\frac{2^33^218^3}{1^36^29^3}$	$\frac{2^222^2}{1^211^2}$	$\frac{2^23^15^130^2}{1^26^110^115^2}$	$\frac{2^146^1}{1^123^1}$
Γ^X	$6C$	$10B$	$14B$	$18C$	$22B$	$30G$	$46AB$

X	E_6^4	E_8^3
ℓ	12+4	30+6,10,15
π^X	$\frac{2^43^412^4}{1^44^46^4}$	$\frac{2^33^35^330^3}{1^36^310^315^3}$
Γ^X	$12B$	$30A$

The main claim is that to each $\mathbf{X}$ we can associate a (vector-valued) mock modular form of weight 1/2 and its shadow, a weight 3/2 unary theta function, $(H^{\mathbf{X}}, S^{\mathbf{X}})$ and that they "exhibit moonshine for $G^{\mathbf{X}}$."

This means, among other things, that for each $g \in G^{\mathbf{X}}$ there should be "twined" analogs $(H_g^{\mathbf{X}}, S_g^{\mathbf{X}})$ with prescribed (mock) modular properties under the congruence subgroup $\Gamma_0(\mathrm{ord} g)$.

The construction of the shadows from $\mathbf{X}$ is concrete while obtaining the mock modular forms is more subtle.

Recall that index m (mock) Jacobi forms admit a theta expansion

$$\phi(\tau,z) = \sum_{r \bmod 2m} h_{m,r}(\tau)\theta_{m,r}(\tau,z) = h_m \cdot \theta_m$$

vector-valued (mock) modular form

$$\sum_{\substack{k\in\mathbb{Z} \\ k=r \bmod 2m}} q^{k^2/2m} y^k$$

Let $\mathcal{T}$, $\mathcal{S}$ be the 2m by 2m matrices specifying the modular transformation of $\theta_{m,r}$ under the generators

$$\theta_m(-1/\tau, -z/\tau) = \sqrt{-i\tau}\, e^{2\pi i m z^2/\tau} \mathcal{S} \theta_m(\tau,z)$$

$$\theta_m(\tau+1,z) = \mathcal{T}\theta_m(\tau,z)$$

and let $\Omega_{r,r'}$ be the components of a 2m by 2m matrix Ω obeying the following conditions:

- $\mathcal{S}^\dagger \Omega \mathcal{S} = \mathcal{T}^\dagger \Omega \mathcal{T} = \Omega$
- $\Omega_{r,r'} - \Omega_{r,-r'}$ = non-negative integer
- $\Omega_{1,1} - \Omega_{1,-1} = 1$

Then $h_m \cdot \Omega \cdot \theta_m = \tilde{h}_m \cdot \theta_m$ is also a (mock) Jacobi form and the components of $\tilde{h}_m$ are positive integer linear combinations of the components of h_m. This transformation can be formulated in terms of Eichler-Zagier involutions acting on Jacobi forms.

(Capelli, Itzykson, Zuber): There is an ADE classification of such matrices in that for X an ADE root system with Coxeter number m(X) there is a 2m by 2m matrix Ω^X such that

$$\Omega^X_{r,r} - \Omega^X_{r,-r} = \text{multiplicity of r as a Coxeter exponent of X}$$

We extend this to unions of A,D,E root systems like $\mathbf{X}$

if $X = \cup_i X_i$ then define $\Omega^X = \sum_i \Omega^{X_i}$

This "folding" procedure also extends to the shadows of the mock modular forms of Umbral Moonshine. These are linear combinations of weight 3/2 unary theta functions

$$S_{m,r}(\tau) = \frac{1}{2\pi i} \frac{\partial}{\partial z} \theta_{m,r}(\tau, z)\Big|_{z=0}$$

and for each $\mathbf{X}$ we define a shadow

$$S^{\mathbf{X}} = \Omega^{\mathbf{X}} \cdot S_m$$

Finally, for each $\mathbf{X}$ we define 2m-component mock modular forms $H^{\mathbf{X}} = \{H_r^{\mathbf{X}}\}$ which appear in the theta decompositions mock Jacobi forms which in turn are given by the decomposition of meromorphic Jacobi forms into its Polar and Finite parts.

$$\psi(\tau, z) = \psi^P(\tau, z) + \psi^F(\tau, z)$$

weight 1 index m meromorphic Jacobi form with first order pole in z

weight one index m mock Jacobi form

The mock modular forms obey a growth condition

$$q^{1/4m} H_r^{\mathbf{X}}(\tau) = O(1), \quad \tau \to i\infty, \quad all\ r$$

However it is not manifest that these forms should exhibit moonshine for $G^{\mathbf{X}}$.

Example: For m=3 we have $\mathbf{X} = A_2^{12},\ G^{\mathbf{X}} = 2.M_{12}$

$$H_1^X = 2q^{-1/12}(-1 + 16q + 55q^2 + 144q^3 + \cdots)$$

$$H_2^X = 2q^{2/3}(10 + 44q + 110q^2 + 280q^3 + \cdots)$$

10,44,110,120,160 are dimensions of faithful irreps

16,55,144 are irreps with trivial Z/2Z action

For each conjugacy class we have MT series $H_{g,r}^{\mathbf{X}}$ and most of these can be identified either with eta functions (when the twined shadow vanishes) or with order 3 mock theta functions of Ramanujan:

$$H_{2B}^{(3,1)} = -2q^{-1/12} f(q^2)$$

$$H_{2B}^{(3,2)} = -4q^{2/3}\omega(-q)$$

Results of Zwegers can be used to verify the existence of a two-dimensional rep $\alpha_2 : \Gamma_0(2) \to GL(2,\mathbb{C})$ such that this pair defines a vector-valued mock modular form for $\Gamma_0(2)$

The main conjecture:

Conjecture 6.1. *Let X be a Niemeier root system and let m be the Coxeter number of X. There exists a naturally defined $\mathbb{Z}/2m\mathbb{Z} \times \mathbb{Q}$-graded super-module*

$$K^X = \bigoplus_{r \bmod 2m} K^X_r = \bigoplus_{r \bmod 2m} \bigoplus_{\substack{D \in \mathbb{Z} \\ D = r^2 \bmod 4m}} K^X_{r,-D/4m} \tag{6.1}$$

for G^X such that the graded super-character attached to an element $g \in G^X$ coincides with the vector-valued mock modular form

C,D,H to appear →

$$\cancel{\alpha}^X H^X_{g,r}(\tau) = \sum_{\substack{D \in \mathbb{Z} \\ D = r^2 \bmod 4m}} \mathrm{str}_{K^X_{r,-D/4m}}(g)\, q^{-D/4m}, \tag{6.2}$$

has been proven for $X = A_1^{24}$ by Gannon and for the remaining cases by Duncan, Griffin and Ono. Explicit constructions of the modules remains an open question except for $X = E_8^3$ (J. Duncan and JH).

Discriminant Property

Umbral Moonshine contains a new element that is not present in Monstrous Moonshine which relates the discriminants of the mock modular forms to the number fields over which the irreducible representations of $G^{\mathbf{X}}$ attached to the discriminants are defined. This discriminant property is perhaps best explained using the Mathieu Moonshine example of Umbral Moonshine.

Table 8: Character table of $G^{(2)} \simeq M_{24}$

$[g]$	FS	1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	23A	23B
$[g^2]$		1A	1A	1A	3A	3B	2A	2A	2B	5A	3A	3B	7A	7B	4B	5A	11A	6A	6B	7A	7B	15A	15B	21A	21B	23A	23B
$[g^3]$		1A	2A	2B	1A	1A	4A	4B	4C	5A	2A	2B	7B	7A	8A	10A	11A	4A	4C	14B	14A	5A	5A	7B	7A	23A	23B
$[g^5]$		1A	2A	2B	3A	3B	4A	4B	4C	1A	6A	6B	7B	7A	8A	2B	11A	12A	12B	14B	14A	3A	3A	21B	21A	23B	23A
$[g^7]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	1A	1A	8A	10A	11A	12A	12B	2A	2A	15B	15A	3B	3B	23B	23A
$[g^{11}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	1A	12A	12B	14A	14B	15B	15A	21A	21B	23B	23A
$[g^{23}]$		1A	2A	2B	3A	3B	4A	4B	4C	5A	6A	6B	7A	7B	8A	10A	11A	12A	12B	14A	14B	15A	15B	21A	21B	1A	1A
χ_1	+	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	+	23	7	−1	5	−1	−1	3	−1	3	1	−1	2	2	1	−1	1	−1	−1	0	0	0	0	−1	−1	0	0
χ_3	∘	45	−3	5	0	3	−3	1	1	0	0	−1	b_7	$\overline{b_7}$	−1	0	1	0	1	$-b_7$	$-\overline{b_7}$	0	0	b_7	$\overline{b_7}$	−1	−1
χ_4	∘	45	−3	5	0	3	−3	1	1	0	0	−1	$\overline{b_7}$	b_7	−1	0	1	0	1	$-\overline{b_7}$	$-b_7$	0	0	$\overline{b_7}$	b_7	−1	−1
χ_5	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	b_{15}	$\overline{b_{15}}$	0	0	1	1
χ_6	∘	231	7	−9	−3	0	−1	−1	3	1	1	0	0	0	−1	1	0	−1	0	0	0	$\overline{b_{15}}$	b_{15}	0	0	1	1
χ_7	+	252	28	12	9	0	4	4	0	2	1	0	0	0	0	2	−1	1	0	0	0	−1	−1	0	0	−1	−1
χ_8	+	253	13	−11	10	1	−3	1	1	3	−2	1	1	1	−1	−1	0	0	1	−1	−1	0	0	1	1	0	0
χ_9	+	483	35	3	6	0	3	3	3	−2	2	0	0	0	−1	−2	−1	0	0	0	0	1	1	0	0	0	0
χ_{10}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	b_{23}	$\overline{b_{23}}$
χ_{11}	∘	770	−14	10	5	−7	2	−2	−2	0	1	1	0	0	0	0	0	−1	1	0	0	0	0	0	0	$\overline{b_{23}}$	b_{23}
χ_{12}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	b_7	$\overline{b_7}$	0	0	0	0	1	b_7	$\overline{b_7}$	0	0	b_7	$\overline{b_7}$	1	1
χ_{13}	∘	990	−18	−10	0	3	6	2	−2	0	0	−1	$\overline{b_7}$	b_7	0	0	0	0	1	$\overline{b_7}$	b_7	0	0	$\overline{b_7}$	b_7	1	1
χ_{14}	+	1035	27	35	0	6	3	−1	3	0	0	2	−1	−1	1	0	1	0	0	−1	−1	0	0	−1	−1	0	0
χ_{15}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2b_7$	$2\overline{b_7}$	−1	0	1	0	−1	0	0	0	0	$-b_7$	$-\overline{b_7}$	0	0
χ_{16}	∘	1035	−21	−5	0	−3	3	3	−1	0	0	1	$2\overline{b_7}$	$2b_7$	−1	0	1	0	−1	0	0	0	0	$-\overline{b_7}$	$-b_7$	0	0
χ_{17}	+	1265	49	−15	5	8	−7	1	−3	0	1	0	−2	−2	1	0	0	−1	0	0	0	0	0	1	1	0	0
χ_{18}	+	1771	−21	11	16	7	3	−5	−1	1	0	−1	0	0	−1	1	0	0	−1	0	0	1	1	0	0	0	0
χ_{19}	+	2024	8	24	−1	8	8	0	0	−1	−1	0	1	1	0	−1	0	−1	0	1	1	−1	−1	1	1	0	0
χ_{20}	+	2277	21	−19	0	6	−3	1	−3	−3	0	2	2	2	−1	1	0	0	0	0	0	0	0	−1	−1	0	0
χ_{21}	+	3312	48	16	0	−6	0	0	0	−3	0	−2	1	1	0	1	1	0	0	−1	−1	0	0	1	1	0	0
χ_{22}	+	3520	64	0	10	−8	0	0	0	0	−2	0	−1	−1	0	0	0	0	0	1	1	0	0	−1	−1	1	1
χ_{23}	+	5313	49	9	−15	0	1	−3	−3	3	1	0	0	0	−1	−1	0	1	0	0	0	0	0	0	0	0	0
χ_{24}	+	5544	−56	24	9	0	−8	0	0	−1	1	0	0	0	0	−1	0	1	0	0	0	−1	−1	0	0	1	1
χ_{25}	+	5796	−28	36	−9	0	−4	4	0	1	−1	0	0	0	0	1	−1	−1	0	0	0	1	1	0	0	0	0
χ_{26}	+	10395	−21	−45	0	0	3	−1	3	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	−1	−1

$q^{n/8}$ $\mathbb{Q}[\sqrt{-7}]$ $\mathbb{Q}[\sqrt{-15}]$ $\mathbb{Q}[\sqrt{-23}]$

Table 48: Decomposition of $K_1^{(2)}$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}	χ_{16}	χ_{17}	χ_{18}	χ_{19}	χ_{20}	χ_{21}	χ_{22}	χ_{23}	χ_{24}	χ_{25}	χ_{26}
-1	−2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0
47	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	0	0	0	2
55	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2	2	0	0	0	2	2	2	2
63	0	0	0	0	0	0	0	0	0	0	0	1	1	0	1	1	2	0	0	2	2	2	4	2	2	6
71	0	0	0	0	0	0	0	0	2	2	2	0	0	2	2	2	0	2	2	2	4	4	4	8	8	10
79	0	0	0	0	2	2	0	2	2	0	0	2	2	2	2	2	4	4	4	6	6	8	12	10	10	24
87	0	0	0	0	0	0	0	0	0	4	4	4	4	6	4	4	2	8	10	8	14	12	22	24	26	40
95	0	2	0	0	2	2	2	4	4	6	6	8	8	4	8	8	12	12	12	18	26	30	40	38	40	80
103	0	0	2	2	2	2	4	2	6	10	10	14	14	18	14	14	16	26	30	28	44	44	70	80	84	136
111	0	0	0	0	8	8	4	6	14	16	16	24	24	22	24	24	34	38	46	58	80	86	128	126	132	254
119	0	0	2	2	8	8	12	8	18	38	38	40	40	46	44	44	46	78	86	88	138	144	218	238	246	424
127	0	2	2	2	18	18	18	22	36	50	50	72	72	68	72	72	100	122	140	170	232	252	378	382	400	742
135	0	2	8	8	25	25	30	26	54	94	94	116	116	130	124	124	140	212	246	262	392	410	630	670	704	1222
143	0	6	6	6	50	50	50	58	100	148	148	194	194	192	202	202	256	342	388	454	654	704	1044	1074	1120	2058
151	0	4	18	18	68	68	80	72	150	252	252	318	318	346	332	332	394	582	664	722	1062	1116	1702	1800	1880	3320
159	0	14	20	20	126	126	128	138	254	390	390	516	516	520	536	536	676	904	1036	1196	1716	1836	2764	2846	2980	5408
167	2	20	40	40	182	182	214	200	396	652	652	814	814	872	860	860	1020	1476	1684	1862	2742	2902	4384	4622	4828	8572
175	2	32	55	55	314	314	328	346	640	988	988	1298	1298	1336	1348	1348	1686	2302	2630	3000	4324	4616	6950	7204	7532	13620
183	2	40	98	98	460	460	512	496	972	1590	1590	2020	2020	2144	2118	2118	2546	3638	4162	4624	6768	7166	10856	11376	11898	21204

$63 = 3^2 \times 7$ $\quad$ $135 = 3^2 \times 15$ $\quad$ $175 = 5^2 \times 7$

The Discriminant property for Umbral Moonshine in detail

$$H^{(2)}(\tau) = 2\left(-1q^{-1/8} + 45q^{7/8} + 231q^{15/8} + 770q^{23/8} + \cdots\right)$$

Proposition 5.7. *Let $\ell \in \Lambda$. If $n > 1$ is an integer satisfying*

1. *there exists an element of $G^{(\ell)}$ of order n, and*
2. *there exists an integer λ that is co-prime to n such that $D = -n\lambda^2$ is a discriminant of $H^{(\ell)}$,*

then there exists at least one pair of irreducible representations ϱ and ϱ^* of $G^{(\ell)}$ and at least one element $g \in G^{(\ell)}$ such that $\mathrm{tr}_\varrho(g)$ is not rational but

$$\mathrm{tr}_\varrho(g), \mathrm{tr}_{\varrho^*}(g) \in \mathbb{Q}(\sqrt{-n}) \tag{5.7}$$

and n divides $o(g)$.

ℓ	n	(ϱ, ϱ^*)
2	$7, 15, 23$	$(\chi_3, \chi_4), (\chi_5, \chi_6), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13}), (\chi_{15}, \chi_{16})$
3	$5, 8, 11, 20$	$(\chi_4, \chi_5), (\chi_{16}, \chi_{17}), (\chi_{20}, \chi_{21}), (\chi_{22}, \chi_{23}), (\chi_{25}, \chi_{26})$
4	$3, 7$	$(\chi_2, \chi_3), (\chi_{13}, \chi_{14}), (\chi_{15}, \chi_{16})$
5	4	$(\chi_8, \chi_9), (\chi_{10}, \chi_{11}), (\chi_{12}, \chi_{13})$
7	3	$(\chi_2, \chi_3), (\chi_6, \chi_7)$
13	4	(χ_3, χ_4)

Table 7: The irreducible representations of type n.

Armed with the preceding discussion we are now ready to state our main observation for the discriminant property of umbral moonshine. For the purpose of stating this we temporarily write $K^{(\ell)}_{r,d}$ for the ordinary representation of $G^{(\ell)}$ with character $g \mapsto c^{(\ell)}_{g,r}(d)$ where the coefficients $c^{(\ell)}_{g,r}(d)$ are assumed to be those given in §C.

Proposition 5.10. *Let n be one of the integers in Table 7 and let λ_n be the smallest positive integer such that $D = -n\lambda_n^2$ is a discriminant of $H^{(\ell)}$. Then $K^{(\ell)}_{r,-D/4\ell} = \varrho_n \oplus \varrho_n^*$ where ϱ_n and ϱ_n^* are dual irreducible representations of type n. Conversely, if ϱ is an irreducible representation of type n and $-D$ is the smallest positive integer such that $K^{(\ell)}_{r,-D/4\ell}$ has ϱ as an irreducible constituent then there exists an integer λ such that $D = -n\lambda^2$.*

Conjecture 5.11. *If D is a discriminant of $H^{(\ell)}$ which satisfies $D = -n\lambda^2$ for some integer λ then the representation $K^{(\ell)}_{r,-D/4\ell}$ has at least one dual pair of irreducible representations of type n arising as irreducible constituents.*

Conjecture 5.12. *For $\ell \in \Lambda = \{2, 3, 4, 5, 7, 13\}$ the representation $K^{(\ell)}_{r,-D/4\ell}$ is a doublet if and only if $D \neq -n\lambda^2$ for any integer λ for any n satisfying the conditions of Proposition 5.7.*

To see some evidence for Conjecture 5.12 one can inspect the proposed decompositions of the representations $K^{(\ell)}_{r,d}$ in the tables in §D for the following discriminants:

- $-D = 7, 15, 23, 63, 135, 175, 207$ for $\ell = 2$,

For $\ell = 2, X = A_1^{24}$ the discriminant property was proved by Creutzig, Hohn and Miezaki

Features of and relations between Moonshines

Classic Moonshine: Monstrous Moonshine and Conway moonshine and relatives. These involve weight zero modular functions, genus zero subgroups of SL(2,R) and have known CFT/VOA constructions with c=24,12.

Umbral Moonshine: Involves weight 1/2 mock modular forms but also characterized by genus zero groups.

Unnamed Moonshine: (B. Rayhaun and JH and JD, JH and BR, to appear) Involves weight 1/2 weakly holomorphic modular forms, genus zero groups

These are all characterized by “modular forms of minimal exponential growth.” For example, modular functions that are polynomials in J have a basis

$J_0(\tau) = 1$ no growth

$J_1(\tau) = q^{-1} + 196884q + \cdots$

$J_2(\tau) = q^{-2} + 42987520q + \cdots$ Exponential growth

$J_m(\tau) = q^{-m} + O(q)$ $c_m(n) \sim \exp(4\pi\sqrt{mn})$

so $J(\tau)$ has “minimal exponential growth” and exhibits moonshine for the Monster group. The theta function

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots$$

It is also part of an infinite family of modular functions investigated by Borcherds and Zagier with the same modular behavior but with different growth conditions:

$$f_0 = 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots$$

no growth

$$f_3 = q^{-3} - 248q + 26752q^4 - 85995q^5 + 1707264q^8 + \cdots$$

minimal exponential growth

$$f_4 = q^{-4} + 492q + 143376q^4 + 565760q^q + 18473000q^8 + \cdots$$

$$f_7 =$$

$$f_8 =$$

...

and exhibits moonshine for the Thompson group:

$$2f_3 + 248f_0 = (1+1)q^{-3} + 248 + (27000 + \overline{27000})q^4 - (85995 + \overline{85995})q^5 + \cdots$$

Decomposition into irreducible representations of the Thompson sporadic group

We saw earlier that the mock modular forms of UM also obey this principle of minimal exponential growth.

Other common characteristics are the prominence of genus zero groups enforcing rigidity and the relation of the coefficients of modular forms to traces of singular moduli (Zagier, Ono&Rolen&Trebat-Leder, Rayhaun and JH).

An important common structural element is that the modular functions of moonshine and their twined versions can be constructed as Rademacher sums (Duncan and Frenkel, Cheng and Duncan) and these have a structure very reminiscent of computations in physics in the context of the AdS/CFT correspondence.

This presence of these common features suggests that there is a unified theory of moonshine to be found which will incorporate all the known examples and probably more as well and may incorporate aspects of physics like the AdS/CFT correspondence and the counting of Black Holes and BPS states.

THANK YOU