Homogenization of a Stationary Maxwell System with Periodic Coefficients

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Plan

- Introduction
- Statement of the problem
- The effective operator
- Main results for the Maxwell system
- Reduction to the second order elliptic operator
- Method of the study of the second order operator

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The traditional results give weak convergence of the solutions to the solution of the homogenized system. Our goal is to obtain approximations for the solutions in the L_2-norm with sharp order remainder estimates.

Statement of the problem

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By $\widetilde{\Gamma}$ we denote the dual lattice. Let $\widetilde{\Omega}$ be the Brillouin zone of $\widetilde{\Gamma}$.

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Example:

$$\Gamma = \mathbb{Z}^3, \quad \Omega = (0,1)^3, \quad \widetilde{\Gamma} = (2\pi\mathbb{Z})^3, \quad \widetilde{\Omega} = (-\pi,\pi)^3.$$

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Suppose that the dielectric permittivity $\eta(\mathbf{x})$ and the magnetic permeability $\mu(\mathbf{x})$ are Γ-periodic symmetric (3×3)-matrix-valued functions with real entries. Assume that

$$c_0\mathbf{1} \leqslant \eta(\mathbf{x}) \leqslant c_1\mathbf{1}, \quad c_0\mathbf{1} \leqslant \mu(\mathbf{x}) \leqslant c_1\mathbf{1}, \quad \mathbf{x} \in \mathbb{R}^3, \quad 0 < c_0 \leqslant c_1 < \infty.$$

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$$L_2(\eta^{-1}) = L_2(\mathbb{R}^3; \mathbb{C}^3; \eta^{-1}), \quad L_2(\mu^{-1}) = L_2(\mathbb{R}^3; \mathbb{C}^3; \mu^{-1})$$

with the inner products

$$(\mathbf{f}, \mathbf{g})_{L_2(\eta^{-1})} := \int_{\mathbb{R}^3} \langle \eta(\mathbf{x})^{-1} \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \rangle \, d\mathbf{x},$$
$$(\mathbf{f}, \mathbf{g})_{L_2(\mu^{-1})} := \int_{\mathbb{R}^3} \langle \mu(\mathbf{x})^{-1} \mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x}) \rangle \, d\mathbf{x}.$$

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We put

$$J := \{ \mathbf{f} \in L_2(\mathbb{R}^3; \mathbb{C}^3) : \operatorname{div} \mathbf{f} = 0 \}.$$

Clearly, J is a closed subspace in L_2 (and also in $L_2(\eta^{-1})$ and $L_2(\mu^{-1})$).

Statement of the problem

Notation

- *Let* $\mathbf{u}(\mathbf{x})$ *be the electric field strength.*
- *Let* $\mathbf{v}(\mathbf{x})$ *be the magnetic field strength.*
- *Then* $\mathbf{w}(\mathbf{x}) = \eta(\mathbf{x})\mathbf{u}(\mathbf{x})$ *is the electric displacement vector,*
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It is convenient to write the Maxwell operator $\mathcal{M} = \mathcal{M}(\eta, \mu)$ in terms of the displacement vectors $\mathbf{w}$, $\mathbf{z}$. Then $\mathcal{M}$ acts in the space $J \oplus J$ viewed as a subspace of $L_2(\eta^{-1}) \oplus L_2(\mu^{-1})$.

Statement of the problem

Definition of the Maxwell operator

The operator $\mathcal{M} = \mathcal{M}(\eta, \mu)$ *acts in the space* $J \oplus J \subset L_2(\eta^{-1}) \oplus L_2(\mu^{-1})$ *and is given by*

$$\mathcal{M} = \begin{pmatrix} 0 & i \,\mathrm{curl}\, \mu^{-1} \\ -i \,\mathrm{curl}\, \eta^{-1} & 0 \end{pmatrix}$$

on the domain

$$\mathrm{Dom}\, \mathcal{M} = \{(\mathbf{w}, \mathbf{z}) \in J \oplus J : \ \mathrm{curl}\, \eta^{-1}\mathbf{w} \in L_2, \ \mathrm{curl}\, \mu^{-1}\mathbf{z} \in L_2\}.$$

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The operator $\mathcal{M}$ is selfadjoint with respect to the weighted inner product.

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with rapidly oscillating coefficients η^ε *and* μ^ε.

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The point $\lambda = i$ is a regular point for $\mathcal{M}_\varepsilon$.

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$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathbf{q} \\ \mathbf{r}\end{pmatrix}, \quad \mathbf{q}, \mathbf{r} \in J. \tag{1}$$

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We also study the corresponding fields

$$\mathbf{u}_\varepsilon = (\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon, \quad \mathbf{v}_\varepsilon = (\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon.$$

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In details, (1) *looks as follows:*

$$\left.\begin{array}{c} \operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon - \mathbf{w}_\varepsilon = -i\mathbf{q} \\ \operatorname{curl}(\eta^\varepsilon)^{-1}\mathbf{w}_\varepsilon + \mathbf{z}_\varepsilon = i\mathbf{r} \\ \operatorname{div}\mathbf{w}_\varepsilon = 0, \quad \operatorname{div}\mathbf{z}_\varepsilon = 0 \end{array}\right\}$$

The effective operator

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Definition of the effective matrix

Let $\mathbf{e}_j$, $j = 1, 2, 3$, *be the standard basis in* $\mathbb{C}^3$. *Let* $\Phi_j(\mathbf{x})$ *be the* Γ*-periodic solution of the problem*

$$\operatorname{div} \eta(\mathbf{x})(\nabla \Phi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \Phi_j(\mathbf{x})\, d\mathbf{x} = 0.$$

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Let $Y_\eta(\mathbf{x})$ *be the matrix with the columns* $\nabla \Phi_j(\mathbf{x})$, $j = 1, 2, 3$. *Denote*

$$\widetilde{\eta}(\mathbf{x}) := \eta(\mathbf{x})(Y_\eta(\mathbf{x}) + \mathbf{1}), \quad \eta^0 := |\Omega|^{-1} \int_\Omega \widetilde{\eta}(\mathbf{x})\, d\mathbf{x}.$$

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Let $\Psi_j(\mathbf{x})$ *be the* Γ*-periodic solution of the problem*

$$\operatorname{div} \mu(\mathbf{x})(\nabla \Psi_j(\mathbf{x}) + \mathbf{e}_j) = 0, \quad \int_\Omega \Psi_j(\mathbf{x})\, d\mathbf{x} = 0.$$

Let $Y_\mu(\mathbf{x})$ *be the matrix with the columns* $\nabla \Psi_j(\mathbf{x})$, $j = 1, 2, 3$. *Denote*

$$\widetilde{\mu}(\mathbf{x}) := \mu(\mathbf{x})(Y_\mu(\mathbf{x}) + \mathbf{1}), \quad \mu^0 := |\Omega|^{-1} \int_\Omega \widetilde{\mu}(\mathbf{x})\, d\mathbf{x}.$$

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Main results for the Maxwell system

So, we study the solutions of the Maxwell system

$$(\mathcal{M}_\varepsilon - iI)\begin{pmatrix}\mathbf{w}_\varepsilon \\ \mathbf{z}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathbf{q} \\ \mathbf{r}\end{pmatrix}, \quad \mathbf{q}, \mathbf{r} \in J, \tag{1}$$

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Classical results

The solutions of (1) weakly converge in L_2 to the solutions of (2):

$$\mathbf{u}_\varepsilon \xrightarrow{w} \mathbf{u}_0, \quad \mathbf{v}_\varepsilon \xrightarrow{w} \mathbf{v}_0, \quad \mathbf{w}_\varepsilon \xrightarrow{w} \mathbf{w}_0, \quad \mathbf{z}_\varepsilon \xrightarrow{w} \mathbf{z}_0, \quad \varepsilon \to 0.$$

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To formulate the results, we consider the "correction Maxwell system"

$$(\mathcal{M}^0 - iI)\begin{pmatrix}\widehat{\mathbf{w}}_\varepsilon \\ \widehat{\mathbf{z}}_\varepsilon\end{pmatrix} = \begin{pmatrix}\mathbf{q}_\varepsilon \\ \mathbf{r}_\varepsilon\end{pmatrix}, \quad \mathbf{q}_\varepsilon := \mathcal{P}_{\eta^0}(Y_\eta^\varepsilon)^*\mathbf{q}, \quad \mathbf{r}_\varepsilon := \mathcal{P}_{\mu^0}(Y_\mu^\varepsilon)^*\mathbf{r}. \quad (3)$$

Here $\mathcal{P}_{\eta^0}$ is the orthogonal projection of the weighted space $L_2((\eta^0)^{-1})$ onto J, and $\mathcal{P}_{\mu^0}$ is the orthogonal projection of $L_2((\mu^0)^{-1})$ onto J.

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Note that $\widehat{\mathbf{u}}_\varepsilon$, $\widehat{\mathbf{v}}_\varepsilon$, $\widehat{\mathbf{w}}_\varepsilon$, and $\widehat{\mathbf{z}}_\varepsilon$ weakly converge to zero in L_2, as $\varepsilon \to 0$.
Finally, we define the auxiliary smoothing operator Π_ε in $L_2(\mathbb{R}^3;\mathbb{C}^3)$:

$$(\Pi_\varepsilon \mathbf{f})(\mathbf{x}) = (2\pi)^{-3/2}\int_{\widetilde{\Omega}/\varepsilon} e^{i\langle \mathbf{x},\boldsymbol{\xi}\rangle}\widehat{\mathbf{f}}(\boldsymbol{\xi})\,d\boldsymbol{\xi},$$

where $\widehat{\mathbf{f}}(\boldsymbol{\xi})$ is the Fourier-image of $\mathbf{f}(\mathbf{x})$.

Main results for the Maxwell system

Our main result is

Theorem 1 [T. Suslina]

For $0 < \varepsilon \leqslant 1$ *we have*

$$\|\mathbf{u}_\varepsilon - (\mathbf{1} + Y_\eta^\varepsilon)(\mathbf{u}_0 + \Pi_\varepsilon \widehat{\mathbf{u}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),$$
$$\|\mathbf{w}_\varepsilon - (\mathbf{1} + G_\eta^\varepsilon)(\mathbf{w}_0 + \Pi_\varepsilon \widehat{\mathbf{w}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),$$
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$$\|\mathbf{z}_\varepsilon - (\mathbf{1} + G_\mu^\varepsilon)(\mathbf{z}_0 + \Pi_\varepsilon \widehat{\mathbf{z}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}).$$

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Theorem 1 [T. Suslina]

For $0 < \varepsilon \leqslant 1$ *we have*

$$
\begin{aligned}
\|\mathbf{u}_\varepsilon - (\mathbf{1} + Y_\eta^\varepsilon)(\mathbf{u}_0 + \Pi_\varepsilon \widehat{\mathbf{u}}_\varepsilon)\|_{L_2} &\leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),\\
\|\mathbf{w}_\varepsilon - (\mathbf{1} + G_\eta^\varepsilon)(\mathbf{w}_0 + \Pi_\varepsilon \widehat{\mathbf{w}}_\varepsilon)\|_{L_2} &\leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),\\
\|\mathbf{v}_\varepsilon - (\mathbf{1} + Y_\mu^\varepsilon)(\mathbf{v}_0 + \Pi_\varepsilon \widehat{\mathbf{v}}_\varepsilon)\|_{L_2} &\leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),\\
\|\mathbf{z}_\varepsilon - (\mathbf{1} + G_\mu^\varepsilon)(\mathbf{z}_0 + \Pi_\varepsilon \widehat{\mathbf{z}}_\varepsilon)\|_{L_2} &\leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}).
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Remark.

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Remark.

1) Estimates of Theorem 1 are order-sharp.

2) The constants depend only on $\|\eta\|_{L_\infty}$, $\|\eta^{-1}\|_{L_\infty}$, $\|\mu\|_{L_\infty}$, $\|\mu^{-1}\|_{L_\infty}$, and the parameters of the lattice.

Remark.
3) All approximations are similar to each other. For instance, we have

$$\mathbf{w}_\varepsilon \sim \mathbf{w}_0 + G_\eta^\varepsilon \mathbf{w}_0 + \Pi_\varepsilon \widehat{\mathbf{w}}_\varepsilon + G_\eta^\varepsilon \Pi_\varepsilon \widehat{\mathbf{w}}_\varepsilon.$$

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3) All approximations are similar to each other. For instance, we have

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4) The result can be formulated in operator terms:

$$\left\| (\mathcal{M}_\varepsilon - iI)^{-1} - (I + G^\varepsilon)(\mathcal{M}^0 - iI)^{-1}(I + Z_\varepsilon) \right\| \leqslant C\varepsilon,$$

where

$$G^\varepsilon = \begin{pmatrix} G^\varepsilon_\eta & 0 \\ 0 & G^\varepsilon_\mu \end{pmatrix}, \quad Z_\varepsilon = \begin{pmatrix} \Pi_\varepsilon \mathcal{P}_{\eta^0} (Y^\varepsilon_\eta)^* & 0 \\ 0 & \Pi_\varepsilon \mathcal{P}_{\mu^0} (Y^\varepsilon_\mu)^* \end{pmatrix}.$$

Main results for the Maxwell system

5) Under some additional assumptions it is possible to replace Π_ε by identity. For instance, this is possible if $\eta \in W^1_{p,\mathrm{per}}(\Omega)$ with $p > 3$ and μ is arbitrary, or if $\mu \in W^1_{p,\mathrm{per}}(\Omega)$ with $p > 3$ and η is arbitrary.

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6) If one of the coefficients (η or μ) is constant, the results are simpler.

Theorem 2 [M. Birman and T. Suslina]

Let $\mu = \mu_0$ be a constant positive matrix. For $0 < \varepsilon \leqslant 1$ we have

$$\|\mathbf{u}_\varepsilon - (\mathbf{1} + Y^\varepsilon_\eta)(\mathbf{u}_0 + \widehat{\mathbf{u}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),$$
$$\|\mathbf{w}_\varepsilon - (\mathbf{1} + G^\varepsilon_\eta)(\mathbf{w}_0 + \widehat{\mathbf{w}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),$$
$$\|\mathbf{v}_\varepsilon - (\mathbf{v}_0 + \widehat{\mathbf{v}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}),$$
$$\|\mathbf{z}_\varepsilon - (\mathbf{z}_0 + \widehat{\mathbf{z}}_\varepsilon)\|_{L_2} \leqslant C\varepsilon(\|\mathbf{q}\|_{L_2} + \|\mathbf{r}\|_{L_2}).$$

Main results for the Maxwell system

Corollary

Let $\mu = \mu_0$ *be a constant positive matrix, and let* $\mathbf{q} = 0$. *Then the "correction fields"* $\widehat{\mathbf{u}}_\varepsilon$, $\widehat{\mathbf{w}}_\varepsilon$, $\widehat{\mathbf{v}}_\varepsilon$, $\widehat{\mathbf{z}}_\varepsilon$ *are equal to zero. For* $0 < \varepsilon \leqslant 1$ *we have*

$$\begin{aligned} \|\mathbf{u}_\varepsilon - (\mathbf{1} + Y_\eta^\varepsilon)\mathbf{u}_0\|_{L_2} &\leqslant C\varepsilon\|\mathbf{r}\|_{L_2}, \\ \|\mathbf{w}_\varepsilon - (\mathbf{1} + G_\eta^\varepsilon)\mathbf{w}_0\|_{L_2} &\leqslant C\varepsilon\|\mathbf{r}\|_{L_2}, \\ \|\mathbf{v}_\varepsilon - \mathbf{v}_0\|_{L_2} &\leqslant C\varepsilon\|\mathbf{r}\|_{L_2}, \\ \|\mathbf{z}_\varepsilon - \mathbf{z}_0\|_{L_2} &\leqslant C\varepsilon\|\mathbf{r}\|_{L_2}. \end{aligned}$$

Reduction to the second order elliptic operator

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$$\mathbf{w}_\varepsilon = \mathbf{w}_\varepsilon^{(\mathbf{q})} + \mathbf{w}_\varepsilon^{(\mathbf{r})}, \quad \mathbf{z}_\varepsilon = \mathbf{z}_\varepsilon^{(\mathbf{q})} + \mathbf{z}_\varepsilon^{(\mathbf{r})},$$

where $(\mathbf{w}_\varepsilon^{(\mathbf{q})}, \mathbf{z}_\varepsilon^{(\mathbf{q})})$ is the solution of system (1) with $\mathbf{r} = 0$, and $(\mathbf{w}_\varepsilon^{(\mathbf{r})}, \mathbf{z}_\varepsilon^{(\mathbf{r})})$ is the solution of system (1) with $\mathbf{q} = 0$.

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Similarly, we represent $\mathbf{u}_\varepsilon$ and $\mathbf{v}_\varepsilon$ as the sum of two terms:

$$\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon^{(\mathbf{q})} + \mathbf{u}_\varepsilon^{(\mathbf{r})}, \quad \mathbf{v}_\varepsilon = \mathbf{v}_\varepsilon^{(\mathbf{q})} + \mathbf{v}_\varepsilon^{(\mathbf{r})}.$$

We study the fields with indices $(\mathbf{q})$ and $(\mathbf{r})$ separately. The cases $\mathbf{q} = 0$ and $\mathbf{r} = 0$ are similar.

Reduction to the second order elliptic operator

The case where $\mathbf{q} = 0$. System (1) with $\mathbf{q} = 0$ takes the form

$$\left.\begin{aligned} \mathbf{w}_\varepsilon^{(\mathbf{r})} = \operatorname{curl} (\mu^\varepsilon)^{-1} \mathbf{z}_\varepsilon^{(\mathbf{r})} \\ \operatorname{curl} (\eta^\varepsilon)^{-1} \mathbf{w}_\varepsilon^{(\mathbf{r})} + \mathbf{z}_\varepsilon^{(\mathbf{r})} = i\mathbf{r} \\ \operatorname{div} \mathbf{w}_\varepsilon^{(\mathbf{r})} = 0, \quad \operatorname{div} \mathbf{z}_\varepsilon^{(\mathbf{r})} = 0 \end{aligned}\right\}$$

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Hence, $\mathbf{z}_\varepsilon^{(\mathbf{r})}$ is the solution of the second order equation

$$\operatorname{curl}(\eta^\varepsilon)^{-1}\operatorname{curl}(\mu^\varepsilon)^{-1}\mathbf{z}_\varepsilon^{(\mathbf{r})} + \mathbf{z}_\varepsilon^{(\mathbf{r})} = i\mathbf{r}, \quad \operatorname{div}\mathbf{z}_\varepsilon^{(\mathbf{r})} = 0. \tag{4}$$

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In order to study equation (4), it is convenient to substitute

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It is convenient to pass from (4) to (5), since the operator in (5) is selfadjoint with respect to the standard inner product in $L_2(\mathbb{R}^3; \mathbb{C}^3)$.

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$$\mathcal{L}_\varepsilon = (\mu^\varepsilon)^{-1/2} \mathrm{curl}\, (\eta^\varepsilon)^{-1} \mathrm{curl}\, (\mu^\varepsilon)^{-1/2} - (\mu^\varepsilon)^{1/2} \nabla \mathrm{div}\, (\mu^\varepsilon)^{1/2}$$

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acting in $L_2(\mathbb{R}^3;\mathbb{C}^3)$. The precise definition of the operator $\mathcal{L}_\varepsilon$ is given in terms of the quadratic form

$$\mathfrak{l}_\varepsilon[\mathbf{f},\mathbf{f}] = \int\limits_{\mathbb{R}^3} \left(\langle(\eta^\varepsilon)^{-1}\mathrm{curl}\,(\mu^\varepsilon)^{-1/2}\mathbf{f}, \mathrm{curl}\,(\mu^\varepsilon)^{-1/2}\mathbf{f}\rangle + |\mathrm{div}\,(\mu^\varepsilon)^{1/2}\mathbf{f}|^2\right)\,d\mathbf{x},$$

$$\mathrm{Dom}\,\mathfrak{l}_\varepsilon = \{\mathbf{f}\in L_2(\mathbb{R}^3;\mathbb{C}^3):\ \mathrm{curl}\,(\mu^\varepsilon)^{-1/2}\mathbf{f}\in L_2,\ \mathrm{div}\,(\mu^\varepsilon)^{1/2}\mathbf{f}\in L_2\}.$$

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This form is closed and nonnegative. By definition, $\mathcal{L}_\varepsilon$ is the selfadjoint operator in $L_2(\mathbb{R}^3; \mathbb{C}^3)$ generated by this form.

Reduction to the second order elliptic operator

The operator

$$\mathcal{L}_\varepsilon = (\mu^\varepsilon)^{-1/2}\mathrm{curl}\,(\eta^\varepsilon)^{-1}\mathrm{curl}\,(\mu^\varepsilon)^{-1/2} - (\mu^\varepsilon)^{1/2}\nabla\mathrm{div}\,(\mu^\varepsilon)^{1/2}$$

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is elliptic and acts in the whole space $L_2(\mathbb{R}^3;\mathbb{C}^3)$.
This operator splits in the orthogonal Weyl decomposition

$$L_2(\mathbb{R}^3;\mathbb{C}^3) = G(\mu^\varepsilon) \oplus J(\mu^\varepsilon),$$

where

$$G(\mu^\varepsilon) = \{\mathbf{g} = (\mu^\varepsilon)^{1/2}\nabla\varphi :\ \varphi \in H^1_{\mathrm{loc}},\ \nabla\varphi \in L_2\},$$
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We are interested in the part of $\mathcal{L}_\varepsilon$ in the subspace $J(\mu^\varepsilon)$. Let $\mathcal{P}(\mu^\varepsilon)$ be the orthogonal projection of $L_2(\mathbb{R}^3; \mathbb{C}^3)$ onto $J(\mu^\varepsilon)$.

Reduction to the second order elliptic operator

Conclusion

The solution of problem (5) can be represented as

$$\mathbf{f}_\varepsilon = \mathcal{P}(\mu^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1}\left(i(\mu^\varepsilon)^{-1/2}\mathbf{r}\right).$$

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So, we have reduced the problem to the study of the resolvent $(\mathcal{L}_\varepsilon + I)^{-1}$ and its "divergence-free part" $\mathcal{P}(\mu^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1}$. We need to find approximations in the $(L_2 \to L_2)$-norm and in the energy norm.

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The fields with index (q) are studied similarly.

Results for the second order elliptic operator

Let $\mathcal{L}^0 = \mathcal{L}(\eta^0, \mu^0)$ be the effective operator. We prove that

$$\|(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1} W^\varepsilon\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leqslant C\varepsilon. \qquad (6)$$

Here $W(\mathbf{x})$ is some periodic matrix-valued function (it will be defined later).

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From here we deduce the required approximations for $\mathbf{v}_\varepsilon^{(r)}$ and $\mathbf{z}_\varepsilon^{(r)}$. The required approximations for $\mathbf{u}_\varepsilon^{(r)}$ and $\mathbf{w}_\varepsilon^{(r)}$ are deduced from

$$\|\mathcal{L}_\varepsilon^{1/2} \left(\mathcal{P}(\mu^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^*\mathcal{P}(\mu^0)(\mathcal{L}^0 + I)^{-1} W^\varepsilon - \varepsilon K(\varepsilon)\right) \|_{L_2\to L_2} \leqslant C\varepsilon, \quad (8)$$

where $K(\varepsilon)$ is appropriate corrector.

Method: the scaling transformation

The operator $\mathcal{L}_\varepsilon$ is studied by the operator-theoretic approach based on the scaling transformation, the Floquet-Bloch theory, and the analytic perturbation theory.

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Let T_ε be the unitary scaling operator in $L_2(\mathbb{R}^3; \mathbb{C}^3)$:

$$(T_\varepsilon \mathbf{f})(\mathbf{x}) = \varepsilon^{3/2} \mathbf{f}(\varepsilon \mathbf{x}).$$

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Scaling transformation. Let $\mathcal{L} = \mathcal{L}(\eta, \mu)$ be the operator

$$\mathcal{L} = \mu^{-1/2}\mathrm{curl}\,\eta^{-1}\mathrm{curl}\,\mu^{-1/2} - \mu^{1/2}\nabla\mathrm{div}\,\mu^{1/2}.$$

Let T_ε be the unitary scaling operator in $L_2(\mathbb{R}^3;\mathbb{C}^3)$:

$$(T_\varepsilon \mathbf{f})(\mathbf{x}) = \varepsilon^{3/2}\mathbf{f}(\varepsilon\mathbf{x}).$$

Then we have

$$(\mathcal{L}_\varepsilon + I)^{-1} = \varepsilon^2 T_\varepsilon^*(\mathcal{L} + \varepsilon^2 I)^{-1} T_\varepsilon$$

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Thus, in order to approximate $(\mathcal{L}_\varepsilon + I)^{-1}$ with error $O(\varepsilon)$, it suffices to approximate $(\mathcal{L} + \varepsilon^2 I)^{-1}$ with error $O(\varepsilon^{-1})$.

Method: factorization

Factorization. It is important that the operator $\mathcal{L}$ admits a factorization of the form

$$\mathcal{L} = \mathcal{X}^* \mathcal{X},$$

where $\mathcal{X}$ is the first order DO given by

$$\mathcal{X}\mathbf{f} = \begin{pmatrix} -i\eta^{-1/2}\mathrm{curl}\,(\mu^{-1/2}\mathbf{f}) \\ -i\mathrm{div}\,(\mu^{1/2}\mathbf{f}) \end{pmatrix}.$$

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Remark. If $\mu = \mu_0$ is constant, then $\mathcal{L}$ can be written as

$$\mathcal{L} = b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D}), \quad g(\mathbf{x}) = \begin{pmatrix} \eta(\mathbf{x})^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \quad b(\mathbf{D}) = \begin{pmatrix} -i\mathrm{curl}\,\mu_0^{-1/2} \\ -i\mathrm{div}\,\mu_0^{1/2} \end{pmatrix}.$$

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The class of operators of the form $b(\mathbf{D})^* g(\mathbf{x}) b(\mathbf{D})$ has been studied by **Birman** and **Suslina**. So, if μ is constant, one can apply general results.

Method: the direct integral expansion

Direct integral. By the Floquet-Bloch theory, the operator $\mathcal{L}$ admits the direct integral expansion

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with periodic boundary conditions. Here

$$\nabla_{\mathbf{k}} \varphi := \nabla \varphi + i\mathbf{k}\varphi, \quad \mathrm{div}_{\mathbf{k}} \mathbf{f} := \mathrm{div}\, \mathbf{f} + i\mathbf{k} \cdot \mathbf{f}, \quad \mathrm{curl}_{\mathbf{k}} \mathbf{f} := \mathrm{curl}\, \mathbf{f} + i\mathbf{k} \times \mathbf{f}.$$

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The precise definition of $\mathcal{L}(\mathbf{k})$ is given in terms of the corresponding quadratic form.

Method: analytic perturbation theory

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The operator $L(t, \boldsymbol{\theta})$ admits a factorization of the form

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$$L(t,\boldsymbol{\theta}) = X(t,\boldsymbol{\theta})^* X(t,\boldsymbol{\theta}),$$

where $X(t,\boldsymbol{\theta})$ is a linear operator pencil:

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Here X_0 is given by

$$X_0\mathbf{f} = \begin{pmatrix} -i\eta^{-1/2}\mathrm{curl}\,(\mu^{-1/2}\mathbf{f}) \\ -i\mathrm{div}\,(\mu^{1/2}\mathbf{f}) \end{pmatrix}$$

with periodic boundary conditions; $X_1(\boldsymbol{\theta})$ is a bounded operator given by

$$X_1(\boldsymbol{\theta})\mathbf{f} = \begin{pmatrix} \eta^{-1/2}\boldsymbol{\theta} \times (\mu^{-1/2}\mathbf{f}) \\ \boldsymbol{\theta} \cdot (\mu^{1/2}\mathbf{f}) \end{pmatrix}.$$

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Consider the kernel of the operator $\mathcal{L}(0)$:

$$\mathfrak{N} = \operatorname{Ker} \mathcal{L}(0) = \operatorname{Ker} X_0.$$

It is given by

$$\mathfrak{N} = \{\mathbf{f}(\mathbf{x}) = \mu(\mathbf{x})^{1/2}(\mathbf{C} + \nabla\Psi_{\mathbf{C}}(\mathbf{x})) : \ \mathbf{C} \in \mathbb{C}^3\},$$

where $\Psi_{\mathbf{C}}(\mathbf{x})$ is periodic solution of the equation

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Method: analytic perturbation theory

By the Kato-Rellich theorem, for $t \leqslant t^0$ there exist real-analytic branches of the eigenvalues $\lambda_l(t, \boldsymbol{\theta})$ and real-analytic branches of the eigenvectors $\varphi_l(t, \boldsymbol{\theta})$ of the operator $L(t, \boldsymbol{\theta})$:

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The vectors $\varphi_l(t, \boldsymbol{\theta})$, $l = 1, 2, 3$, form an orthonormal basis in the eigenspace of $L(t, \boldsymbol{\theta})$ corresponding to the interval $[0, \delta]$. For small $t \leqslant t_*(\boldsymbol{\theta})$ we have the following convergent power series expansions:

$$\lambda_l(t, \boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})t^2 + \mu_l(\boldsymbol{\theta})t^3 + \dots, \quad l = 1, 2, 3,$$
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We have $\gamma_l(\boldsymbol{\theta}) \geqslant c_* > 0$. The vectors $\omega_l(\boldsymbol{\theta})$, $l = 1, 2, 3$, form an orthonormal basis in $\mathfrak{N}$. The coefficients $\gamma_l(\boldsymbol{\theta})$ and the vectors $\omega_l(\boldsymbol{\theta})$, $l = 1, 2, 3$, are called threshold characteristics of $L(t, \boldsymbol{\theta})$.

Method: analytic perturbation theory

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Definition of the spectral germ

The selfadjoint operator $S(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ *such that*

$$S(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}) = \gamma_l(\boldsymbol{\theta})\omega_l(\boldsymbol{\theta}), \quad l = 1, 2, 3,$$

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Thus, the germ contains information about the threshold characteristics. It is possible to calculate the spectral germ.

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$$S^0(\boldsymbol{\theta}) = (\mu^0)^{-1/2} r(\boldsymbol{\theta})^* (\eta^0)^{-1} r(\boldsymbol{\theta}) (\mu^0)^{-1/2} + (\mu^0)^{1/2} \boldsymbol{\theta}\boldsymbol{\theta}^* (\mu^0)^{1/2},$$

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The matrix $S^0(\boldsymbol{\theta})$ is the symbol of the effective operator.

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Let $\mathcal{U} : \mathfrak{N} \to \mathfrak{N}^0$ be the unitary operator, which takes $\mathbf{f} = \mu^{1/2}(\mathbf{C} + \nabla\Psi_{\mathbf{C}})$ to $\mathbf{f}^0 = (\mu^0)^{1/2}\mathbf{C}$, $\mathbf{C} \in \mathbb{C}^3$.

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Remark. If $\mu = \mu_0$ is constant, then $S(\boldsymbol{\theta}) = S^0(\boldsymbol{\theta})$.
Applying abstract results by **Birman** and **Suslina**, it is possible to approximate the resolvent of $L(t, \boldsymbol{\theta})$ by the resolvent of the germ.

Theorem 3 [T. Suslina]

Let P be the orthogonal projection of $L_2(\Omega; \mathbb{C}^3)$ onto $\mathfrak{N}$. Let $S(\boldsymbol{\theta}) : \mathfrak{N} \to \mathfrak{N}$ be the spectral germ of $L(t, \boldsymbol{\theta})$. Then

$$\|(L(t,\boldsymbol{\theta}) + \varepsilon^2 I)^{-1} - (t^2 S(\boldsymbol{\theta}) + \varepsilon^2 I_{\mathfrak{N}})^{-1} P\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon^{-1},$$
$$0 < \varepsilon \leqslant 1, \quad t \leqslant t^0.$$

Method: approximation of the resolvent

Using Theorem 3 and representation $S(\boldsymbol{\theta}) = \mathcal{U}^* S^0(\boldsymbol{\theta}) \mathcal{U}$ for the germ, we arrive at the following result.

Theorem 4 [T. Suslina]

Let $W(\mathbf{x})$ *be the* (3×3)*-matrix with the columns* $\mu(\mathbf{x})^{1/2}(\mathbf{C}_j + \nabla \Psi_{\mathbf{C}_j}(\mathbf{x}))$, $j = 1, 2, 3$, *where* $\mathbf{C}_j = (\mu^0)^{-1/2} \mathbf{e}_j$. *Then for* $0 < \varepsilon \leqslant 1$ *and* $\mathbf{k} \in \widetilde{\Omega}$ *we have*

$$\|(\mathcal{L}(\mathbf{k}) + \varepsilon^2 I)^{-1} - W^*(\mathcal{L}^0(\mathbf{k}) + \varepsilon^2 I)^{-1} W\|_{L_2(\Omega) \to L_2(\Omega)} \leqslant C\varepsilon^{-1}.$$

Method: approximation of the resolvent

Using the direct integral expansion, we obtain

Theorem 5 [T. Suslina]

For $0 < \varepsilon \leqslant 1$ *we have*

$$\|(\mathcal{L} + \varepsilon^2 I)^{-1} - W^*(\mathcal{L}^0 + \varepsilon^2 I)^{-1} W\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leqslant C\varepsilon^{-1}.$$

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Finally, by the scaling transformation, we arrive at the following result.

Theorem 6 [T. Suslina]

Let $\mathcal{L}_\varepsilon = \mathcal{L}(\eta^\varepsilon, \mu^\varepsilon)$ *and* $\mathcal{L}^0 = \mathcal{L}(\eta^0, \mu^0)$. *For* $0 < \varepsilon \leqslant 1$ *we have*

$$\|(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^*(\mathcal{L}^0 + I)^{-1} W^\varepsilon\|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leqslant C\varepsilon.$$

Method: approximation of the resolvent

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The operator $\mathcal{L}$ splits in the orthogonal Weyl decomposition

$$L_2(\mathbb{R}^3;\mathbb{C}^3) = G(\mu) \oplus J(\mu),$$

where

$$G(\mu) = \{\mathbf{g} = \mu^{1/2}\nabla\varphi : \ \varphi \in H^1_{\mathrm{loc}}, \ \nabla\varphi \in L_2\},$$
$$J(\mu) = \{\mathbf{f} \in L_2 : \ \operatorname{div} \mu^{1/2}\mathbf{f} = 0\}.$$

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The operator $\mathcal{L}(\mathbf{k})$ splits in the orthogonal Weyl decomposition

$$L_2(\Omega;\mathbb{C}^3) = G(\mu;\mathbf{k}) \oplus J(\mu;\mathbf{k}),$$

where

$$G(\mu;\mathbf{k}) = \{\mathbf{g} = \mu^{1/2}\nabla_{\mathbf{k}}\varphi : \ \varphi \in H^1_{\mathrm{per}}(\Omega)\},$$
$$J(\mu;\mathbf{k}) = \{\mathbf{f} \in L_2(\Omega;\mathbb{C}^n) : \ \operatorname{div}_{\mathbf{k}}(\mu^{1/2}\mathbf{f}) = 0\}.$$

Method: approximation of the resolvent

Moreover, the germ $S(\boldsymbol{\theta})$ splits in the orthogonal decomposition

$$\mathfrak{N} = G_{\boldsymbol{\theta}} \oplus J_{\boldsymbol{\theta}},$$

where

$$G_{\boldsymbol{\theta}} = \{c\mathbf{f}_{\boldsymbol{\theta}} : \ \mathbf{f}_{\boldsymbol{\theta}} = \mu^{1/2}(\boldsymbol{\theta} + \nabla\Psi_{\boldsymbol{\theta}}), \ c \in \mathbb{C})\}, \quad \dim G_{\boldsymbol{\theta}} = 1,$$
$$J_{\boldsymbol{\theta}} = \{\mathbf{f}_{\perp} = \mu^{1/2}(\mathbf{C}_{\perp} + \nabla\Psi_{\mathbf{C}_{\perp}}) : \ \mu^{0}\mathbf{C}_{\perp} \perp \boldsymbol{\theta}\}, \quad \dim J_{\boldsymbol{\theta}} = 2.$$

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It turns out that two branches $\varphi_1(t, \boldsymbol{\theta})$ and $\varphi_2(t, \boldsymbol{\theta})$ belong to $J(\mu; \mathbf{k})$, and $\varphi_3(t, \boldsymbol{\theta})$ belongs to $G(\mu; \mathbf{k})$. The corresponding "embrios" $\omega_1(\boldsymbol{\theta})$ and $\omega_2(\boldsymbol{\theta})$ belong to $J_{\boldsymbol{\theta}}$, and $\omega_3(\boldsymbol{\theta}) \in G_{\boldsymbol{\theta}}$.

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The part of the germ acting in $J_{\boldsymbol{\theta}}$ corresponds to the "divergence free" part of the operator family $L(t,\boldsymbol{\theta})$.

Method: approximation of the resolvent

These considerations lead to the following result.

Theorem 7 [T. Suslina]

Let $\mathcal{L}_\varepsilon = \mathcal{L}(\eta^\varepsilon, \mu^\varepsilon)$ *and* $\mathcal{L}^0 = \mathcal{L}(\eta^0, \mu^0)$*. Let* $\mathcal{P}(\mu^\varepsilon)$ *be the orthogonal projection of* $L_2(\mathbb{R}^3; \mathbb{C}^3)$ *onto the subspace* $J(\mu^\varepsilon) = \{\mathbf{f} \in L_2 : \operatorname{div}(\mu^\varepsilon)^{1/2}\mathbf{f} = 0\}$*. Let* $\mathcal{P}(\mu^0)$ *be the orthogonal projection of* $L_2(\mathbb{R}^3; \mathbb{C}^3)$ *onto the subspace* $J(\mu^0) = \{\mathbf{f} \in L_2 : \operatorname{div}(\mu^0)^{1/2}\mathbf{f} = 0\}$*. For* $0 < \varepsilon \leqslant 1$ *we have*

$$\|\mathcal{P}(\mu^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^*\mathcal{P}(\mu^0)(\mathcal{L}^0 + I)^{-1}W^\varepsilon\|_{L_2(\mathbb{R}^3)\to L_2(\mathbb{R}^3)} \leqslant C\varepsilon.$$

Method: approximation of the resolvent

Approximation for the resolvent of $\mathcal{L}_\varepsilon$ in the "energy norm":

Theorem 8 [T. Suslina]

For $0 < \varepsilon \leqslant 1$ we have

$$\|\mathcal{L}_\varepsilon^{1/2} \left(\mathcal{P}(\mu^\varepsilon)(\mathcal{L}_\varepsilon + I)^{-1} - (W^\varepsilon)^* \mathcal{P}(\mu^0)(\mathcal{L}^0 + I)^{-1} W^\varepsilon \right.$$
$$\left. -\varepsilon K(\varepsilon)\right) \|_{L_2(\mathbb{R}^3) \to L_2(\mathbb{R}^3)} \leqslant C\varepsilon.$$

Here $K(\varepsilon)$ is a corrector of the form

$$K(\varepsilon) = \sum_{l=1}^{3} \Lambda_l^\varepsilon D_l \mathcal{P}(\mu^0)(\mathcal{L}^0 + I)^{-1} \Pi_\varepsilon W^\varepsilon,$$

and $\Lambda_l(\mathbf{x})$ are appropriate periodic matrix-valued functions.

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The results for the Maxwell system are deduced from Theorems 7 and 8.

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