Electrical Impedance Tomography and Hybrid Imaging

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Electrical impedance tomography (EIT)

Calderón's inverse conductivity problem: Imaging an electrical conductivity $\sigma(x)$ via noninvasive voltage/current measurements at the surface of an object.

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$\hookrightarrow$ Hybrid imaging developed to overcome disadvantages of EIT and other modalities.

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• **Measure resulting current flow** I **across** $\partial\Omega$**.**
Ohm's Law $\implies$

$$I = \sigma \cdot \frac{\partial u}{\partial \nu}$$

Quasi-static regime: **Electric potential** $u(x)$ **satisfies conductivity equation,**

$$\nabla \cdot (\sigma \nabla u)(x) = 0 \textbf{ on } \Omega,$$

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Dirichlet-to-Neumann operator

$$f \longrightarrow \sigma \cdot \frac{\partial u}{\partial \nu} =: \Lambda_\sigma(f) \qquad \textbf{on } \partial\Omega.$$

$\Lambda_\sigma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$ **bounded lin. oper.**

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(iii, . . .) **Stability** of $\Lambda_\sigma \to \sigma$, **numerics, . . .**

A: Yes to (i), (ii), but poor stability.

Progress on isotropic Calderón problem

1980, Calderón: linearization around $\sigma \equiv 1$

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1988, Nachman: reconstruction, $n \geq 3$

1996, Nachman: uniqueness+reconstr., $n = 2$

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2013, Haberman and Tataru: **uniqueness for $\sigma \in C^1$ or Lipschitz close to constant.**

2015, Caro and Rogers: **! for Lipschitz σ.**

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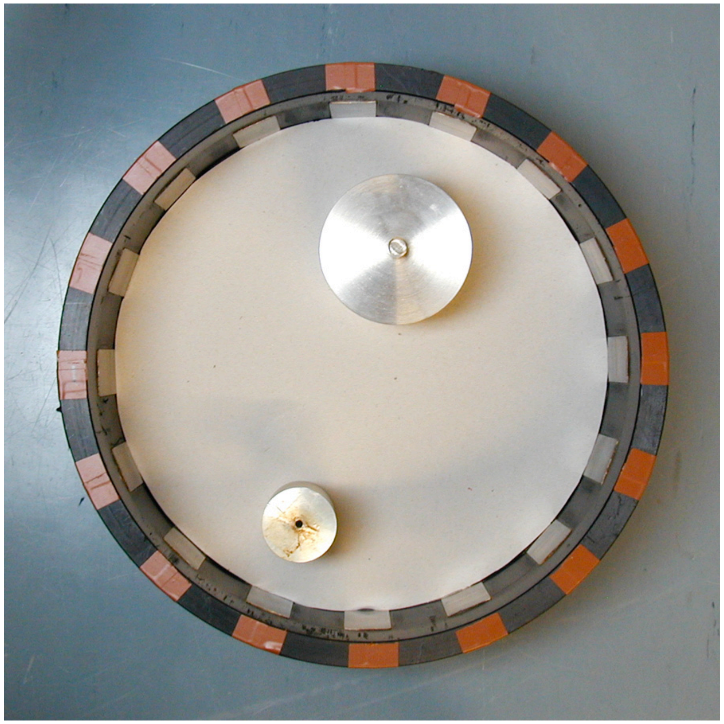
Q.: Does uniqueness hold for $\sigma \in L^\infty$?

Problem: EIT has high contrast sensitivity, but low spatial resolution.



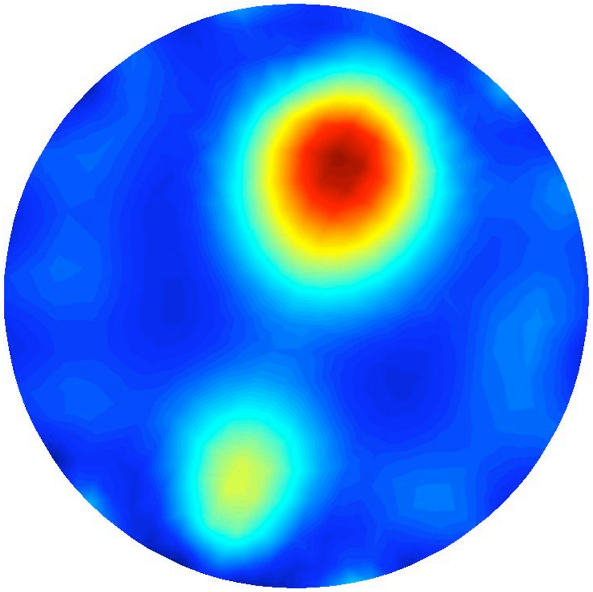


Figure 1: EIT tank and measurements. Source: Kaipio lab, Univ. of Kuopio, Finland

Hybrid inverse problems

Q: **Can one improve imaging by using data from more than one type of wave?**

(i) Image registration, e.g., CT+MRI

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Q: Can one improve imaging by using data from more than one type of wave?

(i) Image registration, e.g., CT+MRI

(ii) Stabilization: collect data for two types of waves, X and Y, simultaneously. Either use

- **Y data to provide a priori information that stabilizes reconstruction from X data;**

or

- **an algorithm using both X and Y data.**

Ex.: Current Density Impedance Imaging (CDI) - Magnetic Resonance EIT (MREIT):

Measure both voltage/current at $\partial\Omega$ and current density $\sigma|\nabla u|$ in the interior (via MRI).

$\hookrightarrow$ J**-substitution algo. of Kwon, Woo, et al.**

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However, want to discuss

(iii) 'Multi-physics' hybrid methods in which two different kinds of waves are physically coupled.

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Mathematically: couple an elliptic PDE with a hyperbolic PDE.

Thermo-acoustic tomography (TAT)

Illuminate object with short microwave pulse. EM energy is absorbed preferentially by subregions of interest, e.g., tumors.

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Thermo-acoustic tomography (TAT)

Illuminate object with short microwave pulse. EM energy is absorbed preferentially by subregions of interest, e.g., tumors.

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Acoustic waves then propagate out to $\partial\Omega$, where measured.

EM governed by diffusion eqn. (elliptic), US by acoustic wave eqn. (hyperbolic).

- **Solve hyperbolic inverse problem for US.**

Reconstructs with good spatial resolution an internal measurement: a functional $F(x, u, \nabla u)$ of the solution $u(x)$ of the elliptic problem for the EM field.

- **Then solve the elliptic inverse problem of finding absorption coefficient in Ω from**
  - u **on** $\partial\Omega$
  - F **on** Ω
  - **Other *a priori* information/assumptions**

Photo-acoustic Tomography (PAT):

**Illumination by infrared EM,
detection by ultrasound.**

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Illumination by infrared EM,
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Ultrasound Modulated Optical Tomography (UMOT):

Illumination by ultrasound,
detection by infrared.

Acousto-Electric Tomography (AET/UMEIT):

Illumination by ultrasound,
detection by EIT.

Model of PAT

• **Illuminate with short pulse.**
Scalar EM field in Ω satisfies

$$-\nabla \cdot (\sigma(x)\nabla u(x)) + a(x)u(x) = 0,$$

$u|_{\partial\Omega}$ **(known)**

$a(x) =$ **absorption coeff. (desired)**

$\sigma(x) =$ **diffusion coeff.**

- **Resulting pressure** $p(x,t)$ **satisfies**

$$\big(\partial_t^2 - c(x)^2\Delta\big)p(x,t) = 0 \textbf{ on } \Omega \times [0,\infty)$$

$$p(x,0) = F(x,u(x)), \quad \partial_t p(x,0) = 0$$

$F = \Gamma(x)a(x)u(x)$**, where** $\Gamma(x) =$ **Grüneisen coeff.**

Then: (1) solve hyperbolic IP and find $F(x,u(x))$ **from** $p|_{\partial\Omega\times[0,T_0]}$

(2) solve problem finding $a(x)$ **from** F, $u|_{\partial\Omega}$, Γ

Real principal type (RPT) operators

$P(x, D) \in \Psi^m(\mathbb{R}^n)$, $n \geq 2$**, is of RPT if**

(i) principal symbol $p_m(x, \xi)$ **is** $\mathbb{R}$**-valued**

(ii) $dp_m(x, \xi) \neq (0, 0)$ **at**

$\Sigma_P = \{(x, \xi) \in T^*\mathbb{R}^n,\ \xi \neq 0 : p_m(x, \xi) = 0\}$

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Thus, Σ_P **is foliated by bicharacteristics** $:=$ **integral curves of**

$$H_{p_m} := \sum \frac{\partial p_m}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial p_m}{\partial x_j} \frac{\partial}{\partial \xi_j}$$

(iii) No bichar is trapped over a compact set $K \subset \mathbb{R}^n$**.**

Duistermaat and Hörmander (FIO II):
constructed parametrices for RPT ops,
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and singularities of $Pu = f$ propagate along the bicharacteristics.

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Thm. For all $f \in \mathcal{E}'(X)$, $Pu = f$ is solvable, and if $(x, \xi) \in WF(u) \setminus WF(f)$, then $WF(u)$ contains the bicharacteristic through (x, ξ).

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Did this by conjugating $P(x, D)$ to model,

$$Q_1(x, D) = \frac{\partial}{\partial x_1} + Q_{-\infty}(x, D)$$

whose Green's function $\frac{\partial}{\partial x_1}$ is $H(x_1) \cdot \delta(x') + \dots$

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After a transformation, singularities of D2N data propagate interior details efficiently from any $x_0 \in \Omega$ to any $y_0 \in \partial\Omega$.

Q.: What kind of PDE have this kind of propagation of singularities?

Complex principal type (CPT) operators

$p_m(x,\xi) = p_m^R(x,\xi) + i\, p_m^I(x,\xi)$ **with**

(i) $\nabla_{x,\xi}\, p^R,\ \nabla_{x,\xi}\, p^I$ **linearly indep. at**

$$\Sigma = \{(x,\xi) :\, p_m(x,\xi) = 0\}$$ **(codim 2)**

(ii) Poisson bracket $\{p_m^R, p_m^I\} :=$

$$(\nabla_\xi\, p_m^R)\cdot(\nabla_x\, p_m^I) - (\nabla_x\, p_m^R)\cdot(\nabla_\xi\, p_m^I) \equiv 0 \text{ on } \Sigma$$

(i), (ii) $\Longleftrightarrow$ Σ **is a codimension 2 coisotropic submanifold of** T^*X**.** $\Longrightarrow$

Σ is foliated by 2-dim bicharacteristic leaves, which project to characteristic surfaces in X.

(iii) a nontrapping assumption.

Thm. (D.-H.) $P(x, D)$ **is locally solvable and if** $Pu = f$**, then**

$$WF(u) \setminus WF(f)$$

is a union of bicharacteristic leaves.

Virtual hybrid edge detection: **Exploit CPT operator structure underlying EIT to extract information about interior singularities of the conductivity.**

Singularities propagate efficiently along 2D characteristics to $\partial\Omega$**.**

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Thank you!