The Brascamp–Lieb inequality in modern harmonic analysis and PDE

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Plan

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Part 2: Some recent incarnations of the Brascamp–Lieb inequality in harmonic analysis, and links with PDE.

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$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} (f_j \circ L_j)^{p_j} \leq C \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}, \qquad \text{(BL)}$$

where $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ is a linear surjection and $p_j \in [0,1]$ for each $1 \leq j \leq m$;

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Notice that (BL) is equivalent to

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} g_j \circ L_j \leq C \prod_{j=1}^{m} \|g_j\|_{L^{q_j}(\mathbb{R}^{n_j})},$$

where $q_j = 1/p_j \in [1, \infty]$.

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- **Young's convolution inequality:** If $k \in \mathbb{N}$ and $p_1 + p_2 + p_3 = 2$ then

$$\int_{\mathbb{R}^{2k}} f_1(x)^{p_1} f_2(x-y)^{p_2} f_3(y)^{p_3} \mathrm{d}x\mathrm{d}y \leq C_{\mathbf{p}} \left(\int_{\mathbb{R}^k} f_1 \right)^{p_1} \left(\int_{\mathbb{R}^k} f_2 \right)^{p_2} \left(\int_{\mathbb{R}^k} f_3 \right)^{p_3};$$

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  (Sharp constant $C_{\mathbf{p}}$ obtained by testing on centred gaussians; Beckner/Brascamp–Lieb 1975.)

- **The Loomis–Whitney inequality:** For $1 \leq j \leq n$ let $\pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be given by

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$$|\Omega| \geq \prod_{j=1}^{n} \frac{|\Omega|}{|\pi_j(\Omega)|}.$$

- **The affine-invariant Loomis–Whitney inequality:** For $1 \leq j \leq n$ let $L_j : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a linear map, and $X(L_j) \in \mathbb{R}^n$ be the wedge product of the rows of L_j.

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  Follows from the standard Loomis–Whitney inequality just by changes of variables.

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Theorem (Lieb 1990)

For any Brascamp–Lieb datum $(\mathbf{L}, \mathbf{p})$ *the constant* $BL(\mathbf{L}, \mathbf{p})$ *is exhausted by centred gaussian inputs; i.e.*

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Even with Lieb's formula for $\mathrm{BL}(\mathbf{L}, \mathbf{p})$, it is still far from clear when it is finite...

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Easy necessary condition 1: by scaling (replacing f_j with $f_j(\lambda\cdot)$ for each j and $\lambda > 0$),

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since the integrand

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$$\dim V \leq \sum_{j=1}^{m} p_j \dim L_j V \qquad \textit{for all } V \leq \mathbb{R}^n.$$

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- the *continuity* of the constant $\mathbf{L} \mapsto \mathrm{BL}(\mathbf{L}, \mathbf{p})$ (B–Bez–Cowling–Flock 2016);
- a polynomial time algorithm for determining whether $\mathrm{BL}(\mathbf{L}, \mathbf{p}) < \infty$ and more (Garg–Gurvits–Oliveira–Wigderson 2016).

Part 2: Some recent variants of the Brascamp–Lieb inequality in harmonic analysis, and links with PDE.

The so-called *nonlinear Brascamp–Lieb inequality* replaces the linear surjections $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ with *local submersions* $B_j : U \to \mathbb{R}^{n_j}$, defined on a neighbourhood U of a point $x_0 \in \mathbb{R}^n$.

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Conjecture (Nonlinear Brascamp–Lieb)

If $dB_j(x_0) = L_j$ with $BL(\mathbf{L}, \mathbf{p}) < \infty$, then provided U is taken sufficiently small,

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{p_j} \lesssim \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}.$$

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Theorem (Nonlinear Loomis–Whitney; B–Carbery–Wright 2005)

If $dB_j(x_0) = L_j$ where $L_1, \ldots, L_n : \mathbb{R}^n \to \mathbb{R}^{n-1}$, then provided U is taken sufficiently small,

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Variant 1: A nonlinear Brascamp–Lieb inequality

The so-called *nonlinear Brascamp–Lieb inequality* replaces the linear surjections $L_j : \mathbb{R}^n \to \mathbb{R}^{n_j}$ with *local submersions* $B_j : U \to \mathbb{R}^{n_j}$, defined on a neighbourhood U of a point $x_0 \in \mathbb{R}^n$.

Conjecture (Nonlinear Brascamp–Lieb)

If $dB_j(x_0) = L_j$ *with* $BL(\mathbf{L}, \mathbf{p}) < \infty$, *then provided* U *is taken sufficiently small,*

$$\int_U \prod_{j=1}^m (f_j \circ B_j)^{p_j} \lesssim \prod_{j=1}^m \left(\int_{\mathbb{R}^{n_j}} f_j \right)^{p_j}.$$

This is true for the Loomis–Whitney inequality; i.e.

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Generalises to "block Loomis–Whitney", whereby $\bigoplus_j \ker L_j = \mathbb{R}^n$ (B–Bez 2010).

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For this it is convenient to first recall the multilinear formulation of the Brascamp–Lieb inequality:

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Theorem (B–Bez–Flock–Lee 2015)

Suppose $dB_j(x_0) = L_j$ with $BL(\mathbf{L}, \mathbf{p}) < \infty$ and U is sufficiently small. Then for every $\varepsilon > 0$ there is a constant $C_\varepsilon < \infty$ such that

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Proceed by induction on δ, the scale at which the g_j are "constant"...

Such nonlinear Brascamp–Lieb inequalities may be recast as Radon-like transform estimates of the type

$$\int_{\mathbb{R}^{n_1}\times\cdots\times\mathbb{R}^{n_m}} f_1(y_1)\cdots f_m(y_m)\delta(F(y))dy \lesssim \|f_1\|_{L^{q_1}(\mathbb{R}^{n_1})}\cdots\|f_m\|_{L^{q_m}(\mathbb{R}^{n_m})}$$

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Multilinear Radon-like transforms

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A trilinear example in the plane:

Corollary (B–Bez–Gutiérrez 2013)

If $F:\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^3$ is smooth in a neighbourhood of a point y_0 and satisfies

$$\det(\partial_{y_{11}}F\times\partial_{y_{12}}F \quad \partial_{y_{21}}F\times\partial_{y_{22}}F \quad \partial_{y_{31}}F\times\partial_{y_{32}}F)\neq 0$$

there, then there is a neighbourhood $V\ni y_0$ such that

$$\int_V f_1(y_1)f_2(y_2)f_3(y_3)\delta(F(y))dy \lesssim \|f_1\|_{L^2(\mathbb{R}^2)}\|f_2\|_{L^2(\mathbb{R}^2)}\|f_3\|_{L^2(\mathbb{R}^2)}.$$

Proof. Parametrise the action of the distribution $\delta\circ F$ by $x\in\mathbb{R}^3$, reducing it to the nonlinear Loomis–Whitney inequality in $\mathbb{R}^3$...

Example from obstacle scattering (Born series). The error in approximating a potential $q : \mathbb{R}^2 \to \mathbb{R}$ by its Born approximation q_B is comprised of a series of multilinear operators. The main term involves, for example, the bilinear operator $S(q)$ defined by

$$\widehat{S(q)}(x) = \frac{i\pi}{|x|} \int_{\Gamma(x)} \widehat{q}(x-y)\widehat{q}(y)\,d\sigma_x(y),$$

where $\Gamma(x)$ is the circle centred at $x/2$ of radius $|x|/2$ in $\mathbb{R}^2$, and $d\sigma_x$ is arc-length measure on $\Gamma(x)$.

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By duality, L^2 Sobolev bounds on $S(q)$ may be recast as L^2 bounds on an associated trilinear form, which may be expressed in terms of

$$\Lambda(f_1,f_2,f_3):=\int_{(\mathbb{R}^2)^3}f_1(y_1)f_2(y_2)f_3(y_3)\delta(F(y))dy,$$

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$$F(y)=\left(y_1-y_2-y_3,\left|y_2-\frac{y_1}{2}\right|-\left|\frac{y_1}{2}\right|\right).$$

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$$F(y) = \left(y_1 - y_2 - y_3, \left|y_2 - \frac{y_1}{2}\right| - \left|\frac{y_1}{2}\right|\right).$$

Another example: well-posedness of the Zakharov system (plasma physics), Bejenaru–Herr–Holmer–Tataru 2009–2011.

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If S is the *paraboloid* then this becomes $\|u\|_{L^{2(n+1)/(n-1)}_{x,t}} \lesssim \|\widehat{g}\|_2 = \|g\|_2$ – the classical *Strichartz estimate* for the Schrödinger equation (Strichartz 1978).

Now suppose $\Sigma_1, \ldots, \Sigma_m$ parametrise $n_1, \ldots, n_m$ dimensional submanifolds $S_1, \ldots, S_m$ of $\mathbb{R}^n$, and $E_1, \ldots, E_m$ are their associated Fourier extension operators; i.e. that

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on setting $f_j = |\widehat{g}_j|^2$, maybe written as

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We conjecture that the linearity requirement on the submanifolds S_j can be relaxed here, leading to certain "Fourier-analytic Brascamp–Lieb inequalities"...

Theorem (B–Carbery–Tao 2006; B–Bez–Flock–Lee 2015)

Suppose $BL(\mathbf{L}, \mathbf{p}) < \infty$, *where* $L_j := (\mathrm{d}\Sigma_j(0))^*$. *Then for each* $\varepsilon > 0$

$$\int_{B(0;R)} \prod_{j=1}^m |E_j g_j|^{2p_j} \lesssim_\varepsilon R^\varepsilon \prod_{j=1}^m \|g_j\|_2^{2p_j}.$$

Let us restrict attention to n codimension-1 submanifolds $S_1, \ldots, S_n$ of $\mathbb{R}^n$.

Definition (Transversality)

We say that $S_1, \ldots, S_n$ are *transversal* if there exists $\nu > 0$ such that whenever $v_1, \ldots, v_n$ are unit normal vectors to $S_1, \ldots, S_n$ respectively, then $|\det(v_1\ v_2\ \cdots\ v_n)| \geq \nu$.

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Corollary (B–Carbery–Tao 2006)

If $E_1, \ldots, E_n$ are extension operators associated with transversal compact submanifolds $S_1, \ldots, S_n$ of $\mathbb{R}^n$, then

$$\|E_1 g_1 \cdots E_n g_n\|_{L^{2/(n-1)}(B(0;R))} \lesssim_\varepsilon R^\varepsilon \|g_1\|_2 \cdots \|g_n\|_2.$$

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In the context of transversal patches of *paraboloid*, this as a *Strichartz estimate*...

Corollary

Let $u_1, \ldots, u_n : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}$ be solutions of $i\partial_t u = \Delta u$ with initial data $g_1, \ldots, g_n$ respectively.

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We say that $S_1, \ldots, S_n$ are *transversal* if there exists $\nu > 0$ such that whenever $v_1, \ldots, v_n$ are unit normal vectors to $S_1, \ldots, S_n$ respectively, then $|\det(v_1\ v_2\ \cdots\ v_n)| \geq \nu$.

Corollary (B–Carbery–Tao 2006)

If $E_1, \ldots, E_n$ are extension operators associated with transversal compact submanifolds $S_1, \ldots, S_n$ of $\mathbb{R}^n$, then

$$\|E_1 g_1 \cdots E_n g_n\|_{L^{2/(n-1)}(B(0;R))} \lesssim_\varepsilon R^\varepsilon \|g_1\|_2 \cdots \|g_n\|_2.$$

In the context of transversal patches of *paraboloid*, this as a *Strichartz estimate*...

Corollary

Let $u_1, \ldots, u_n : \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{C}$ be solutions of $i\partial_t u = \Delta u$ with initial data $g_1, \ldots, g_n$ respectively. If supp$(\widehat{g}_1), \ldots,$ supp$(\widehat{g}_n) \subseteq \mathbb{R}^{n-1}$ meet no affine hyperplane, then

$$\|u_1 \cdots u_n\|_{L^{2/(n-1)}_{t,x}(|x|,|t| \leq R)} \lesssim_\varepsilon R^\varepsilon \|g_1\|_2 \cdots \|g_n\|_2.$$

Multilinear Strichartz estimates

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(The corresponding linear inequality $\|u\|_{L^{2n/(n-1)}} \lesssim \|g\|_2$ is false.)

Applications

- Progress on *Stein's Fourier restriction conjecture*: deeper $L^p \to L^q$ estimates for E (Bourgain–Guth 2011).

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$$(Eg)^n = \sum_{\alpha_1,\ldots,\alpha_n} E(g\chi_{U_{\alpha_1}}) \cdots E(g\chi_{U_{\alpha_n}}),$$

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Thank you for listening!