Introduction to nonlinear PDEs on graphs

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Summary

Background: from potential wells to nonlinear graphs

In many problems (BECs, photonics, optics), wave dynamics is modeled with the nonlinear Schrödinger (Gross–Pitaevskii) equation

$$iu_t = -u_{xx} + V(x)u \pm |u|^{2p}u,$$

where $p > 0$ is the nonlinearity power and $V(x) : \mathbb{R} \mapsto \mathbb{R}$ is a confining potential. The upper sign is defocusing (repelling) and the lower sign is focusing (attractive).

- Double-well potentials such as

  $$V(x; s) = \frac{1}{2}\left(V_0(x-s) + V_0(x+s)\right), \quad s \geq 0,$$

  where V_0 is a single-well potential such as $V_0(x) = -\mathrm{sech}^2(x)$.

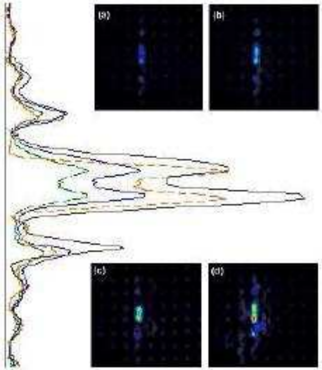
- Periodic potentials (lattices)

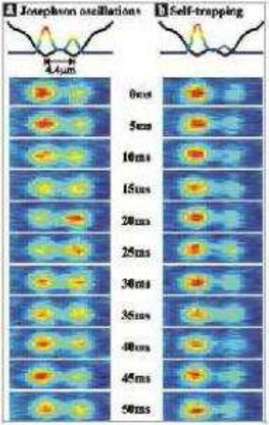
  $$V(x+L) = V(x), \quad L > 0,$$

  such as $V(x) = \sin^2(x)$.

Experiments on symmetry-breaking bifurcations

- M.Obertaler’s group in Heidelberg, Germany (BECs)
- Z. Chen’s group at San Francisco, USA (photonics)



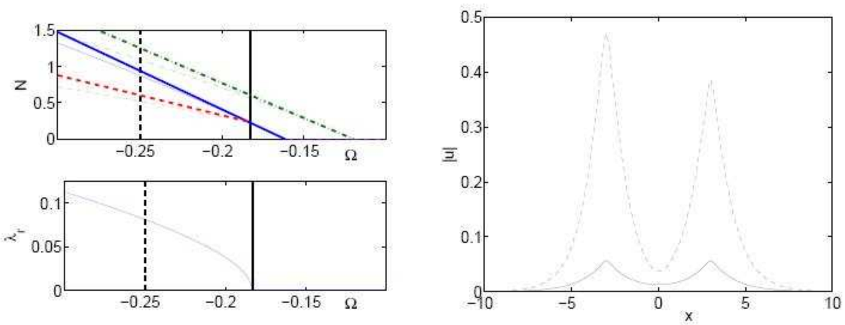


Double-well potentials

Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a double-well potential

$$\omega\phi = -\phi_{xx} + V(x;s)\phi - |\phi|^2\phi.$$

Let V_0 support exactly one negative eigenvalue of $L_0 = -\partial_x^2 + V_0(x)$ and s be large. The operator $L = -\partial_x^2 + V(x;s)$ has two negative eigenvalues with symmetric and anti-symmetric eigenfunctions. In the focusing case, the bifurcation diagram looks as



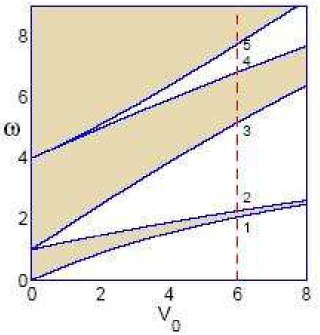
A. Sacchetti; E. Kirr; P.G. Kevrekidis; J. Marzuola; M. Weinstein;

Periodic potentials (lattices)

Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a periodic potential

$$\omega\phi = -\phi_{xx} + V(x)\phi \pm |\phi|^2\phi$$

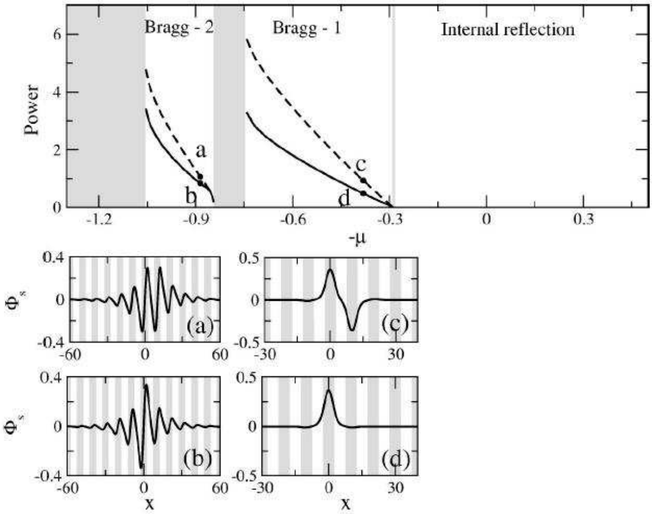
Spectrum of $L = -\partial_x^2 + V(x)$ for $V(x) = V_0 \sin^2(x)$ and $N = 1$:



J. Yang; M. Weinstein; T. Dohnal; G. Schneider; V. Konotop; G. Alfimov;

Gap solitons

For $V(x) = V_0 \sin^2(x)$ and the defocusing case, the bifurcation diagram is



Asymptotic reductions in the periodic potentials

The Gross–Pitaevskii equation with a periodic potential can be homogeonized and reduced to one of the three models with spatially independent coefficients.

- Coupled-mode (Dirac) equations for *small-amplitude* potentials

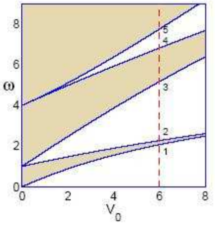
$$\left\{\begin{array}{l} i(a_t + a_x) + b = (|a|^2 + 2|b|^2)a \\ i(b_t - b_x) + a = (2|a|^2 + |b|^2)b \end{array}\right.$$

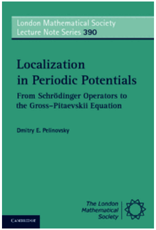
- Envelope (NLS) equations *near band edges*

$$ia_t + a_{xx} + |a|^2 a = 0$$

- Lattice (DNLS) equations for *large-amplitude* potentials

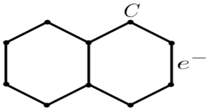
$$i\dot{a}_n + \alpha(a_{n+1} + a_{n-1}) + |a_n|^2 a_n = 0.$$





Introduction

Graph models for the dynamics of constrained quantum particles were first suggested by Pauling and then used by Ruedenberg and Scherr in 1953 to study the spectrum of aromatic hydrocarbons.

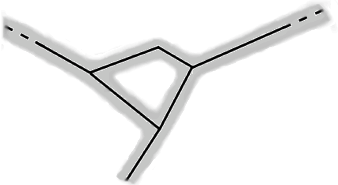


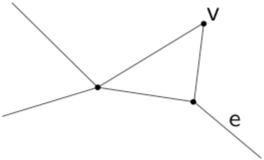
Nowadays graph models are widely used in the modeling of quantum dynamics of thin graph-like structures (quantum wires, nanotechnology, large molecules, periodic arrays in solids, photonic crystals...).

- G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs* (AMS, Providence, 2013).
- P. Exner and H. Kovarík, *Quantum Waveguides*, (Springer, Heidelberg, 2015).

Metric Graphs

Graphs are one-dimensional approximations for constrained dynamics in which transverse dimensions are small with respect to longitudinal ones.



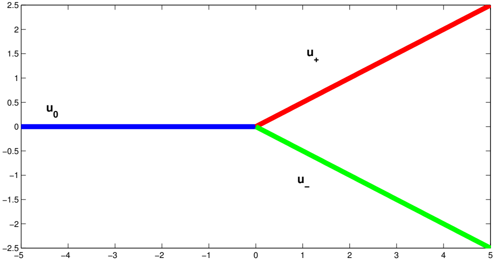


A metric graph Γ is given by a set of edges and vertices, with a metric structure on each edge. Proper boundary conditions are needed on the vertices to ensure that certain differential operators defined on graphs are self-adjoint.

Kirchhoff boundary conditions:

- Functions in each edge have the same value at each vertex.
- Sum of fluxes (signed derivatives of functions) is zero at each vertex.

Example: Y junction graph



The Laplacian operator on the graph Γ is defined by

$$\Delta\Psi = \left[\begin{array}{ll} u_0''(x), & x \in (-\infty, 0), \\ u_\pm''(x), & x \in (0, \infty) \end{array} \right],$$

acting on functions in the form

$$\Psi = \left[\begin{array}{ll} u_0(x), & x \in (-\infty, 0) \\ u_\pm(x), & x \in (0, \infty) \end{array} \right],$$

in the domain

$$\mathcal{D}(\Gamma) = \left\{ \begin{array}{l} (u_0, u_+, u_-) \in H^2(\mathbb{R}^-) \times H^2(\mathbb{R}^+) \times H^2(\mathbb{R}^+) : \\ u_0(0) = u_+(0) = u_-(0), \quad u_0'(0) = u_+'(0) + u_-'(0) \end{array} \right\}.$$

Laplacian on the Y junction graph

Lemma

The operator $\Delta : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint.

The Kirchhoff boundary conditions are symmetric:

$$\begin{aligned} \langle \Phi, \Delta\Psi \rangle - \langle \Delta\Phi, \Psi \rangle &= \left[\bar{v}_0' u_0 - \bar{v}_0 u_0' \right]_{x=0} - \left[\bar{v}_+' u_+ - \bar{v}_+ u_+' \right]_{x=0} - \left[\bar{v}_-' u_- - \bar{v}_- u_-' \right]_{x=0} \\ &= 0, \end{aligned}$$

where $\Phi = (v_0, v_+, v_-)$ and $\Psi = (u_0, u_+, u_-)$ satisfy the Kirchhoff conditions:

$$\left\{ \begin{array}{l} u_0(0) = u_+(0) = u_-(0), \\ u_0'(0) = u_+'(0) + u_-'(0). \end{array} \right.$$

Moreover, Δ is self-adjoint under generalized Kirchhoff boundary conditions

$$\left\{ \begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0) = \alpha_- u_-(0) \\ \alpha_0^{-1} u_0'(0) = \alpha_+^{-1} u_+'(0) + \alpha_-^{-1} u_-'(0), \end{array} \right.$$

where $\alpha_0, \alpha_+, \alpha_-$ are arbitrary nonzero parameters.

NLS on the Y junction graph

So far, $\alpha_0, \alpha_+, \alpha_-$ are arbitrary. Let us connect these parameters with the nonlinear coefficients of a nonlinear Schrödinger equation defined on the graph Γ:

$$
\begin{aligned}
i\partial_t u_0 + \partial_x^2 u_0 + \alpha_0^2 |u_0|^2 u_0 &= 0, \quad x < 0, \\
i\partial_t u_\pm + \partial_x^2 u_\pm + \alpha_\pm^2 |u_\pm|^2 u_\pm &= 0, \quad x > 0,
\end{aligned}
$$

subject to the generalized Kirchhoff boundary conditions at $x = 0$.

The charge (power) functional

$$
Q = \int_{-\infty}^{0} |u_0|^2 dx + \int_0^{+\infty} |u_+|^2 dx + \int_0^{+\infty} |u_-|^2 dx
$$

is constant in time t (related to the gauge symmetry).

The Hamiltonian (energy) functional

$$
E = \int_{-\infty}^{0} \left(|\partial_x u_0|^2 - \frac{\alpha_0^2}{2} |u_0|^4 \right) dx + \text{similar terms for } u_\pm,
$$

is constant in time t (related to the time translation symmetry).

NLS on the Y junction graph

The momentum functional

$$P = i \int_{-\infty}^{0} \left(\bar{u}_0 \partial_x u_0 - u_0 \partial_x \bar{u}_0 \right) dx + \text{similar terms for } u_{\pm},$$

is no longer constant in time t because the spatial translation is broken. As a result, the solitary wave scatters at the Y junction point with a nonzero reflection coefficient.

NLS on the Y junction graph

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Let us consider the constraint

$$\frac{1}{\alpha_0^2} = \frac{1}{\alpha_+^2} + \frac{1}{\alpha_-^2}$$

on the generalized Kirchhoff boundary conditions:

$$\left\{ \begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0) = \alpha_- u_-(0) \\ \alpha_0^{-1} u_0'(0) = \alpha_+^{-1} u_+'(0) + \alpha_-^{-1} u_-'(0). \end{array} \right.$$

Then, the momentum is decreasing function of time:

$$\frac{dP}{dt} = -\frac{2\alpha_0^2}{\alpha_+^2 \alpha_-^2} \left| \alpha_+ \partial_x u_+(0) - \alpha_- \partial_x u_-(0) \right|^2 \leq 0.$$

Reflectionless scattering of solitary waves

If the initial data satisfy the reduction

$$\alpha_+ u_+(x) = \alpha_- u_-(x), \quad x \in \mathbb{R}^+,$$

then, the reduction is invariant with respect to the time evolution of the NLS equation. In this case, $\frac{dP}{dt} = 0$ and the spatial translation invariance is restored.

The solitary wave is transmitted through the Y junction without any reflection. In this case, the Kirchhoff boundary conditions imply

$$\left\{ \begin{array}{l} \alpha_0 u_0(0) = \alpha_+ u_+(0), \\ \alpha_0 \partial_x u_0(0) = \alpha_+ \partial_x u_+(0), \end{array} \right.$$

and the function

$$U(x,t) = \left\{ \begin{array}{ll} \alpha_0 u_0(x,t), & x < 0, \\ \alpha_\pm u_\pm(x,t), & x > 0, \end{array} \right.$$

satisfies the integrable cubic NLS equation on the infinite line

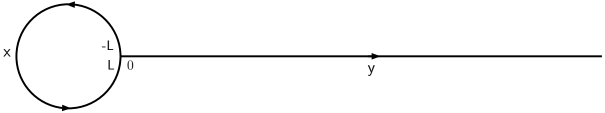
$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad x \in \mathbb{R},$$

where the vertex $x = 0$ does not appear as an obstacle in the time evolution.
D. Matrasulov; K. Sabirov; H. Uecker; D. Dytukh; J.G. Caputo;

NLS equation on star graphs

- Gnutzmann-Smilansky-Derevyanko, Phys. Rev. A **83** (2011), 033831: a complex set of resonances after inserting a single nonlinear edge in a linear quantum graph; rigorous analysis by L.Tentarelli, arXiv:1503.00455.
- Series of papers on star graphs by Adami-Cacciapuoti-Finco-Noja: Scattering of solitons; Standing waves and stability (2011-2014).
- Recent works by Adami-Serra-Tilli on nonexistence of ground states in networks with closed cycles (2014-2016). Variational principle and concentration compactness principle are used.
- Recent papers by Sabirov-Uecker on soliton propagation in fat graphs and graphs with other boundary conditions (2014-2016).
- Classification of standing waves and computations of the bifurcation diagram on tadpole graphs by C.Cacciapuoti, D.Finco, D.Noja, Phys. Rev. E **91** (2015), 013206; rigorous results on existence, bifurcations, and stability by D.Noja, D.P., and G.Shaikhova, Nonlinearity **28** (2015), 2343.
- Further exploration of bifurcation methods for stationary states on bounded and unbounded graphs: J. Marzuola and D.P., Applied Math. Research Express **2016**, 98–145; D.P. and G. Schneider, arXiv:1603.05463 (2016).

NLS equation on a tadpole graph



The ring is placed on the interval $[-L, L]$ and the semi-infinite line is $[0, \infty)$. The Laplacian operator on Γ acts on the functions in the form

$$\Psi = \left[\begin{array}{ll} u(x), & x \in (-L, L) \\ v(y), & y \in (0, \infty) \end{array} \right],$$

defined in the domain

$$\mathcal{D}(\Gamma) = \left\{ \begin{array}{l} (u, v) \in H^2(-L, L) \times H^2(0, \infty) : \\ v(0) = u(L) = u(-L), \quad v'(0) = u'(L) - u'(-L) \end{array} \right\}.$$

The generalized NLS equation is taken in the form:

$$i\partial_t \Psi + \partial_x^2 \Psi + (p+1)|\Psi|^{2p}\Psi = 0, \quad x \in \Gamma,$$

subject to the Kirchhoff boundary conditions at the vertex point.

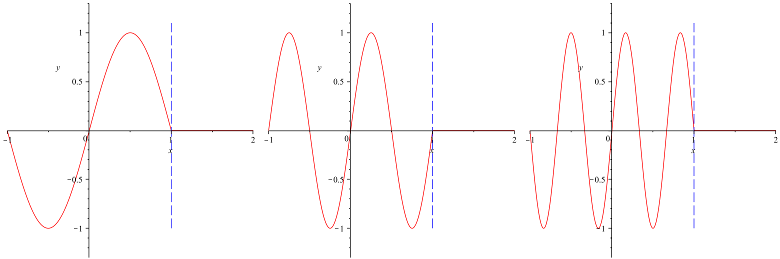
Existence of standing waves on the tadpole graph

$$\begin{cases} -u''(x) - (p+1)|u|^{2p}u = \omega u, & x \in (-L, L)\ , \\ -v''(y) - (p+1)|v|^{2p}v = \omega v, & y \in (0, \infty)\ , \\ u(L) = u(-L) = v(0)\ , \\ u'(L) - u'(-L) = v'(0)\ . \end{cases}$$

Linear spectrum:

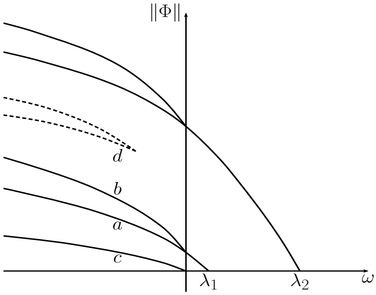
- Essential spectrum: $\sigma_{ess}(-\Delta) = [0, \infty)$ with resonance at 0.
- Embedded eigenvalues: $\left\{\lambda_n = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{N}\right\} \subset \sigma_{ess}(-\Delta)$
  The corresponding (normalized) eigenfunctions are:

$$\Upsilon_n = \frac{1}{\sqrt{L}}\left(\sin\left(\frac{n\pi x}{L}\right), 0\right) \qquad n = 1, 2, 3, \ldots$$



Bifurcation diagram

The following bifurcation diagram has been computed for $p = 1$ (Cacciapuoti *et al.*):



The diagram describes the families of stationary states and their possible relation with the spectrum of $-\Delta$.

The model, although simple, exhibits a surprisingly rich behavior

- branches of standing waves bifurcating from the embedded eigenvalues
- pitchfork bifurcation at threshold $\omega = 0$: edge solitons
- branches of non linearly related standing waves (dashed lines)

Standing waves bifurcating from the zero resonance

Let $\omega = -\epsilon^2$ and consider small values of ϵ. For the solution on the tail of the tadpole, we can scale

$$v(y) = \epsilon^{\frac{1}{p}} \phi(z), \quad z = \epsilon y,$$

where ϕ is a decaying solution of the second-order equation

$$-\phi''(z) + \phi - (p+1)|\phi|^{2p}\phi = 0, \quad z > 0.$$

Let $\phi_0(z) = \operatorname{sech}^{\frac{1}{p}}(pz)$ be the unique symmetric solitary wave. Then, $\phi(z) = \phi_0(z+a)$ for unknown parameter a.

Bifurcation problem:

$$\left\{ \begin{array}{l} -u''(x) + \epsilon^2 u - (p+1)|u|^{2p}u = 0, \quad x \in (-L, L), \\ u(L) = u(-L) = \epsilon^{\frac{1}{p}} \phi_0(a), \\ u'(L) - u'(-L) = \epsilon^{1+\frac{1}{p}} \phi_0'(a). \end{array} \right. \tag{1}$$

- Primary branch bifurcating from zero solution $(u, v) = (0, 0)$.
- Higher branches bifurcating from the solutions $(u, v) = (u_{n,\omega}, 0)$.

Primary branch

Using the scaling transformation

$$u(x) = \epsilon^{\frac{1}{p}} \psi(z), \quad z = \epsilon x,$$

we can write the bifurcation problem as

$$\begin{cases} -\psi''(z) + \psi - (p+1)|\psi|^{2p}\psi = 0, \quad z \in (-\epsilon L, \epsilon L), \\ \psi(\epsilon L) = \psi(-\epsilon L) = \phi_0(a), \\ \psi'(\epsilon L) - \psi'(-\epsilon L) = \phi_0'(a). \end{cases}$$

Primary branch

Using the scaling transformation

$$u(x) = \epsilon^{\frac{1}{p}} \psi(z), \quad z = \epsilon x,$$

we can write the bifurcation problem as

$$\left\{ \begin{array}{l} -\psi''(z) + \psi - (p+1)|\psi|^{2p}\psi = 0, \quad z \in (-\epsilon L, \epsilon L), \\ \psi(\epsilon L) = \psi(-\epsilon L) = \phi_0(a), \\ \psi'(\epsilon L) - \psi'(-\epsilon L) = \phi_0'(a). \end{array} \right.$$

Construction of positive solution:

- An even solution is defined near the origin: $\psi(0) = \psi_0$, $\psi'(0) = 0$.
- The continuity boundary condition

  $$\phi_0(a) = \psi(\epsilon L) = \psi_0 + \frac{1}{2}\psi''(0)\epsilon^2 L^2 + \mathcal{O}(\epsilon^4),$$

  hence $\psi_0 = \phi_0(a) + \mathcal{O}(\epsilon^2)$ is uniquely defined for every $a \in \mathbb{R}$.
- The flux boundary condition

  $$\phi_0'(a) = 2\psi'(\epsilon L) = 2\psi''(0)\epsilon L + \mathcal{O}(\epsilon^3).$$

  where $\phi_0'(0) = 0$ and $\phi_0''(0) \neq 0$. Hence, $a = 2\epsilon L + \mathcal{O}(\epsilon^3)$ is unique.
- The small solution is unique and positive: $u = \epsilon^{\frac{1}{p}}(1 + \mathcal{O}_{C^\infty(-L,L)}(\epsilon^2))$.

Numerical solutions for $p = 1$

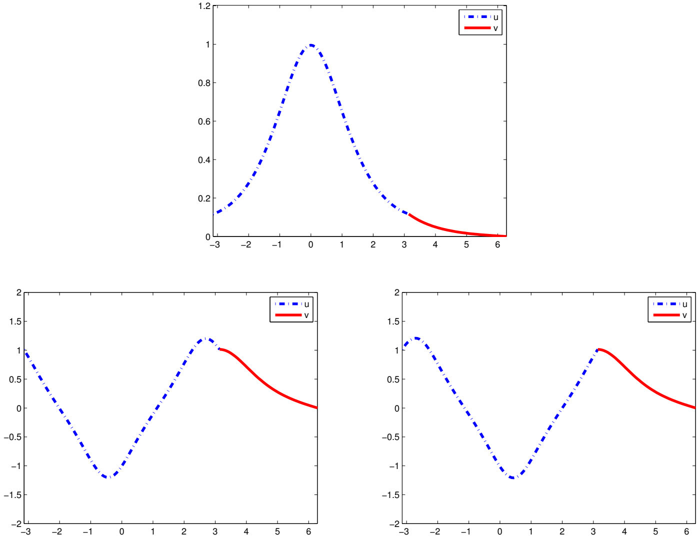


Figure : Standing wave solutions (u, v) versus x for $\omega = -1$ along the primary branch.

Orbital stability

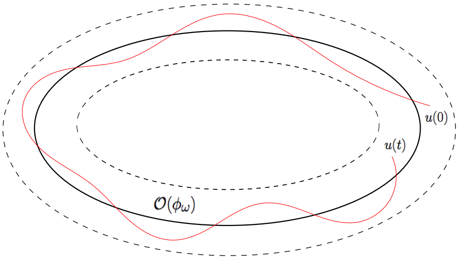
Recall that NLS has $U(1)$, or phase, symmetry. No stability of equilibrium *points* Φ can hold, but stability of equilibrium *orbits* $\left\{e^{i\theta}\Phi\right\}_{\theta\in\mathbb{R}}$ may be attained sometimes.

Definition

We say that $e^{i\omega t}\Phi$ is *orbitally stable* in a Banach space V if $\forall\epsilon > 0\ \exists\delta > 0$ such that $\forall u_0 \in V$ with $\|u_0 - \Phi\|_V < \delta$, NLS has a global solution $u(t) \in V$ with initial datum u_0 satisfying

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\Phi\|_V < \epsilon,$$

for every $t \in \mathbb{R}$.



Spectral stability as a way to orbital stability

Linearization of NLS with $\Psi = e^{i\omega t}\left(\Phi(x) + U(x,t) + iW(x,t)\right)$ and the separation of variables $U(x,t) = \tilde{U}(x)e^{\lambda t}$ results in the spectral problem

$$L_+ \tilde{U} = -\lambda \tilde{W}, \quad L_- \tilde{W} = \lambda \tilde{U},$$

associated with the Schrödinger operators L_+ and L_-. For the tadpole graph, this yields two self-adjoint problems

$$L_- : \quad \left\{ \begin{array}{ll} -U''(x) - \omega U - (p+1)|u|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (p+1)|v|^{2p} V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0), & \end{array} \right.$$

and

$$L_+ : \quad \left\{ \begin{array}{ll} -U''(x) - \omega U - (2p+1)(p+1)|u|^{2p} U = \lambda U, & x \in (-L, L), \\ -V''(y) - \omega V - (2p+1)(p+1)|v|^{2p} V = \lambda V, & y \in (0, \infty), \\ U(L) = U(-L) = V(0), & \\ U'(L) - U'(-L) = V'(0). & \end{array} \right.$$

Definition

We say that the standing wave is spectrally stable if no eigenvectors $\tilde{U}, \tilde{W} \in \mathcal{D}(-\Delta)$ exist for an eigenvalue with $\mathrm{Re}(\lambda) > 0$.

Criteria for spectral stability

Denote the number of negative eigenvalues of $L_\pm$ by $n(L_\pm)$ and assume that L_+ is invertible and $\mathrm{Ker}(L_-)$ is one-dimensional.

Consider a constrained space associated with the $U(1)$, or phase, symmetry:

$$L_c^2 := \left\{ U \in L^2 : \quad \langle U, \Phi \rangle_{L^2} = 0 \right\}.$$

Constraint can reduce the number of negative eigenvalues of L_+.

The following criteria summarize the results from Shatah–Strauss, Weinstein, Grillakis, Jones, Kapitula–Kevrekidis–Stanstede, Pelinovsky, etc.

- If $n(L_+) = 1$ and $n(L_-) = 0$, then
  - Φ is spectrally and orbitally stable if $n(L_+|_{L_c^2}) = 0$
  - Φ is spectrally and orbitally unstable if $n(L_+|_{L_c^2}) = 1$.
- If $n(L_+|_{L_c^2}) - n(L_-)$ is nonzero, then Φ is unstable.
- If $n(L_+|_{L_c^2}) + n(L_-)$ is odd, then Φ is unstable.

Stability of the primary branch for $\omega = -\epsilon^2$

Recall that

$$\Phi = \epsilon^{\frac{1}{p}}(\psi(z), \phi(z)), \quad z = \epsilon x,$$

where $\psi(z) = 1 + \mathcal{O}(z^2)$ and $\phi(z) = \phi_0(z + a)$.

- $n(L_-) = 0$ because $L_-\Phi = 0$ and Φ is strictly positive.
- $n(L_+) = 1$ according to the asymptotic analysis below.

The spectral problem for L_+ with $\lambda = \epsilon^2\Lambda$ is

$$\left\{ \begin{array}{l} -U''(z) + U(z) - (2p+1)(p+1)|\psi(z)|^{2p}U(z) = \Lambda U(z), \quad z \in (-\epsilon L, \epsilon L), \\ -V''(z) + V(z) - (2p+1)(p+1)|\phi(z)|^{2p}V(z) = \Lambda V(z), \quad z \in (0,\infty), \\ U(\epsilon L) = U(-\epsilon L) = V(0), \\ U'(\epsilon L) - U'(-\epsilon L) = V'(0), \end{array} \right.$$

The leading-order problem is related to the Schrödinger equation on half-line

$$\left\{ \begin{array}{l} -V''(z) + V(z) - (2p+1)(p+1)\mathrm{sech}^2(pz)V(z) = \Lambda V(z), \quad z \in (0,\infty), \\ V'(0) = 0, \end{array} \right.$$

which has only one negative eigenvalue $\Lambda_0 < 0$.

Stability of the primary branch

- $n(L_+|_{L^2_c}) = 0$ if the slope condition is satisfied

$$\frac{d}{d\omega}\|\Phi\|^2 < 0$$

This can be checked directly from asymptotic solutions:

$$\|u\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\|\psi(\epsilon\cdot)\|^2_{L^2(-L,L)} = \epsilon^{\frac{2}{p}}\left(2L + \mathcal{O}(\epsilon^2)\right)$$

and

$$\|v\|^2_{L^2(0,\infty)} = \epsilon^{\frac{2}{p}-1}\|\phi_0\|^2_{L^2(a,\infty)} = \epsilon^{\frac{2}{p}-1}\left(\|\phi_0\|^2_{L^2(0,\infty)} + \mathcal{O}(\epsilon)\right).$$

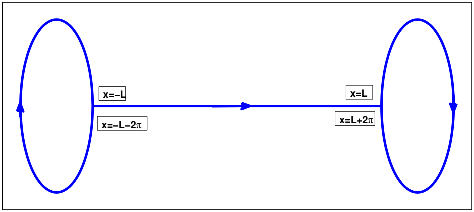
Theorem

For $\omega = -\epsilon^2$ with $\epsilon > 0$ sufficiently small, the primary branch is orbitally stable for every $p \in (0,2)$ and orbitally unstable for every $p \in (2,\infty)$.

Conjecture

For $p = 1$, Φ is expected to be a constrained minimizer of energy for all $\omega < 0$.

Dumbbell Graph



Let the line segment be placed on $I_0 := [-L, L]$ and the end rings are placed on $I_- := [-L-2\pi, -L]$ and $I_+ := [L, L+2\pi]$. Then, Δ acts piecewise on

$$\Psi = \left[\begin{array}{c} u_-(x),\ x \in I_-, \\ u_0(x),\ x \in I_0, \\ u_+(x),\ x \in I_+, \end{array} \right],$$

subject to the Kirchhoff boundary conditions at the two junctions

$$\left\{ \begin{array}{l} u_+(L+2\pi) = u_+(L) = u_0(L), \\ u_+'(L) - u_+'(L+2\pi) = u_0'(L) \end{array} \right.$$

J. Marzuola and D.P., Applied Math. Research Express **2016**, 98–145.

Existence of standing waves on the dumbbell graph

Ground state is the standing wave of smallest energy E at a fixed value of Q,

$$E_0 = \inf\{E(\Psi) : \quad \Psi \in \mathcal{E}(\Delta), \quad Q(\Psi) = Q_0\}.$$

Euler–Lagrange equation is

$$-\Delta\Phi - 2|\Phi|^2\Phi = \Lambda\Phi \qquad \Lambda \in \mathbb{R}\,,\; \Phi \in \mathcal{D}(\Delta).$$

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Linear spectrum of isolated eigenvalues:

- Double eigenvalues $\{n^2\}_{n\in\mathbb{N}}$ with eigenfunctions supported in each ring.
- Simple eigenvalues $\{\omega_n^2\}_{n\in\mathbb{N}}$ with eigenfunctions symmetric on graph.
- Simple eigenvalues $\{\Omega_n^2\}_{n\in\mathbb{N}}$ with eigenfunctions anti-symmetric on graph.

For both $L < \pi$ and $L \geq \pi$, we have the following ordering of eigenvalues:

$$0 < \Omega_1 < \omega_1 < 1 < \Omega_2 < \dots$$

This ordering gives ordering of bifurcations from the constant solution

$$\Phi(x) = p, \quad \Lambda = -2p^2, \quad Q_0 = 2(L + 2\pi)p^2,$$

which exists for any $Q_0 > 0$.

Bifurcation diagram

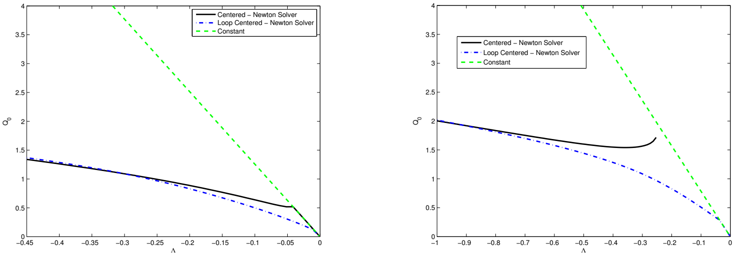


Figure : The bifurcation diagram for $L = 2\pi$ (left) and $L = \pi/2$ (right).

Theorem

There exist Q_0^* and Q_0^{**} ordered as $0 < Q_0^* < Q_0^{**} < \infty$ such that the ground state for $Q_0 \in (0, Q_0^*)$ is given by the constant solution, which undertakes

- *the symmetry breaking bifurcation at Q_0^* due to Ω_1,*
- *the symmetry preserving bifurcation at Q_0^{**} due to ω_1.*

Numerical approximations of the ground states

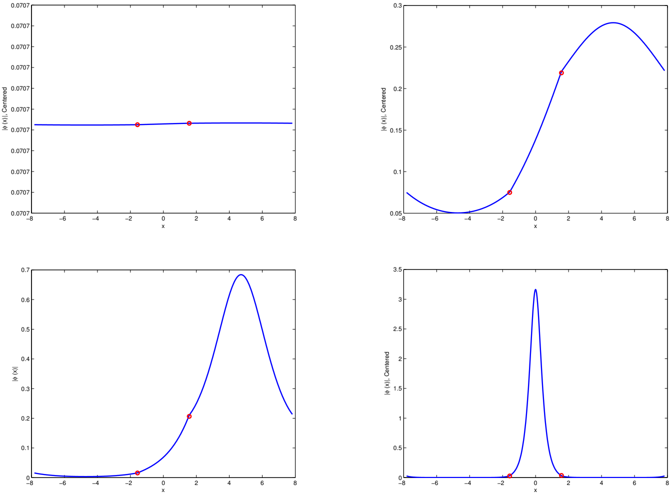


Figure : Ground states for $L = \pi/2$ and $\Lambda = -0.01$ (top left), $\Lambda = -0.1$ (top right), $\Lambda = -1.5$ (bottom left), and $\Lambda = -10.0$ (bottom right). The values at $\pm L$ are marked with a red circle.

Standing waves in the limit of large energy

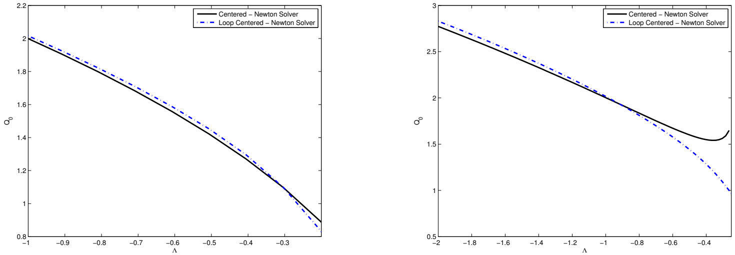


Figure : The bifurcation diagram for $L = 2\pi$ (left) and $L = \pi/2$ (right).

Standing waves in the limit of large energy

Theorem

In the limit of large negative Λ*, there exist two standing wave solutions. One solution is a positive asymmetric wave localized in the ring:*

$$\Phi(x) = |\Lambda|^{1/2}\text{sech}(|\Lambda|^{1/2}(x - L - \pi)) + \tilde{\Phi}(x), \quad Q_0 = 2|\Lambda|^{1/2} + \tilde{Q}_0,$$

and the other solution is a positive symmetric wave localized in the central line segment:

$$\Phi(x) = |\Lambda|^{1/2}\text{sech}(|\Lambda|^{1/2}x) + \tilde{\Phi}(x), \quad Q_0 = 2|\Lambda|^{1/2} + \tilde{Q}_0,$$

where $\|\tilde{\Phi}\|_{H^2(I_- \cup I_0 \cup I_+)} \to 0$ *and* $|\tilde{Q}_0| \to 0$ *as* $\Lambda \to -\infty$ *in both cases. The positive symmetric wave is a ground state for* Q_0 *sufficiently large.*

Remarks:

- The energy difference between the two solitary waves is exponentially small.
- The asymmetric wave is a local constrained minimizer. It is also orbitally stable.

Numerical approximations in the limit of large energies

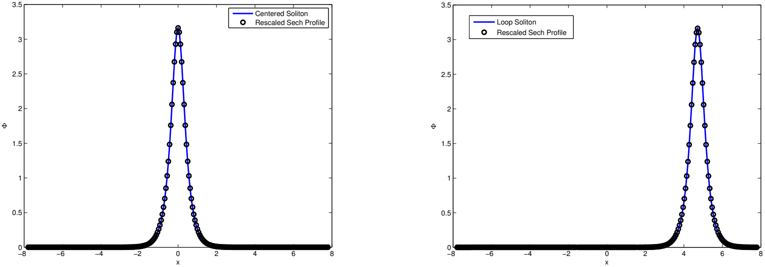
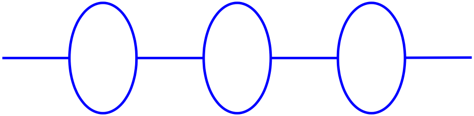


Figure : Comparison of the standing waves (solid line) localized in the central link (left) and in one of the rings (right) to the rescaled solitary wave profile (dots) for $L = \pi/2$ and $\Lambda = -10.0$.

Summary on the ground state in the dumbbell graph:

- $\Lambda \to -0$ - constant state;
- $\Lambda < 0$ decreases - asymmetric state concentrated in a ring;
- $\Lambda \to -\infty$ - symmetric state concentrated in a central segment.

Periodic Graph



Let the periodic graph Γ consist of the circles of the normalized length 2π and the horizontal links of the length L. Writing the periodic graph as

$$\Gamma = \oplus_{n \in \mathbb{Z}} \Gamma_n, \qquad \text{with} \qquad \Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-},$$

we parameterize $\Gamma_{n,0} := [nP, nP + L]$ and $\Gamma_{n,\pm} := [nP + L, (n+1)P]$, where $P = L + \pi$ is the graph period.

D.P. and G. Schneider, arXiv: 1603.05463

Conclusion

- We have defined the NLS evolution equations on graphs and considered the role of Kirchhoff boundary conditions in the energy conservation.
- We have classified ground state solutions for the NLS equation on tadpole, dumbbell, and periodic graphs.
- We have used various bifurcation methods compared to variational methods used in the work of R. Adami and others.
- We have analyzed both existence and stability properties of nonlinear bound states on graphs.
- Nonlinear dynamics of traveling solitary waves on graphs is the next problem for further analysis.

Thank you!