On the gaps in spectrum of the Maxwell Operator: case of Photonic Crystal Fibres

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Photonic crystal fibres

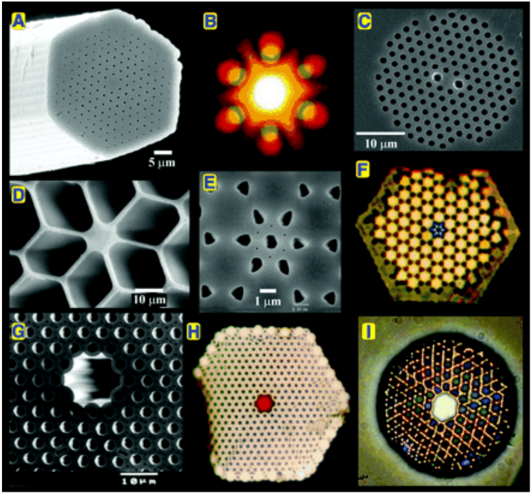
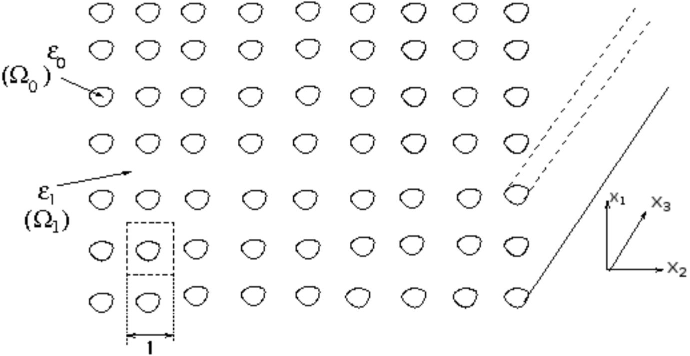


Figure: Taken from "Photonic Crystal Fibres" Phillip Russell, Science, 2003

Photonic crystals: Problem Formulation



$$\nabla \times E = -\mu \frac{\partial H}{\partial t}, \qquad \nabla \times H = \epsilon \frac{\partial E}{\partial t},$$
$$\nabla \cdot (\epsilon E) = 0, \qquad \nabla \cdot H = 0,$$

$$\epsilon = \epsilon_0 \chi_0(x) + \epsilon_1 \chi_1(x), \quad \epsilon_0 \neq \epsilon_1, \quad \mu \text{ constant} \quad (\mu = 1)$$

$$E = E(x_1, x_2) \exp\big(\mathrm{i}(kx_3 + \omega t)\big), \qquad H = H(x_1, x_2) \exp\big(\mathrm{i}(kx_3 + \omega t)\big)$$

k=0

If $k = 0$ then

$$-\Delta E_3(x) = \epsilon(x)\omega^2 E_3(x), \quad x \in \mathbb{R}^2,$$
$$-\nabla \epsilon(x)^{-1} \nabla H_3(x) = \omega^2 H_3(x), \quad x \in \mathbb{R}^2.$$

We have spectral problem for

$$A[E_3, H_3] = \int_{\mathbb{R}^2} |\nabla E_3|^2 + \epsilon(x)^{-1} |\nabla H_3|^2 dx$$

and

$$B[E_3, H_3] = \int_{\mathbb{R}^2} \epsilon(x)|E_3|^2 + |H_3|^2 dx.$$

Gaps due to high-contrast
A.Figotin, P. Kuchment 1993, R.Hempel, K Lienau 2000, V. Zhikov, 2000 and many others.
Spectral problem for

$$a[u] = \int_{\Omega_0} |\nabla u|^2 dx + t^2 \int_{\Omega_1} |\nabla u|^2 dx,$$

and

$$b[u] = \int_{\mathbb{R}^2} |u|^2 dx.$$

Gaps appear as $t \to \infty$. Hight-contrast.

No high contrast in $\epsilon(x), \mu(x)$ for Photonic Crystal Fibers.

Gaps are not expected unless $\epsilon_1 << \epsilon_2$ or $\epsilon_2 << \epsilon_1$.

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No high contrast in $\epsilon(x), \mu(x)$ for Photonic Crystal Fibers.

Gaps are not expected unless $\epsilon_1 << \epsilon_2$ or $\epsilon_2 << \epsilon_1$ for $k = 0$.

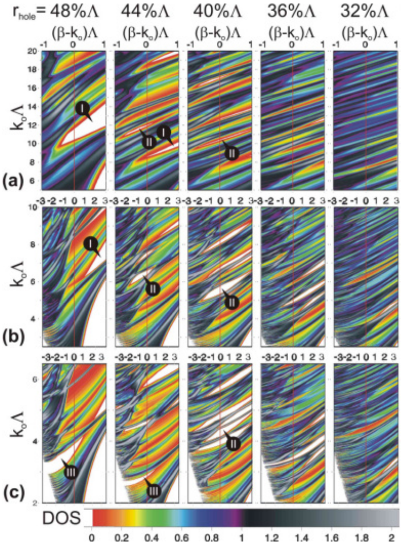
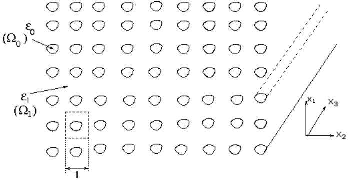


Figure: From J.M.Pottage, D.M.Bird, T.D.Hedley, T.A.Birks, J.C.Knight and P.St.J. Russell, Optics Express, 2003

Maxwell equations for plane waves in PCF, Oblique incidence (Case of PCF): $k \neq 0$



In each phase E_3 and H_3 satisfy the following equations

$$\Delta E_3 + (\omega^2 \epsilon_1 - k^2) E_3 = 0, \quad \Delta H_3 + (\omega^2 \epsilon_1 - k^2) H_3 = 0 \quad \text{in } \Omega_1$$

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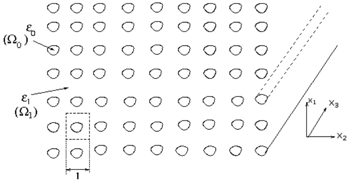
Maxwell equations for plane waves in PCF, Oblique incidence (Case of PCF): $k \neq 0$



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E_3 and H_3 coupled across interface $\Gamma = \partial\Omega_0$:

$$\omega \left[\frac{\epsilon}{a} \nabla E_3 \cdot n \right] = -k \left[\frac{1}{a} \nabla H_3 \cdot n^\perp \right], \qquad k \left[\frac{1}{a} \nabla E_3 \cdot n^\perp \right] = \omega \left[\frac{1}{a} \nabla H_3 \cdot n \right]$$

where $a = \omega^2 \epsilon(x) - k^2$ discontinuous on Γ.

$$\partial_1\left(\frac{\omega\epsilon}{a}\partial_1 E_3\right)+\partial_2\left(\frac{\omega\epsilon}{a}\partial_2 E_3\right)+\partial_1\left(\frac{k}{a}\partial_2 H_3\right)-\partial_2\left(\frac{k}{a}\partial_1 H_3\right)=-\omega\epsilon E_3,$$
$$-\partial_1\left(\frac{k}{a}\partial_2 E_3\right)+\partial_2\left(\frac{k}{a}\partial_1 E_3\right)+\partial_1\left(\frac{\omega}{a}\partial_1 H_3\right)+\partial_2\left(\frac{\omega}{a}\partial_2 H_3\right)=-\omega H_3,$$

where $a(x)=\omega^2\epsilon(x)-k^2$.
Find $u=(E_3,H_3)$ such that

$$\int_{\mathbb{R}^2}\frac{\omega}{a}\left(\epsilon\nabla u_1\cdot\overline{\nabla\phi_1}+\nabla u_2\cdot\overline{\nabla\phi_2}\right)+\frac{k}{a}\left(\{u_2,\overline{\phi_1}\}-\{u_1,\overline{\phi_2}\}\right)\,\mathrm{d}x$$
$$=\omega\int_{\mathbb{R}^2}\epsilon u_1\overline{\phi_1}+u_2\overline{\varphi_2}\,\mathrm{d}x\quad\forall\phi\in C_0^\infty(\mathbb{R}^2)$$

$\{f,g\}:=f_{x_2}g_{x_1}-f_{x_1}g_{x_2}$.
The above form is symmetric, and positive if $k^2<\omega^2\min\{\epsilon_0,\epsilon_1\}$.

Oblique incidence: $k = \omega\kappa$

If $k = \omega\kappa$, $\kappa \geq 0$ then we have **usual spectral problem**: Find u such that

$$\int_{\mathbb{R}^2} \frac{1}{\epsilon(x) - \kappa^2} \left(\epsilon(x)\nabla u_1 \cdot \overline{\nabla\phi_1} + \nabla u_2 \cdot \overline{\nabla\phi_2}\ \right) +$$

$$+ \int_{\mathbb{R}^2} \frac{\kappa}{\epsilon(x) - \kappa^2} \left(\left\{u_2, \overline{\phi_1}\right\} - \left\{u_1, \overline{\phi_2}\right\}\right)\ \mathrm{d}x = \omega^2 \int_{\mathbb{R}^2} \epsilon(x) u_1 \overline{\phi_1} + u_2 \overline{\phi_2}\ \mathrm{d}x,$$

$$\forall \phi \in C_0^\infty(\mathbb{R}^2).$$

The above form is symmetric, and positive if $\kappa^2 < \min\{\epsilon_0, \epsilon_1\}$.

Anti-resonant reflecting optical waveguide (ARROW)

Assume $\epsilon_0 > \epsilon_1 = 1$.

Spectral problem

$$A_\kappa(u,\phi) = \lambda B(u,\phi), \qquad \forall \phi \in C_0^\infty(\mathbb{R}^2),$$
$$\lambda = (\epsilon_0 - \kappa^2)\omega^2.$$

$$A_\kappa[u] := \int_{\Omega_1} \kappa \frac{\epsilon_0 - 1}{1-\kappa^2}|\partial u|^2 + \frac{\epsilon_0 + \kappa}{1+\kappa}|\nabla u|^2 dx + \int_{\Omega_0} \epsilon_0 |\nabla u_1|^2 + |\nabla u_2|^2 dx$$

where

$$|\partial u|^2 = |\partial_{x_1} u_1 + \partial_{x_2} u_2|^2 + |\partial_{x_2} u_1 - \partial_{x_1} u_2|^2$$

Scalar product is

$$B[u] := \int_{\Omega_1} |u|^2 dx + \int_{\Omega_0} \epsilon_0 |u_1|^2 + |u_2|^2 dx$$

If $\kappa < 1$: then A_κ is positive.
If $\kappa \to 1$ then there is high contrast.

Floquet-Bloch decomposition

Periodicity. Floquet transform

$$x \to (x,\theta), x \in \square = [-\pi,\pi]^2, \theta \in [-1/2, 1/2)^2.$$

Find $u \in H^1_\theta(\square)$ ($u(y) = e^{i\theta\cdot x} v(x)$, v $\square$-periodic) such that

$$A_\kappa(u,\phi) = \lambda B(u,\phi), \qquad \forall \phi \in H^1_\theta(\square).$$

Spectrum:

$$0 \le \lambda_1(\kappa,\theta) \le \lambda_2(\kappa,\theta) \le \ldots \le \lambda_n(\kappa,\theta) \le \ldots$$

Behaviour of spectra near $\kappa = 1$

Theorem

$$\lim_{\kappa \nearrow 1} \lambda_n(\kappa, \theta) = \lambda_n(\theta).$$

Here $\lambda_n(\theta)$ are eigenvalues of the problem:
Find λ and $u \in V_\theta = \{u \in H^1_\theta(\Box) : \partial u = 0 \text{ in } Q_1\}$ such that

$$B(u, \phi) = \lambda A(u, \phi), \qquad \forall \phi \in V_\theta,$$

where

$$A[u] := \int_\Box \epsilon_0 |\nabla u_1|^2 + |\nabla u_2|^2 dx,$$

and

$$B[u] := \int_\Box |u|^2 dx + (\epsilon_0 - 1) \int_{Q_0} |u_1|^2 dx.$$

Lemma

There exists a constant $c > 0$ such that for any $u \in H^1_\theta(\Box)$ there is $v \in V_\theta$ such that $\|u - v\|_{H^1(\Box)} \leq c\|\partial u\|_{L_2(Q_1)}$.

Find $\lambda \in \mathbb{C}$ and $u \in V_\theta = \{u \in H^1_\theta(\Box) : \partial u = 0 \text{ in } Q_1\}$ such that

$$B(u,\phi) = \lambda A(u,\phi), \quad \forall \phi \in V_\theta,$$

where

$$A[u] := \int_\Box |\nabla u|^2 + (\epsilon_0 - 1)|\nabla u_1|^2 dx,$$

and

$$B[u] := \int_\Box |u|^2 dx + (\epsilon_0 - 1)\int_{Q_0} |u_1|^2 dx.$$

What can we say about $\lambda_n(\theta)$?

Limit problem, $\kappa = 1$, First gap

Let
$$B_\delta := \{x : |x| < \delta\} \subset Q_0.$$

Theorem

$$\lambda_2(\theta) < 8\epsilon_0\delta^{-2}\left(1 + 4\ln\frac{\pi}{\delta}\right)^{-1},$$
$$\lambda_3(\theta) > \epsilon_0^{-1}\Lambda_2(Q_0),$$

where $\Lambda_2(Q_0)$ is 2nd eigenvalue of Neumann Laplacian in Q_0.

Corollary

If $8\epsilon_0^2 \leq \Lambda_2(Q_0)\delta^2\left(1 + 4\ln\frac{\pi}{\delta}\right)$, then there is a gap.

Limit problem, $\kappa = 1$

Find $\lambda \in \mathbb{C}$ and $u \in V_\theta = \{u \in H^1_\theta(\square) : \partial u = 0 \text{ in } Q_1\}$ such that

$$B(u,\phi) = \lambda A(u,\phi), \quad \forall \phi \in V_\theta,$$

where

$$A[u] := \int_\square |\nabla u|^2 + (\epsilon_0 - 1)|\nabla u_1|^2 dx,$$

and

$$B[u] := \int_\square |u|^2 dx + (\epsilon_0 - 1)\int_{Q_0} |u_1|^2 dx.$$

Toy problem

Find $\lambda \in \mathbb{C}$ and $u \in V_\theta = \{u \in H^1_\theta(\square) : \partial u = 0 \text{ in} Q_1\}$ such that

$$B(u, \phi) = \lambda A(u, \phi), \quad \forall \phi \in V_\theta,$$

where

$$A[u] := \int_\square |\nabla u|^2 dx,$$

and

$$B[u] := \int_\square |u|^2 dx.$$

Potentials

Let $\theta \neq 0$. Then for any $u \in H^1_\theta(\Box)$ there is $f \in L_2(\Box)$ such that

$$u = \partial \Delta_\theta^{-1} f,$$

where

$$\partial = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_2 & -\partial_1 \end{pmatrix}$$

Then constrain $\partial u = 0 \text{ in} Q_1$ is equivalent to $f = 0 \text{ in} Q_1$.
Problem. $\lambda \in \mathbb{C}$ and $f \in L_2(\Box), \operatorname{supp} f \subset \bar{Q}_0$ such that

$$a(f, \phi) = \lambda b(u, \phi), \qquad \forall \phi \in L_2(\Box), \operatorname{supp} \phi \subset \bar{Q}_0$$

where

$$a(f, \phi) := \langle f, \phi \rangle := \int_\Box f \overline{\phi} dx = \int_{Q_0} f \overline{\phi} dx,$$

and

$$b[f] := -\int_{Q_0} \overline{f} \Delta_\theta^{-1} f dx.$$

"Small inclusions"

Assume $\overline{Q_0} \subset B_\delta = \{x : |x| < \delta\}, \delta < \pi$.
Find $\lambda \in \mathbb{C}$ and $f \in L_2(Q_0)$ such that

$$a(f, \phi) = \lambda b(f, \phi), \quad \forall \phi \in L_2(Q_0),$$

where

$$a(f, f) := \langle f, f \rangle, \quad b(f, f) := \langle -\Delta_\theta^{-1} f, f \rangle.$$

Aim is to "replace" $-\Delta_\theta^{-1} f$ by $-\Delta^{-1} f$, where

$$(-\Delta^{-1} f)(x) = -\frac{1}{2\pi} \int_{Q_0} \ln|x - y| f(y) dy.$$

$$(-\Delta_\theta^{-1} f)(x) = (-\Delta^{-1} f)(x) + \int_{Q_0} g_\theta(x, y) f(y) dy.$$

Then

$$\langle -\Delta_\theta^{-1} f, f \rangle \approx \langle -\Delta^{-1} f, f \rangle + g_\theta(0, 0) \left| \int_{Q_0} f(y) dy \right|^2.$$

"Small" inclusions

Lemma

Let $f \in L_2(\square)$, $\operatorname{supp} f \subset B_\delta$ and $\delta < \pi$. Then

$$\left|\langle(-\Delta_\theta)^{-1}f, f\rangle - \langle(-\Delta)^{-1}f, f\rangle - g_\theta\left|\langle f, 1\rangle\right|^2\right| < \frac{3\delta}{\pi}\left(\langle -\Delta^{-1}f, f\rangle + g_\theta\left|\langle f, 1\rangle\right|^2\right),$$

where $g_\theta = g_\theta(0,0) > (2\pi)^{-1}\ln\pi$.

Problem 1. Find $\lambda^{(1)} \in \mathbb{C}$ and $f \in L_2(Q_0)$ such that

$$a^{(1)}(f, \phi) = \lambda^{(1)} b^{(1)}(f, \phi), \quad \forall \phi \in L_2(Q_0),$$

where

$$a^{(1)}(f, f) := \langle f, f\rangle, \quad b^{(1)}(f, f) := \langle -\Delta^{-1}f, f\rangle + g_\theta\left|\langle f, 1\rangle\right|^2.$$

$$\lambda_n^{(1)}\left(1 + \frac{3\delta}{\pi}\right)^{-1} < \lambda_n < \lambda_n^{(1)}\left(1 - \frac{3\delta}{\pi}\right)^{-1}$$

Consider $Q_0 = \delta\Omega$, where $\Omega \subset B_1$. After rescaling $x = \delta y$ we obtain the following problem.

Problem 2. Find $\lambda^{(2)} \in \mathbb{C}$ and $f \in L_2(\Omega)$ such that

$$a^{(2)}(f, \phi) = \lambda^{(2)} b^{(2)}(f, \phi), \quad \forall \phi \in L_2(\Omega),$$

where

$$a^{(2)}(f, f) := \int_\Omega |f|^2 dy, \quad b^{(2)}(f, f) := -\int_\Omega \overline{f} \Delta^{-1} f dy + \left(-(2\pi)^{-1} \ln \delta + g_\theta\right) \left|\int_\Omega f dy\right|^2,$$

and

$$\lambda_n^{(1)} = \delta^{-2} \lambda_n^{(2)}.$$

Here

$$\nu := -(2\pi)^{-1} \ln \delta + g_\theta$$

is a big positive parameter.

Consider representation

$$f(y) = \alpha + h(y),$$

where $\alpha \in \mathbb{C}$ and $h \in L_2(\Omega)$, $\int_\Omega h dy = 0$.

Problem 3. Find $\lambda^{(3)} \in \mathbb{C}$ and $\alpha \in \mathbb{C}, h \in L_2(\Omega), \int_\Omega h dy = 0$ such that

$$a^{(3)}(\alpha, h, \beta, \psi) = \lambda b^{(3)}(\alpha, h, \beta, \psi), \qquad \forall \phi = \beta + \psi \in L_2(\Omega), \int_\Omega \psi dy = 0,$$

where

$$a^{(3)}[\alpha, h] := |\alpha|^2 |\Omega| + \int_\Omega |h|^2 dy, \quad b^{(3)}[\alpha, h] := -\int_\Omega \overline{(\alpha + h)} \Delta^{-1}(\alpha + h) dy + \nu |\alpha|^2 |\Omega|^2,$$

and

$$\lambda_n^{(2)} = \lambda_n^{(3)}.$$

Problem 4. Find $\lambda^{(4)} \in \mathbb{C}$ and $\alpha \in \mathbb{C}, h \in L_2(\Omega), \int_\Omega h dy = 0$ such that

$$a^{(4)}(\alpha, h, \beta, \psi) = \lambda b^{(4)}(\alpha, h, \beta, \psi), \qquad \forall \phi \in L_2(\Omega),$$

where

$$a^{(4)}[\alpha, h] := |\alpha|^2 |\Omega| + \int_\Omega |h|^2 dy, \quad b^{(4)}[\alpha, h] := -\int_\Omega \overline{h} \Delta^{-1} h dy + \nu |\alpha|^2 |\Omega|^2,$$

and

$$\lambda_n^{(4)} \left(1 + c\nu^{-1/2}\right)^{-1} \leq \lambda_n^{(3)} \leq \lambda_n^{(4)} \left(1 - c\nu^{-1/2}\right)^{-1}.$$

Finally

$$\lambda_1^{(4)} = 2\pi |\Omega|^{-1} \left(2\pi g_\theta - \ln \delta\right)^{-1},$$

and $\lambda_2^{(4)}, \lambda_3^{(4)}, ..$ are eigenvalues of the problem for

$$a^{(5)}[h] := \int_\Omega |h|^2 dy, \quad b^{(5)}[h] := -\int_\Omega \overline{h} \Delta^{-1} h dy,$$

with domain $h \in L_2(\Omega), \int_\Omega h dy = 0$.

Back to PCF

We have similar results for PCF.

$$\lambda_{1,2}(\theta) = \delta^{-2}\,(2\pi g_\theta - \ln\delta)^{-1}\,\Lambda_{1,2} + O(\delta^{-2}\,(2\pi g_\theta - \ln\delta)^{-3/2}),$$

where $\Lambda_{1,2}$ are some positive numbers, and

$$\lambda_n(\theta) = \Lambda_n + O(\delta^{-2}\,(2\pi g_\theta - \ln\delta)^{-1/2}), \quad n = 3, 4\ldots,$$

where $\Lambda_3, \Lambda_4, \ldots$ are the eigenvalues of operator generated by quadratic forms

$$a[f] := \int_{Q_0} |f|^2 dx + (\epsilon_0 - 1)\int_{Q_0} \overline{f}\,\mathrm{grad}\,\mathrm{div}\Delta^{-1} f dx,$$

and

$$b[f] := \int_{Q_0} \overline{f}\Delta^{-1} f dx + (\epsilon_0 - 1)\int_{Q_0} \overline{\mathrm{div}\Delta^{-1} f}\,\mathrm{div}\Delta_\theta^{-1} f dx.$$

with domain $f \in L_2(Q_0), \int_{Q_0} f dx = 0$.

Thank you