Inverse Scattering Transform and Nonlinear Evolution Equations

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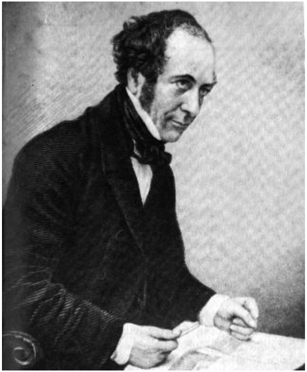
Outline

- I. Introduction, background, Solitons
- II. Compatible linear systems, Lax pairs, connection to nonlinear evolution equations
- III. Inverse Scattering Transform (IST): KdV; KdV is related to the time independent Schrödinger scattering problem
- IV. IST: NLS, mKdV, SG, nonlocal NLS... These eq are related to 2x2 scattering problem with two potentials (q, r) and suitable symmetry

I. Introduction–Background

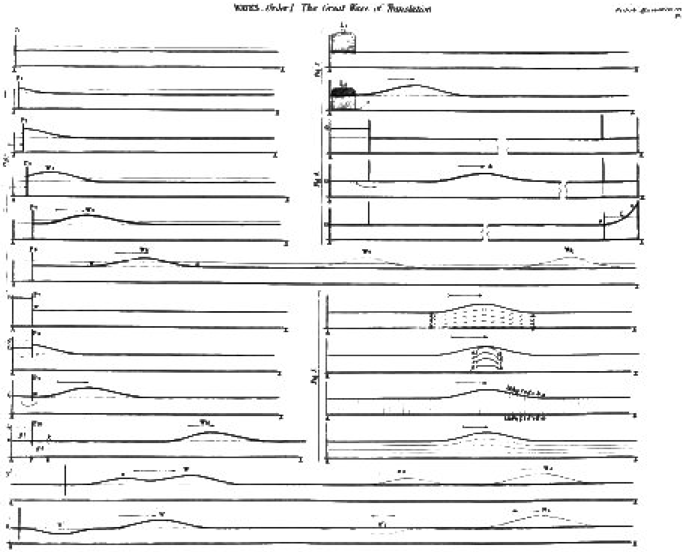
- 1837–British Association for the Advancement of Science (BAAS) sets up a "Committee on Waves"; one of two members was J. S. Russell (Naval Scientist).
- 1837, 1840, 1844 (Russell's major effort): "Report on Waves" to the BAAS–describes a remarkable discovery



Russell-Wave of Translation

- Russell observed a localized wave: “rounded smooth...well-defined heap of water”
- Called it the “Great Wave of Translation” – later known as the solitary wave
- “ Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon...”

Russell Experiments



Recreation: July 1995 [I]

Recreation: July 1995 [II]

Russell: to Mathematicians, Airy

Russell: "... it now remained for the mathematician to predict the discovery after it had happened..."
Leading British fluid dynamics researchers doubted the importance of Russell's solitary wave. G. Airy (below): believed Russell's wave was linear

Stokes

1847–G. Stokes : Stokes worked with nonlinear water wave equations and found a traveling periodic wave where the speed depends on amplitude; he was ambivalent w/r Russell. Stokes made many other critical contributions to fluid dynamics –"Navier-Stokes equations"

Boussinesq, Korteweg-deVries

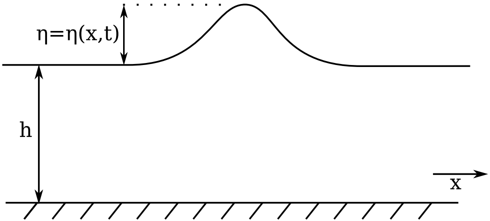
- 1871-77 – J. Boussinesq (left): new nonlinear eqs. and solitary wave solution for shallow water waves
- 1895 –D. Korteweg (right) & G. deVries: also shallow water waves ("KdV" eq.); NL periodic sol'n: "cnoidal" wave; limit case: the solitary wave (also see E. deJager '06: comparison Boussinesq – KdV)
- Russell's work was (finally) confirmed


KdV Equation –1895

KdV eq –1895

$$\frac{1}{\sqrt{gh}}\eta_t + \eta_x + \frac{3}{2h}\eta\eta_x + \frac{h^2}{2}(\frac{1}{3} - \hat{T})\eta_{xxx} = 0$$

where $\eta(x,t)$ is wave elevation above mean height h; g is gravity and $\hat{T}$ is normalized surface tension ($\hat{T} = \frac{T}{\rho g h^2}$)



KdV Eq.–con't

- nondimensional KdV eq.

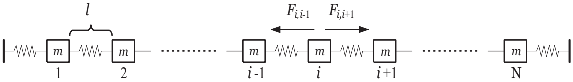
$$u_t + 6uu_x + u_{xxx} = 0$$

- solitary wave:

$$u = 2\kappa^2 \mathrm{sech}^2 \kappa(x - 4\kappa^2 t - x_0), \;\; \kappa, x_0 \;\; const$$

KdV –Modern Times

- 1895-1960 – Korteweg & deVries (KdV): water waves...
- 1960's – mathematicians developed approx methods to find reduced eq governing physical systems; KdV is an important "universal" eq
- 1960s M. Kruskal: 'FPU' (Fermi-Pasta-Ulam, 1955) problem



with force law: $F(\Delta) = -k(\Delta + \alpha\, \Delta^2), \alpha$ const; M.K. finds KdV eq in the continuum limit

KdV –Modern Times–con't

- 1965 –computation on KdV eq.

$$u_t + uu_x + \delta^2 u_{xxx} = 0$$

N. Zabusky, M. Kruskal introduced the term: *Solitons*

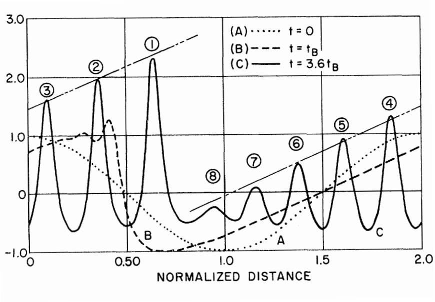


Figure: Calculations of the KdV Eq. with $\delta^2 \approx 0.02$ — from numerical calculations of ZK 1965

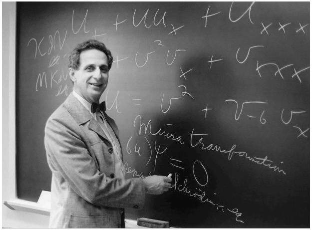
KdV –Modern Times–con't

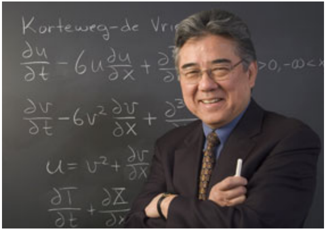
Kruskal and Miura study cons laws of KdV eq & modified KdV (mKdV) eq. Below KdV eq. left; mKdV eq right:

$$u_t + 6uu_x + u_{xxx} = 0, \quad v_t - 6v^2 v_x + v_{xxx} = 0$$

Miura finds a transformation between KdV and mKdV:

$$u = -(v_x + v^2)$$





KdV leads the way to IST

- Miura Transf leads to scattering problem and linearization of KdV: $v = \phi_x/\phi$

$$\phi_{xx} + (k^2 + u(x,t))\phi = 0, \quad \phi_t = \mathcal{M}\phi$$

$k_t = 0, \quad k$ constant

- 1967 – Method to find solution of KdV for decaying data: Gardner, Greene, Kruskal, Miura
- 1970's-present – KdV developments led to new methods & results in math physics
- Termed Inverse Scattering Transform (IST)–find solitons as special solutions –connect to e-values

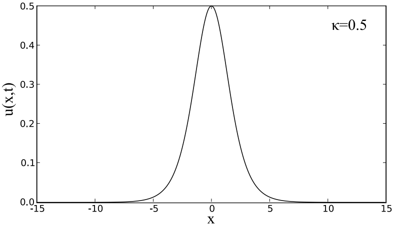
KdV Solitary Wave -Soliton

Normalized equation:

$$u_t + 6uu_x + u_{xxx} = 0$$

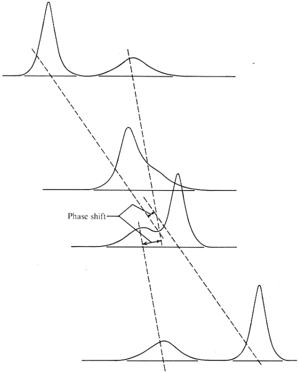
Soliton: $\quad u_s(x,t) = 2\kappa^2 \operatorname{sech}^2 \kappa(x - 4\kappa^2 t - x_0)$

One eigenvalue: $u_{max} = 2\kappa^2$; speed $= 2u_{max}$, $x_0 = 0$



KdV –Two Soliton Interaction

KdV eq. with two eigenvalues: two solitons



Solitons: speed and amplitude preserved upon interaction

NLS is Integrable

Another important integrable eq. is the nonlinear Schrödinger eq. –NLS (Zakharov & Shabat, 1972)

$$iq_t = q_{xx} \pm 2q^2 q^* = q_{xx} + Vq; \;\; V = \pm 2qq^*(x,t), \quad * = cc$$

Related to

$$\phi_x = X\phi = \begin{pmatrix} -ik & q(x,t) \\ r(x,t) & ik \end{pmatrix} \phi \;\; \text{with} \;\; r(x,t) = \mp q^*(x,t)$$

$$\phi_t = T\phi, \qquad T = T[q,r], 2x2$$

$k_t = 0$, k is constant

Key IST Equations

With suitable 'time' operator T, AKNS (1973) find many compatible eq–including:

$r = \mp q^* \in \mathbb{C}$: $\quad iq_t = q_{xx} \pm 2q^2 q^*$: NLS

$r = \mp q \in \mathbb{R}$: $\quad q_t + q_{xxx} \pm 6q^2 q_x = 0$: mKdV

$r = -q = u_x/2 \in \mathbb{R}$: $\quad u_{xt} = \sin u$: Sine-Gordon eq (SinG)

$r = q = u_x/2 \in \mathbb{R}$: $\quad u_{xt} = \sinh u$: Sinh-Gordon eq (SinhG)

$r = -1$: $\quad q_t + q_{xxx} \pm 6qq_x = 0$: KdV

–Plus a class of other eqs

AKNS carry out linearization/solution; term the method "IST"

Solitons correspond to eigenvalues of: $\phi_x = X\phi$; can travel; can remain stationary and oscillate in time (breather)...

NLS Soliton

$r = -q^* \in \mathbb{C}$: $\qquad iq_t = q_{xx} + 2q^2 q^*$: NLS

Soliton solution to NLS: $k = \xi + i\eta$ eigenvalue of $\phi_x = X\phi$

$$q(x,t) = 2\eta e^{-2i\xi x + 4i(\xi^2 - \eta^2) - i\psi_0} \mathrm{sech}(2\eta(x - 4\xi - \delta_0))$$

Here ψ_0, δ_0 are related to the phase/amplitude of the 'norming' constant

Soliton solution has amplitude 2η and travels with velocity 4ξ

‘Nonlocal NLS’ is Integrable

Motivated by studies in PT symmetric systems

$$iq_t = q_{xx} + V(x)q; \quad V(x) = V^*(-x)$$

MJA & Z. Musslimani 2013 find a new symmetry of AKNS system:

$$r(x,t) = \mp q^*(-x,t)$$

and an integrable ‘nonlocal NLS’ eq

$$iq_t = q_{xx} + V(x,t)q; \quad V(x,t) = \pm 2q(x,t)q^*(-x,t) = V^*(-x,t)$$

Nonlocal mKdV, Sine-Gordon Equations

Also find: integrable nonlocal mKdV, SinGordon eq & others

Real nonlocal mKdV, $r(x,t) = \mp q(-x,-t)$ $q \in \mathbb{R}$:

$$q_t(x,t) + q_{xxx}(x,t) \pm 6q(x,t)q(-x,-t)q_x(x,t) = 0$$

MJA, Z. Musslimani, 2016

II. Compatible linear systems, Lax Pairs $1+1$d

Lax (1968) considered two operators; i.e. operator 'pair'– in general:

$$\mathcal{L}v = \lambda v$$
$$v_t = \mathcal{M}v$$

Here $\mathcal{L}, \mathcal{M}$ depend on 'potential' u

Find KdV from compatibility of:

$$\mathcal{L} = \partial_x^2 + u$$
$$\mathcal{M} = u_x + \gamma - (2u + 4\lambda)\partial_x = \gamma - 3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3}$$

where γ is const and λ is a spectral parameter with $\lambda_t = 0$: 'isospectral flow'

Lax Pairs –con't

Take $\partial/\partial t$ of $\mathcal{L}v = \lambda v$:

$$\mathcal{L}_t v + \mathcal{L}v_t = \lambda_t v + \lambda v_t;$$

Use $v_t = \mathcal{M}v$

$$\begin{aligned}\mathcal{L}_t v + \mathcal{L}\mathcal{M}v = \lambda_t v + \lambda\mathcal{M}v &= \lambda_t v + \mathcal{M}\lambda v \\ &= \lambda_t v + \mathcal{M}\mathcal{L}v \quad => \end{aligned}$$

$$[\mathcal{L}_t + (\mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L})]\, v = \lambda_t v$$

Hence to find nontrivial ef $v(x,t)$

$$(*) \;\; \mathcal{L}_t + [\mathcal{L},\mathcal{M}] = 0 \;\; \text{where} \;\; [\mathcal{L},\mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L}$$

if and only if $\lambda_t = 0$; (*) called Lax eq

Compatible Matrix Systems

Extension:

$$v_x = \mathbf{X} v, \qquad v_t = \mathbf{T} v$$

where v is an n-d vector and $\mathbf{X}$ and $\mathbf{T}$ are $n \times n$ matrices : $\mathbf{X} = \mathbf{X}[\mathbf{u}; \lambda], \mathbf{T} = \mathbf{T}[\mathbf{u}; \lambda]$; $\mathbf{u}$ is the potential

Require compatibility: $v_{xt} = v_{tx}$, then

$$\mathbf{X}_t - \mathbf{T}_x + [\mathbf{X}, \mathbf{T}] = 0$$

and require e-value dependence to be *isospectral*: $\lambda_t = 0$. Above eq more general than original Lax pair: allows more gen'l e-value dependence than $\mathcal{L}v = \lambda v$

2×2 Matrix Systems

After KdV developments and Lax' ideas, ZS (1972) found compatible pair and method of sol'n of NLS. AKNS (1973) generalized this to class of eq including NLS, mKdV, SG, KdV etc

AKNS: E-value prob- or scattering problem $(\mathcal{L})$:

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= ikv_2 + r(x,t)v_1 \end{aligned}$$

Time dependence $(\mathcal{M})$:

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2 \\ v_{2,t} &= Cv_1 + Dv_2 \end{aligned}$$

with $k_t = 0$ and A, B, C and D functionals of $q(x,t)$, $r(x,t)$ and k

2×2 Matrix Systems–Special Cases

Note when $r(x,t) = -1$, then from

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= \quad ikv_2 + r(x,t)v_1 = ikv_2 - v_1 \end{aligned}$$

we can solve for v_1 in terms of v_2; find v_2 satisfies:

$$v_{2,xx} + (k^2 + q)v_2 = 0$$

i.e the time independent Schrödinger e-value prob–which is related to KdV

Method described below (AKNS) yields physically interesting NL evolution eq when $r = -1$, $r = \mp q^*, q \in \mathbb{C}$, $r = \mp q$, $q \in \mathbb{R}$

2×2 Matrix Systems–con't

Consider the 2×2 compatible matrix system

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= ikv_2 + r(x,t)v_1 \end{aligned}$$

$$\begin{aligned} v_{1,t} &= Av_1 + Bv_2 \\ v_{2,t} &= Cv_1 + Dv_2 \end{aligned}$$

Namely require $v_{j,xt} = v_{j,tx}, \;\; j = 1,2$, and $dk/dt = 0$: isospectral flow

This yields two eq of form: $\Gamma_j^1 v_1 + \Gamma_j^2 v_2 = 0, \;\; j = 1,2;$ we take $\Gamma_j^1 = \Gamma_j^2 = 0$

2×2 Matrix Systems–con't

This leads to $D = -A$ and three eq for A, B, C

$$\begin{aligned} A_x &= qC - rB \\ B_x + 2ikB &= q_t - 2Aq \\ C_x - 2ikC &= r_t + 2Ar \end{aligned}$$

Note the e-value dependence k in coef of B, C in 2nd 3rd eq
Look for sol'ns A, B, C in finite powers of k

$$A = \sum_{j=0}^{n} A_j k^j, \qquad B = \sum_{j=0}^{n} B_j k^j, \qquad C = \sum_{j=0}^{n} C_j k^j$$

Substitution yields eq which determine A_j, B_j, C_j and leave two additional constraints: NL evolution eq

2×2 Matrix Systems–Example

$$
\begin{aligned}
A_x &= qC - rB \\
B_x + 2ikB &= q_t - 2Aq \\
C_x - 2ikC &= r_t + 2Ar
\end{aligned}
$$

Prototype: $n = 2,\ A = A_2 k^2 + A_1 k + A_0$ etc.
The coefficients of k^3 give $B_2 = C_2 = 0$; at order k^2, we obtain $A_2 = a =$ const etc.
Find after some algebra: coupled NL evoln eq (constraint on sol'ns of A, B, C eq)

$$
\begin{aligned}
-\frac{1}{2} a q_{xx} &= q_t - a q^2 r \\
\frac{1}{2} a r_{xx} &= r_t + a q r^2
\end{aligned}
$$

2×2 Matrix Systems–NLS

If $r = \mp q^*$ and $a = 2i$, then find:

$$iq_t = q_{xx} \pm 2q^2 q^* \qquad \text{NLS}$$

Both focusing $(+)$ and defocusing $(-)$ cases inlcuded
Summary $n = 2$ with $r = \mp q^*$ find

$$\begin{aligned} A &= 2ik^2 \mp iqq^* \\ B &= 2qk + iq_x \\ C &= \pm 2q^* k \mp iq_x^* \end{aligned}$$

provided that $q(x, t)$ satisfies the NLS eq
and recall: $dk/dt = 0$: isospectral flow

2×2 –KdV,mKdV

For $n = 3$: with $r = -1$, obtain the KdV eq:

$$q_t + 6qq_x + q_{xxx} = 0$$

If $r = \mp q$, real, obtain the mKdV eq

$$q_t \pm 6q^2 q_x + q_{xxx} = 0$$

2×2 –Sine-Gordon, Sinh-Gordon Eq

Another system: $n = -1$, let

$A = \frac{a(x,t)}{k}, \qquad B = \frac{b(x,t)}{k}, \qquad C = \frac{c(x,t)}{k}$

Find eq for a, b, c; special cases are

$$(i): \quad a = \frac{i}{4}\cos u, \qquad b = -c = \frac{i}{4}\sin u, \qquad q = -r = -\frac{1}{2}u_x$$

and u satisfies the Sine–Gordon eq:

$$u_{xt} = \sin u$$

(ii): $\quad a = \frac{i}{4}\cosh u, \qquad b = -c = -\frac{i}{4}\sinh u, \qquad q = r = \frac{1}{2}u_x$

and u satisfies the Sinh–Gordon eq

$$u_{xt} = \sinh u$$

2×2– New Symmetry

If $r(x,t) = \mp q^*(-x,t)$ then using method described earlier find

$$iq_t = q_{xx} \pm 2q^2(x,t)q^*(-x,t) \qquad \text{Nonlocal NLS}$$

or written as

$$iq_t = q_{xx} \pm V[q]q(x,t), \qquad V[q] = q(x,t)q^*(-x,t)$$

2×2–General Class of NL Eq

A, B, C eq are linear eq that be solved for decaying q, r subject to constraint; find:

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t + 2A_\infty(L)\begin{pmatrix} r \\ q \end{pmatrix} = 0$$

where $A_\infty(k) = \lim_{|x|\to\infty} A(x,t,k)$; $A_\infty(k)$ can be the ratio of two entire functions; L is

$$L = \frac{1}{2i}\begin{pmatrix} \partial_x - 2r(I_- q) & 2r(I_- r) \\ -2q(I_- q) & -\partial_x + 2q(I_- r) \end{pmatrix}$$

where $\partial_x \equiv \partial/\partial x$ and $(I_- f)(x) \equiv \int_{-\infty}^{x} f(y)dy$

2×2–General Class of NL Eq–con't

Ex. $A_\infty(k) = 2ik^2$ find:

$$\begin{pmatrix} r \\ -q \end{pmatrix}_t = -4iL^2 \begin{pmatrix} r \\ q \end{pmatrix} = -2L \begin{pmatrix} r_x \\ q_x \end{pmatrix} = i \begin{pmatrix} r_{xx} - 2r^2 q \\ q_{xx} - 2q^2 r \end{pmatrix}$$

With $r = \mp q^*$ we have the NLS eq

$$iq_t = q_{xx} \pm 2q^2 q^* \qquad \text{NLS}$$

Other Eigenvalue Problems

There have been numerous applications and generalizations of these method. For example the matrix generalization of 2×2 system; to $N \times N$ systems i.e.

$$\frac{\partial v}{\partial x} = ik\mathbf{J}v + \mathbf{Q}v, \quad \frac{\partial v}{\partial t} = \mathbf{T}v$$

where $\mathbf{Q}$ is an $N \times N$ matrix with $Q^{ii} = 0$,
$\mathbf{J} = \mathrm{diag}(J^1, J^2, \ldots, J^N)$, $\quad v(x,t)$ is an N-dimensional vector

$\mathbf{T}$ is an $N \times N$ matrix and $k_t = 0$

$\mathbf{T}$ can be expanded in powers of k to find NL Evol Eq

Find numerous interesting compatible NL evol eq such as N wave eq, Boussinesq eq, vector NLS system etc.

$2 + 1$d 'Scattering' Problems

There are compatible systems in $2 + 1$d e.g. $N \times N$ linear system:

$$\frac{\partial v}{\partial x} = \mathbf{J}\frac{\partial v}{\partial y} + \mathbf{Q}v, \quad \frac{\partial v}{\partial t} = \mathbf{T}v$$

NL wave eq are obtained by expanding $\mathbf{T}$ in powers of $\frac{\partial}{\partial y}$
Find N wave eq, Davey-Stewartson (2×2 system with $r = \mp q^*$), and Kadomstsev-Petviashvili (KP) eq (2×2 system with $r = -1$):

$$(q_t + 6qq_x + q_{xxx})_x + \sigma^2 q_{yy} = 0 \qquad \text{KP}$$

where $\sigma^2 = \mp 1$: so called KP I,II eq

2×2 system with $r = -1$ yields:

$$\sigma v_y + v_{xx} + uv = 0$$

Discrete Eigenvalue Problems

Recall the continuous 2×2 system

$$\begin{aligned} v_{1,x} &= -ikv_1 + q(x,t)v_2 \\ v_{2,x} &= \quad ikv_2 + r(x,t)v_1 \end{aligned}$$

Discretizing $v_{j,x} \approx \frac{v_{j,n+1} - v_{j,n}}{h}$ and calling $z = e^{\pm ikh} \approx 1 \pm ikh + \cdots$ and $Q_n(t) = hq_n, R_n(t) = hr_n$ leads to the following discrete 2×2 eigenvalue problem

$$\begin{aligned} v_{1,n+1} &= zv_{1,n} + Q_n(t)v_{2,n} \\ v_{2,n+1} &= \frac{1}{z}v_{2,n} + R_n(t)v_{1,n} \end{aligned}$$

Discrete Eigenvalue Problems–con't

To

$$v_{1,n+1} = zv_{1,n} + Q_n(t)v_{2,n}$$
$$v_{2,n+1} = \frac{1}{z}v_{2,n} + R_n(t)v_{1,n}$$

we add time dependence

$$\frac{dv_{1,n}}{dt} = Av_{1,n} + Bv_{2,n}$$
$$\frac{dv_{2,n}}{dt} = Cv_{1,n} + Dv_{2,n}$$

Taking $z_t = 0$ requiring compatiblity and expanding A_n, B_n, C_n, D_n in finite Laurent series in z yields NL Evol eq as constraints

Discrete Eigenvalue Problems–con't

Ex. Expanding
$A_n = \sum_{j=-2}^{2} A_{j,n} z^j$ similar for B_n, C_n, D_n eventually yields

$$\begin{aligned} i\frac{d}{dt}Q_n &= Q_{n+1} - 2Q_n + Q_{n-1} - Q_n R_n \left(Q_{n+1} + Q_{n-1}\right) \\ -i\frac{d}{dt}R_n &= R_{n+1} - 2R_n + R_{n-1} - Q_n R_n \left(R_{n+1} + R_{n-1}\right) \end{aligned}$$

With $R_n = \mp Q_n^*$ we have the integrable discrete NLS eq

$$i\frac{d}{dt}Q_n = Q_{n+1} - 2Q_n + Q_{n-1} \pm |Q_n|^2 \left(Q_{n+1} + Q_{n-1}\right)$$

or with $Q_n(t) = hq_n(t)$

$$i\frac{d}{dt}q_n = \frac{1}{h^2}\left(q_{n+1} - 2q_n + q_{n-1}\right) \pm |q_n|^2 \left(q_{n+1} + q_{n-1}\right)$$

Conclusions II.

- NL Evolution Eq are obtained from compatible linear systems
- Lax Pair:

$$\mathcal{L}v = \lambda v$$
$$v_t = \mathcal{M}v$$

with $\lambda_t = 0$. Or extension:

$$v_x = \mathbf{X}v, \qquad v_t = \mathbf{T}v$$

- From 2×2 systems find NLS, mKdV, SG, KdV, nonlocal NLS, ...
- May generalize to find higher order compatible matrix systems, multidimensional eq; discrete eq....

III. Inverse Scattering Transform (IST) for KdV

Motivation: linear Fourier Transform (FT)

Consider the linear evol eq

$$u_t = \sum_{j=0}^{N} a_j \partial_x^j u = a_0 u + a_1 u_x + a_2 u_{xx} + \cdots , \quad a_j \in \mathbb{R} \;\; \text{const}$$

The soln $u(x,t)$ can be found via FT as

$$u(x,t) = \frac{1}{2\pi} \int b(k,t) e^{ikx} dk \qquad \text{(FT)}$$

Note: $\int = \int_{-\infty}^{\infty}$ and it is assumed that $|u| \to 0$ rapidly as $|x| \to \infty$, and u is sufficiently smooth

Note: $\int = \int_{-\infty}^{\infty}$

Fourier Transform–con't

Substituting FT into linear eq yields (assume interchanges etc)

$$\int \mathrm{e}^{ikx}\{b_t - b\sum_{j=0}^{N}(ik)^j a_j\}\, dk = 0 \quad \text{or} \quad b_t = b\sum_{j=0}^{N}(ik)^j a_j$$

So

$$b(k,t) = b_0(k)\mathrm{e}^{-i\omega(k)t}, \quad \omega(k) = i\sum_{j=0}^{N}(ik)^j a_j$$

When $\omega(k) \in \mathbb{R}$ ($a_{2j} = 0, j = 0, 1...$), it is called the dispersion relation. Thus the soln is given by

$$u(x,t) = \frac{1}{2\pi}\int b_0(k)e^{i[kx-\omega(k)t]}dk$$

where: $b_0(k) = \int u_0(x)e^{-ikx}dx$

$u(x,t) \in \mathbb{R}$ symmetry: $b_0(k) = b_0^*(-k)$

FourierTransform–Linearized KdV

The previous result shows that for the linearized KdV eq

$$u_t + u_{xxx} = 0$$

The FT soln is

$$u(x,t) = \frac{1}{2\pi}\int b_0(k)e^{i[kx+k^3t]}dk; \quad b_0(k) = \int u_0(x)e^{-ikx}dx\,dk$$

Schematically the soln process via FT is:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct FT}} & b(k,0) = b_0(k) \\ & & \Big\downarrow t\text{: time evolution} \\ u(x,t) & \xleftarrow{\text{inverse FT}} & b(k,t) = b_0(k)e^{ik^3t} = b_0(k)e^{-i\omega(k)t} \end{array}$$

IST for KdV

Compatibility of the following system

$\mathcal{L}v: \; v_{xx}+(\lambda+u(x,t))v=0$ and $\mathcal{M}v: \; v_t=(\gamma+u_x)v+(4\lambda-2u)v_x$

where γ=const and $\lambda_t=0$ yields the KdV eq

$$u_t+6uu_x+u_{xxx}=0 \quad \text{KdV}$$

Soln via IST for decaying data: $\mathcal{S}(k)$ denotes scattering data

$$\begin{tikzcd} u(x,0) \arrow[r, "\text{Direct Scattering}"] & \mathcal{L}:\mathcal{S}(k,0) \arrow[d, "t\text{: time evolution: } \mathcal{M}"] \\ u(x,t) & \mathcal{S}(k,t) \arrow[l, "\text{Inverse Scattering}"'] \end{tikzcd}$$

Direct Scattering–con't

Begin with discussion of direct scattering problem. Let $\lambda = k^2$, then $\mathcal{L}$ (scattering) operator is:

$$\mathcal{L}: \quad v_{xx} + \left(u(x) + k^2\right)v = 0$$

note suppression the time dependence in u. Assume that $u(x) \in \mathbb{R}$ and decays sufficiently rapidly, e.g. u lies in the space of functions

$$L_2^1: \quad \int_{-\infty}^{\infty} (1 + |x|^2)|u(x)|dx < \infty$$

Associated with operator $\mathcal{L}$ are efcns for real k that are bounded for all values of x, and have appropriate analytic extensions into UHP–k, LHP–k

Direct Scattering–con't

Appropriate efcns associated with operator $\mathcal{L}$ are defined from their BCs; i.e. identify 4 efcns defined by the following asymptotic BCs

$$\phi(x;k) \sim e^{-ikx}, \quad \bar{\phi}(x;k) \sim e^{ikx} \quad \text{as} \quad x \to -\infty$$
$$\psi(x;k) \sim e^{ikx}, \quad \bar{\psi}(x;k) \sim e^{-ikx} \quad \text{as} \quad x \to \infty$$

So, e.g. $\phi(x,k)$ is a soln of the eq which tends to e^{-ikx} as $x \to -\infty$ etc. Note: $\bar{\phi}$ does not represent cc; rather *=cc
From BCs and $u(x) \in \mathbb{R}$ have symmetries:

$$\phi(x;k) = \bar{\phi}(x;-k) = \phi^*(x,-k)$$

$$\psi(x;k) = \psi(x;-k) = \psi^*(x,-k)$$

Direct Scattering–con't

The *Wronskian* of 2 solns ψ, ϕ is defined as

$$W(\phi, \psi) = \phi\psi_x - \phi_x\psi$$

and from Abel's Theorem, the Wronskian is const.
Hence from $\pm\infty$:

$$W(\psi, \bar{\psi}) = -2ik = -W(\phi, \bar{\phi})$$

Since $\mathcal{L}$ is a linear 2nd order ODE, from linear independence of its solutions we obtain the following completeness relationships between the efcns

$$\begin{aligned} \phi(x;k) = & \quad a(k)\bar{\psi}(x;k) + b(k)\psi(x;k) \\ \bar{\phi}(x;k) = & \quad \bar{a}(k)\psi(x;k) + \bar{b}(k)\bar{\psi}(x;k) \end{aligned}$$

For $u(x) \in \mathbb{R}$ only need first eq

Direct Scattering–con't

$a(k), b(k)$ can be expressed in terms of Wronskians:

$$a(k) = \frac{W(\phi(x;k), \psi(x;k))}{2ik}, \qquad b(k) = -\frac{W(\phi(x;k), \bar{\psi}(x;k))}{2ik}$$

Thus $\phi, \psi, \bar{\psi}$ determine $a(k), b(k)$ which are part of the 'scattering data'
When $u(x) \in \mathbb{R}$ find symmetries: $a(-k) = a^*(k); b(-k) = b^*(k)$ and unitarity:

$$|a(k)|^2 - |b(k)|^2 = 1, \quad k \in \mathbb{R}$$

Direct Scattering–con't

It is more convenient to work with modified efcns:

$$M(x;k) = \phi(x;k)e^{ikx} \sim 1 \quad \text{as} \quad x \to -\infty$$
$$\bar{N}(x;k) = \bar{\psi}(x;k)e^{ikx} \sim 1 \quad \text{as} \quad x \to +\infty$$

and also $N(x,k) = \psi(x;k)e^{ikx}$. Completeness of efcns implies

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)N(x;k)$$

where $\quad \rho(k) = \frac{b(k)}{a(k)}$

$\tau(k) = 1/a(k)$ and $\rho(k)$ are called the **transmission** and **reflection** coefs

Direct Scattering–con't

$\psi(x;k) = \bar{\psi}(x;-k)$ implies $N(x;k) = \bar{N}(x;-k)e^{2ikx}$

Due to this symmetry will only need 2 efcns. Namely, from completeness:

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

where $\rho(k) = \frac{b(k)}{a(k)}$

(*) will be a fundamental eq. Later will show that (*) is a generalized **Riemann-Hilbert boundary value problem** (RHBVP)

Analyticity of Efcns

Theorem

For $u \in L_2^1 : \int_{-\infty}^{\infty}(1+|x|^2)|u| < \infty$

(i) $M(x;k)$ *and* $a(k)$ *are analytic fcns of* k *for* $Imk > 0$ *and tend to unity as* $|k| \to \infty$*; they are continuous on* $Imk = 0$*;*

(ii) $\bar{N}(x;k)$ *and* $\bar{a}(k)$ *are analytic fcns of* k *for* $Imk < 0$ *and tend to unity as* $|k| \to \infty$*; they are continuous on* $Imk = 0$

Using Green's fcn techniques may show that $M(x;k), \bar{N}(x;k)$ satisfy the following Volterra integral eq

$$\begin{aligned} M(x;k) &= 1 + \frac{1}{2ik}\int_{-\infty}^{x}\left\{1 - e^{2ik(x-\xi)}\right\}u(\xi)M(\xi;k)d\xi \\ \bar{N}(x;k) &= 1 - \frac{1}{2ik}\int_{x}^{\infty}\left\{1 - e^{-2ik(\xi-x)}\right\}u(\xi)\bar{N}(\xi;k)d\xi \end{aligned}$$

Proof: Convergence of Neumann series

Potential and Efcns

From efcn can determine potential u

Using

$$\bar{N}(x;k) = 1 - \frac{1}{2ik}\int_x^\infty \left\{1 - e^{-2ik(\xi - x)}\right\} u(\xi)\bar{N}(\xi;k)d\xi$$

then as $k \to \infty$, Riemann-Lesbegue Lemma implies:

$$\bar{N}(x;k) \sim 1 - \frac{1}{2ik}\int_x^\infty u(\xi)d\xi \qquad (**)$$

Analyticity, RH Problem and Scattering Data

We will work with

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

where $\rho(k) = \frac{b(k)}{a(k)}$

Note: LHS: $\frac{M(x;k)}{a(k)}$ is analytic UHP-k/[zero's of $a(k)$]; RHS: $\bar{N}(x;k)$ is analytic LHP-k;

We will consider remaining term as the 'jump' (change) in analyticity across Real k axis

Required Scattering Data

Scattering data that will be needed: $\rho(k)$ and information about zero's of $a(k)$:

For real $u(x)$ from operator $\mathcal{L}$ can show:

$a(k)$ has a **finite number of simple zero's on img axis**:
$a(k_j) = 0$; $\{k_j = i\kappa_j\}, j = 1, ...J; \quad \kappa_j > 0$;

At every zero $k_j = i\kappa_j$ there are L^2 bound states:

$\phi_j = \phi(x, k_j), \psi_j = \psi(x, k_j)$ such that $\phi_j = b_j\psi_j => M_j = b_j N_j$;
for inverse problem we will need: $C_j = b_j/a'(k_j)$; $j = 1, ...J$

C_j are called 'norming constants'

Next: Inverse Problem

Scattering data: $\mathcal{S}(k) = \{\rho(k), \ \ \{\kappa_j, C_j\}, \ j = 1, 2, ...J \ \}$

Recall scheme:

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\
 & & \Big\downarrow {\scriptstyle t:\ \text{time evolution: } \mathcal{M}} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t)
\end{array}
$$

Next consider inverse problem at fixed time

Inverse Scattering–Projection Operators

Recall

$$\frac{M(x;k)}{a(k)} = \bar{N}(x;k) + \rho(k)e^{2ikx}\bar{N}(x;-k) \quad (*)$$

(*) is fundamental eq.
Apart from poles at $a(k_j) = 0$, $\frac{M(x;k)}{a(k)}$ is anal UHP; and $\bar{N}(x;k)$ is anal in LHP. $(*)$ a generalized (RH) problem; it leads to an integral eq for $N(x;k)$

Use projection operators
Consider the $\mathcal{P}^{\pm}$ projection operator defined by

$$(\mathcal{P}^{\pm}f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\zeta)d\zeta}{\zeta-(k\pm i0)} = \lim_{\varepsilon\downarrow 0}\left\{\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\zeta)d\zeta}{\zeta-(k\pm i\varepsilon)}\right\}$$

Projection Operators–con't

If $f(k) = f_\pm(k)$ is anal in the UHP/LHP-k and $f_\pm(k) \to 0$ as $|k| \to \infty$ (for Im $k \gtrless 0$), then from contour integration:

$$(\mathcal{P}^\pm f_\mp)(k) = 0$$
$$(\mathcal{P}^\pm f_\pm)(k) = \pm f_\pm(k),$$

To most easily explain ideas, 1st assume that there are no poles, that is $a(k) \neq 0$. Then operating on (*) with $\mathcal{P}^-$:

$$\mathcal{P}^-\left[\left(\frac{M(x;k)}{a(k)} - 1\right)\right] = \mathcal{P}^-\left[(\bar{N}(x;k) - 1) + \rho(k)e^{2ikx}\bar{N}(x;-k)\right]$$

From Proj: LHS=0 (since assumed no zero's of $a(k)$); and $\mathcal{P}^-\left[(\bar{N}(x;k) - 1)\right] = -(\bar{N}(x;k) - 1)$ implies

Inverse Problem: no poles

$$\bar{N}(x;k) = 1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta - (k - i0)} \qquad \text{(E1)}$$

Symmetry: $\quad N(x;k) = e^{2ikx}\bar{N}(x;-k) =>$ an integral eq

$$N(x;k) = e^{2ikx}\left\{1 + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + k + i0}\right\}$$

Reconstruction of the potential u; As $k \to \infty$ (E1) implies

$$\bar{N}(x;k) \sim 1 - \frac{1}{2\pi ik}\int_{-\infty}^{\infty} \rho(\zeta)N(x;\zeta)d\zeta \qquad \text{(E2)}$$

From direct integral eq (**): $\quad \bar{N}(x;k) \sim 1 - \frac{1}{2ik}\int_x^{\infty} u(\xi)d\xi$;
comparing (**) & (E2):

$$u(x) = -\frac{\partial}{\partial x}\left\{\frac{1}{\pi}\int_{-\infty}^{\infty} \rho(\zeta)N(x;\zeta)d\zeta\right\}$$

Inverse Problem: Including Poles

For the case when $a(k)$ has zeros, one can extend the above result; suppose

$$a(k_j = i\kappa_j) = 0, \quad \kappa_j > 0, \quad j = 1, \cdots J$$

We define

$$N_j(x) = N(x; k_j)$$

Evaluating the pole contributions and carrying out similar calculations as before leads to

$$N(x;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{J} \frac{C_j N_j(x)}{k + i\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + k + i0}\right\}$$

Inverse Problem: Including Poles–con't

To complete the system, evaluate at $k = k_p = i\kappa_p$

$$N(x;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{J}\frac{C_j N_j(x)}{k + i\kappa_j} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + k + i0}\right\}$$

$$N_p(x) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J}\frac{C_j N_j(x)}{i(\kappa_p + \kappa_j)} + \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(\zeta)N(x;\zeta)d\zeta}{\zeta + i\kappa_p}\right\}$$

for $p = 1, \ldots J$. Above is a coupled system of integral eq for $N(x,k); \{N_j(x) = N(x, k_j = i\kappa_j)\}, j = 1, \cdots, J$
From these eq $u(x)$ is reconstructed from

$$u(x) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j N_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\rho(\zeta)N(x;\zeta)d\zeta\right\}$$

IST – So Far

So far in the IST process direct and inverse problem have been discussed

Direct problem (from operator $\mathcal{L}$): $u(x) \to \mathcal{S}(k)$

Inverse problem: $\mathcal{S}(k) = \{\rho(k), \ \{\kappa_j, C_j\} \ \} \to u(x)$

Direct and inverse problems are the NL analogues of the direct and inverse Fourier transform

Next need time dependence; recall:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\ & & \Big\downarrow {\scriptstyle t:\ \text{time evolution:}\ \mathcal{M}} \\ u(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t) \end{array}$$

IST: Time Dependence

For time dependence use associated time evolution operator: $\mathcal{M}$ which for the KdV eq is

$$v_t = \mathcal{M}v = (u_x + \gamma)v + (4k^2 - 2u)v_x$$

with γ const. With $v = \phi(x,k)$ and using

$$\phi(x,t;k) = M(x,t;k)e^{-ikx},$$

M then satisfies

$$M_t = (\gamma - 4ik^3 + u_x + 2iku)M + (4k^2 - 2u)M_x$$

Also recall

$$M(x,t;k) = a(k,t)\bar{N}(x,t;k) + b(k,t)N(x,t;k)$$

IST: Time Dependence

The asymptotic behavior of $M(x,t;k)$ is given by

$$\begin{aligned} M(x,t;k) &\to 1, && \text{as} \quad x \to -\infty \\ M(x,t;k) &\to a(k,t) + b(k,t)e^{2ikx} && \text{as} \quad x \to \infty \end{aligned}$$

From

$$M_t = (\gamma - 4ik^3 + u_x + 2iku)M + (4k^2 - 2u)M_x$$

and using the fact that $u \to 0$ rapidly as $x \to \pm\infty$, find

$$\gamma - 4ik^3 = 0, \quad x \to -\infty$$

$$a_t + b_t e^{2ikx} = 8ik^3 b e^{2ikx}, \quad x \to +\infty$$

and by equating coef of e^0, e^{2ikx} find

$$a_t = 0, \qquad b_t = 8ik^3 b$$

IST: Time Dependence–con't

Solving a, b eq yields

$$a(k,t) = a(k,0), \qquad b(k,t) = b(k,0)\exp(8ik^3t) \quad \text{so}$$

$$\rho(k,t) = \frac{b(k,t)}{a(k,t)} = \rho(k,0)e^{8ik^3t}$$

$a(k_j) = 0$ implies

$$k_j = i\kappa_j = \mathrm{constant}$$

Since the e-values are const in time, this is an "isospectral flow"
Also find the time dependence of the $C_j(t)$ is given by

$$C_j(t) = C_j(0)e^{8ik_j^3t} = C_j(0)e^{8\kappa_j^3t} \qquad j = 1, \ldots J$$

IST– With Time Dependence

Thus we have the time dependence scattering data:

$\mathcal{S}(k,t) = \{\rho(k,t), \ \ \{\kappa_j, C_j(t)\} \ \ j = 1, \ldots, J\}$; with

$\rho(k,t) = \rho(k,0)e^{8ik^3t}; \kappa_j = \text{const}; C_j(t) = C_j(0)e^{8\kappa_j^3 t}, \ j = 1, \ldots J$

Next we will add time dependence to inverse problem

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\
 & & \Big\downarrow t\text{: time evolution: } \mathcal{M} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t)
\end{array}
$$

Inverse Problem: Including Poles–time included

Inegral-algebraic system with time included:

$$N(x,t;k) = e^{2ikx}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{k+i\kappa_j} + \int_{-\infty}^{\infty} \frac{\rho(\zeta,t)N(x,t;\zeta)d\zeta}{2\pi i(\zeta+k+i0)}\right\}$$

$$N_p(x,t) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{i(\kappa_p+\kappa_j)} + \int_{-\infty}^{\infty} \frac{\rho(\zeta,t)N(x,t;\zeta)d\zeta}{2\pi i(\zeta+i\kappa_p)}\right\}$$

for $p = 1, \ldots, J$

From these eq $u(x)$ is reconstructed from

$$u(x,t) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j(t)N_j(x,t) - \frac{1}{\pi}\int_{-\infty}^{\infty} \rho(\zeta,t)N(x,t;\zeta)d\zeta\right\}$$

‘Pure’ Solitons–Reflectionless Potls

‘Pure’ solitons are obtained by assuming $\rho(k,0) = 0$ ‘reflectionless’ potentials. From IST–need only the discrete contributions

$$N_p(x,t) = e^{-2\kappa_p x}\left\{1 - \sum_{j=1}^{J} \frac{C_j(t)N_j(x,t)}{i(\kappa_p + \kappa_j)}\right\}, \quad p = 1, \cdots, J$$

Above is a linear algebraic system for
$\{N_p(x,t) = N(x,t,k_p)\}, p = 1, .., J$
From these eq $u(x,t)$ is reconstructed from

$$u(x,t) = \frac{\partial}{\partial x}\left\{2\sum_{j=1}^{J} C_j(t)N_j(x,t)\right\}$$

IST–One Soliton

When there is only one ev ($J = 1$) find

$$N_1(x,t) - \frac{iC_1(0)}{2\kappa_1} e^{-2\kappa_1 x + 8\kappa_1^3 t} N_1(x,t) = e^{-2\kappa_1 x}$$

which yields $N_1(x,t)$ and $u(x,t)$:

$$N_1(x,t) = \frac{2\kappa_1 e^{-2\kappa_1 x}}{2\kappa_1 - iC_1(0) e^{-2\kappa_1 x + 8\kappa_1^3 t}}$$

$$u(x,t) = 2\frac{\partial}{\partial x}\left\{ e^{8\kappa_1^3 t} iC_1(0) N_1(x,t) \right\}$$

which leads to the familiar one soliton soln:

$$u(x,t) = 2\kappa_1^2 \operatorname{sech}^2\left\{ \kappa_1 (x - 4\kappa_1^2 t - x_1) \right\}$$

where x_1 is defined via $-iC_1(0) = 2\kappa_1 \exp(2\kappa_1 x_1)$

Conserved Quantities

May relate $a(k)$, which is a constant of motion, to an infinite number of conserved quantities from

$$a(k) = \frac{1}{2ik} W(\phi, \psi)$$
$$= \frac{1}{2ik}(\phi\psi_x - \phi_x\psi) = \lim_{x\to+\infty} \frac{1}{2ik}\left(\phi ik \mathrm{e}^{ikx} - \phi_x \mathrm{e}^{ikx}\right)$$

and developing large k expn for $\phi(x, t; k)$ as a functional of u
The first few nontrivial conserved quantities are found to be:

$$C_1 = \int_{-\infty}^{\infty} u dx, \quad C_3 = \int_{-\infty}^{\infty} u^2 dx, \quad C_5 = \int_{-\infty}^{\infty} (2u^3 - u_x^2) dx, \ldots$$

May use similar ideas to find conservation laws:

$$\partial_t T_j + \partial_x F_j = 0, \quad j = 1, 2 \ldots$$

IST–via Gel'fand-Levitan-Marchenko (GLM) Eq

The GLM eq may be derived from the RHBVP formulation
$N(x,t;k)$ is written in terms of a triangular kernel:

$$N(x,t;k) = e^{2ikx}\left\{1 + \int_x^\infty K(x,s;t)e^{ik(s-x)}ds\right\}$$

Subst above into inverse integral eq and take a FT yields

$$K(x,y;t)+F(x+y;t)+\int_x^\infty K(x,s;t)\,F(s+y;t)\,ds = 0 \qquad y \geq x$$

$$\text{where} \quad F(x;t) = \sum_{j=1}^{J}(-i)C_j(t)e^{-\kappa_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(k,t)e^{ikx}dk$$

and also find: $\quad u(x,t) = 2\partial_x K(x,x;t)$
May get soliton solns from GLM; Rigorous inverse pb:
Deift-Trubowitz ('79); Marchenko ('86); ...

Conclusions– III: IST for KdV

- Steps of IST are analogous to Fourier transforms
- Method:

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\
 & & \Big\downarrow t\text{: time evolution: } \mathcal{M} \\
u(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t)
\end{array}
$$

- Direct Scattering from $u(x,0)$ formulate a RHBVP in terms of known scattering data:
  $\mathcal{S}(k,0) = \{\rho(k,0), \ \ \{\kappa_j, C_j(0)\} \ \ j = 1, \ldots, J\}$
- Find how scattering data evolve:

  $$\rho(k,t) = \rho(k,0)e^{8ik^3t}; \quad \kappa_j = \text{const}; \quad C_j(t) = C_j(0)e^{8\kappa_j^3 t}$$
- Inverse scattering: convert RHBVP to a set of integral eq defined from above data; reconstruct $u(x,t)$

IV. IST: 2×2 Systems

Next study following 2×2 compatible systems:

$$v_x = \mathcal{L}v = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

$$v_t = \mathcal{M}v = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v$$

IST–2 $\times$ 2 Systems Direct Scattering

Recall: Soln process via IST:

$$\begin{array}{ccc} q(x,0), r(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\ & & \Big\downarrow t\text{: time evolution: } \mathcal{M} \\ q(x,t), r(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t) \end{array}$$

Summary of IST –Main Steps

- Direct scattering:
  - Define 4 efcns and scattering data

    Required scattering data at $t = 0$:
    $\mathcal{S}(k,0): \ \{\rho(k,0), \bar{\rho}(k,0); k_j, C_j(0); \bar{k}_j, \bar{C}_j(0)\}$
  - Establish analytic properties of efcns
  - Construct a RHBVP relating efcns and scattering data –from direct problem
  - Find all symmetries on scattering side

Summary of IST –Main Steps–con't

- Inverse scattering:
  - Use analyticity to reformulate RHBVP in terms of a closed system of integral-algebraic eqs
- Find time dependence of scattering data: $\mathcal{S}(k,t): \ \{\rho(k,t), \bar{\rho}(k,t); k_j, C_j(t); \bar{k}_j, \bar{C}_j(t)\}$
- Reconstruct potentials $q(x,t), r(x,t)$ from efcns via linear integral eq from RHBVP
- Special case pure solitons: $\rho(k,t) = \bar{\rho}(k,t) = 0$; inverse problem reduces to algebraic system of eq
- Can also reformulate inverse integral eq in terms of GLM eq

IST–2 $\times$ 2 Systems Direct Scattering

For

$$v_x = \ = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v$$

when $q, r \to 0$ sufficiently rapidly as $x \to \pm\infty$ the efcns are asymptotic to the solns of

$$v_x \sim \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix} v$$

Efcns–2×2 Systems

Key efcns defined by the following BCs:

$$\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \qquad \bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \qquad \text{as } x \to -\infty$$

$$\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \qquad \bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \qquad \text{as } x \to +\infty$$

Convenient to work with efcns which have const BCs at infinity:

As $x \to -\infty$:

$$M(x,k) = e^{ikx}\phi(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{M}(x,k) = e^{-ikx}\bar{\phi}(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

As $x \to \infty$:

$$N(x,k) = e^{-ikx}\psi(x,k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \bar{N}(x,k) = e^{ikx}\bar{\psi}(x,k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Wronskian and Lin Indepence of Efcns

Let $\quad u(x,k) = \left(u^{(1)}(x,k), u^{(2)}(x,k)\right)^T \qquad$ and

$\qquad v(x,k) = \left(v^{(1)}(x,k), v^{(2)}(x,k)\right)^T \qquad$ be 2 solns of $\mathcal{L}$ eq

The Wronskian of u and v is

$$W(u,v) = u^{(1)}v^{(2)} - u^{(2)}v^{(1)}$$

which satisfies

$$\frac{d}{dx}W(u,v) = 0 => W(u,v) = W_0 \quad \text{const}$$

From the asymptotic behavior of the efcns find:

$$\begin{aligned} W\left(\phi,\bar{\phi}\right) &= \lim_{x\to-\infty} W\left(\phi(x,k),\bar{\phi}(x,k)\right) = 1 \\ W\left(\psi,\bar{\psi}\right) &= \lim_{x\to+\infty} W\left(\psi(x,k),\bar{\psi}(x,k)\right) = -1 \end{aligned}$$

Thus the solns ϕ and $\bar{\phi}$ are linearly independent, as are ψ and $\bar{\psi}$

Efcns and Scattering Data

Completeness of efcns implies

$$\begin{aligned} \phi(x,k) &= b(k)\psi(x,k) + a(k)\bar{\psi}(x,k) \\ \bar{\phi}(x,k) &= \bar{a}(k)\psi(x,k) + \bar{b}(k)\bar{\psi}(x,k) \end{aligned}$$

It follows that $a(k), \bar{a}(k), b(k), \bar{b}(k)$ (scatt data) satisfy:

$$\begin{aligned} a(k) &= W(\phi,\psi), \qquad \bar{a}(k) = W(\bar{\psi},\bar{\phi}) \\ b(k) &= W(\bar{\psi},\phi), \qquad \bar{b}(k) = W\left(\bar{\phi},\psi\right) \end{aligned}$$

Efcns and Scattering Data–con't

In terms of $M, N, \bar{M}, \bar{N}$ completeness implies:

$$\begin{aligned} \frac{M(x,k)}{a(k)} &= \bar{N}(x,k) + \rho(k)e^{2ikx}N(x,k) \\ \frac{\bar{M}(x,k)}{\bar{a}(k)} &= N(x,k) + \bar{\rho}(k)e^{-2ikx}\bar{N}(x,k) \end{aligned}$$

where the *reflection coefficients* are

$$\rho(k) = b(k)/a(k), \qquad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k)$$

The above eqs will be considered as generalized Riemann-Hilbert (RH) pbs. Need analyticity–next

Analyticity of Efcns

By converting DEs to integral eq, and using Neumann series, may prove

Theorem

If $q, r \in L^1(\mathbb{R})$, then $\{M(x,k), N(x,k), a(k)\}$ are analytic functions of k for $\mathrm{Im} k > 0$ and continuous for $\mathrm{Im} k = 0$, and $\{\bar{M}(x,k), \bar{N}(x,k), \bar{a}(k)\}$ are analytic functions of k for $\mathrm{Im} k < 0$ and continuous for $\mathrm{Im} k = 0$.

Proof: Convergence of Neumann series

RHBVP-Direct Side

In terms of $M, N, \bar{M}, \bar{N}$ we found:

$$\begin{aligned} \frac{M(x,k)}{a(k)} &= \bar{N}(x,k) + \rho(k) e^{2ikx} N(x,k) \\ \frac{\bar{M}(x,k)}{\bar{a}(k)} &= N(x,k) + \bar{\rho}(k) e^{-2ikx} \bar{N}(x,k) \end{aligned}$$

where the *reflection coefficients* are

$$\rho(k) = b(k)/a(k), \qquad \bar{\rho}(k) = \bar{b}(k)/\bar{a}(k)$$

We have that $M(x,k), N(x,k), a(k)$ are analytic in UHP and $\bar{M}(x,k)\bar{N}(x,k), \bar{a}(k)$ are analytic in LHP

The above eqs can be considered as generalized RHBVP pb

Summary of IST –Next Steps

- Additional scattering data needed: eigenvalues, ‘norming const’
- Find all symmetries on scattering side
- Use analyticity to reformulate RHBVP in terms of integral eq
- Find time dependence of scattering data: $\mathcal{S}(k,t): \ \{\rho(k,t), \bar{\rho}(k,t); k_j, C_j(t); \bar{k}_j, \bar{C}(t)_j\}$
- Reconstruct potentials $q(x,t), r(x,t)$ from efcns via integral eq with t dependence from RHBVP
- Special case pure solitons: $\rho(k,t) = \bar{\rho}(k,t) = 0$; inverse problem reduces to algebraic system of eq
- Can also reformulate inverse integral eq in terms of GLM(t) eq

Required Scattering Data

Scattering data that will be needed–in general position: $\rho(k), \bar{\rho}(k)$ and information about zero's (evalues) of $a(k), \bar{a}(k)$

For general $q(x), r(x)$ **proper** e-values correspond to L^2 bound states; they are assumed simple and not on the real k axis

At: $a(k_j) = 0, k_j = \xi_j + i\eta_j, \eta_j > 0, \quad j = 1, 2, ..., J$ with

$$\phi_j(x) = b_j \psi_j(x) \;\text{ where }\; \phi_j(x) = \phi(x, k_j) \;\text{ etc}$$

Similarly at: $\bar{a}(\bar{k}_j) = 0, \bar{k}_j = \bar{\xi}_j - i\bar{\eta}_j, \; \bar{\eta}_j > 0, \quad j = 1, 2, ..., \bar{J}$ with

$$\bar{\phi}_j(x) = \bar{b}_j \bar{\psi}_j(x)$$

Required Scattering Data–con't

In terms of $M, N, \bar{M}, \bar{N}$ proper e-values correspond to

$$M_j(x) = b_j e^{2ik_jx} N_j(x)\,, \qquad \bar{M}_j(x) = \bar{b}_j\, e^{-2i\bar{k}_jx} \bar{N}_j(x)$$

For the inverse problem need 'norming const":
$C_j = b_j/a'(k_j)$, $\bar{C}_j = \bar{b}_j/\bar{a}'(\bar{k}_j)$

Scattering data that will be needed:

$$\mathcal{S}(k) = \{\rho(k), \{k_j, C_j\}, j = 1, ..., J;\ \bar{\rho}(k), \{\bar{k}_j, \bar{C}_j\}, j = 1, ..., \bar{J}\}$$

Symmetry Reductions

When $r(x) = \mp q^*(x)$:

$$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,k^*) \\ \mp N^{(1)}(x,k^*) \end{pmatrix}^* , \qquad \bar{M}(x,k) = \begin{pmatrix} \mp M^{(2)}(x,k^*) \\ M^{(1)}(x,k^*) \end{pmatrix}^*$$

$$\bar{a}(k) = a^*(k^*) , \qquad \bar{b}(k) = \mp b^*(k^*) ,$$

Thus the zeros of $a(k)$ and $\bar{a}(k)$ are paired, equal in number: $\bar{J} = J$

$$\bar{k}_j = k_j^* , \qquad \bar{b}_j = -b_j^* \qquad j = 1, \dots, J$$

Only have e-values when $r(x) = -q^*(x)$: no e-values when $r(x) = +q^*(x)$

Symmetry Reductions–con't

For $r(x) = \mp q(x)$

$$\bar{N}(x,k) = \begin{pmatrix} N^{(2)}(x,-k) \\ \mp N^{(1)}(x,-k) \end{pmatrix}, \qquad \bar{M}(x,k) = \begin{pmatrix} \mp M^{(2)}(x,-k) \\ M^{(1)}(x,-k) \end{pmatrix}$$

$$\bar{a}(k) = a(-k)\,, \qquad \bar{b}(k) = \mp b(-k)\,,$$

Thus zeros of $a(k)$ and $\bar{a}(k)$ are paired, equal in number: $\bar{J} = J$

$$\bar{k}_j = -k_j\,, \qquad \bar{b}_j = -b_j^* \qquad j = 1, \ldots, J$$

Only have e-values when $r(x) = -q(x)$ no e-values when $r(x) = +q(x)$

Since $r(x) = -q(x) \in \mathbb{R}$ satisfies $r(x) = -q(x)^*$ **both symmetry conditions hold**; so when k_j is an e-value so is $-k_j^*$; i.e. either the e-values come in pairs: $\{k_j, -k_j^*\}$ or they are pure Img

Symmetry Reductions–con't

For $r(x) = \mp q^*(-x)$

$$N(x,k) = \begin{pmatrix} \pm M^{(2)}(-x,-k^*) \\ M^{(1)}(-x,-k^*) \end{pmatrix}^*, \quad \bar{N}(x,k) = \begin{pmatrix} \pm \bar{M}^{(2)}(-x,-k^*) \\ \bar{M}^{(1)}(-x,-k^*) \end{pmatrix}^*$$

and the scattering data satisfies

$$a(k) = a^*(-k^*), \quad \bar{a}(k) = \bar{a}^*(-k^*), \quad \bar{b}(k) = \mp b^*(-k^*)$$

It follows that if $k_j = \xi_j + i\eta_j$ is a zero of $a(k)$ in UHP-k then $-k_j^* = -\xi_j + i\eta_j$ is also a zero of $a(k)$ in UHP-k etc

Inverse Problem

Recall: Soln process via IST:

$$
\begin{array}{ccc}
q(x,0), r(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\
 & & \Big\downarrow t\text{: time evolution: } \mathcal{M} \\
q(x,t), r(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t)
\end{array}
$$

– –

Operating with projection operators on the completeness relations after subtracting behavior at infinity and pole contributions

$$
\begin{aligned}
\frac{M(x,k)}{a(k)} &= \bar{N}(x,k) + \rho(k) e^{2ikx} N(x,k) \\
\frac{\bar{M}(x,k)}{\bar{a}(k)} &= N(x,k) + \bar{\rho}(k) e^{-2ikx} \bar{N}(x,k)
\end{aligned}
$$

yields integral eqs

Inverse Problem–Integral Eq

Genl $q(x), r(x)$:

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x}}{k - k_j} N_j(x) + \int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x} N(x,\zeta) d\zeta}{2\pi i(\zeta - (k - i0))}$$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2i\bar{k}_j x}}{k - \bar{k}_j} \bar{N}_j(x) - \int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}(x,\zeta) d\zeta}{2\pi i(\zeta - (k + i0))}$$

where $N_j(x) = N(x, k_j)$, $\bar{N}_j(x) = \bar{N}(x, \bar{k}_j)$ We close the system by evaluating above eq at k_p and $\bar{k}_p$

By considering large k behavior from above eq and from direct Volterra integral eq we find reconstruction formulae for $r(x), q(x)$

Inverse Problem–Reconstruction Formulae

Genl $q(x), r(x)$:

$$r(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j N_j^{(2)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \rho(\zeta) e^{2i\zeta x} N^{(2)}(x,\zeta) d\zeta$$

$$q(x) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j \bar{N}_j^{(1)}(x) + \frac{1}{\pi} \int_{-\infty}^{+\infty} \bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}^{(1)}(x,\zeta) d\zeta$$

Summary of IST –Next Steps

- Find time dependence of scattering data:
  $\mathcal{S}(k,t): \ \{\rho(k,t), \bar{\rho}(k,t); k_j, C_j(t); \bar{k}_j, \bar{C}_j(t)\}$
- Can now use integral equations from inverse side to reconstruct potentials $q(x,t), r(x,t)$
- Special case pure solitons: $\rho(k,t) = \bar{\rho}(k,t) = 0$; inverse problem reduces to algebraic system of eq
- Can also reformulate inverse integral eq in terms of GLM(t) eq

IST: Next Time Dependence

Soln process via IST:

$$
\begin{array}{ccc}
q(x,0), r(x,0) & \xrightarrow{\text{Direct Scattering}} & \mathcal{L} : \mathcal{S}(k,0) \\
 & & \Big\downarrow t\text{: time evolution: } \mathcal{M} \\
q(x,t), r(x,t) & \xleftarrow{\text{Inverse Scattering}} & \mathcal{S}(k,t)
\end{array}
$$

IST: 2×2 Time Dependence

The associated M operator determines the evolution of the efcns
Taking into account BCs $\phi(x,k,t)$ satisfies

$$\partial_t \phi = \begin{pmatrix} A - A_\infty & B \\ C & -A - A_\infty \end{pmatrix} \phi \qquad (E)$$

$$\text{where} \quad A_\infty = \lim_{|x|\to\infty} A(x,k)$$

Using completeness and evaluating $x \to \infty$:

$$\phi(x,k,t) = b(k,t)\psi(x,k,t) + a(k,t)\bar{\psi}(x,k,t) \sim \begin{pmatrix} a(t)e^{-ikx} \\ b(t)e^{ikx} \end{pmatrix}$$

Then as $x \to \infty$, (E) yields:

$$\begin{pmatrix} a_t e^{-ikx} \\ b_t e^{ikx} \end{pmatrix} = \begin{pmatrix} 0 \\ -2A_\infty b e^{ikx} \end{pmatrix}$$

IST: 2×2 Time Dependence–con't

Doing the same for $\bar{\phi}(x, k, t)$ find

$$\partial_t a = 0, \qquad \partial_t \bar{a} = 0$$
$$\partial_t b = -2A_\infty b, \qquad \partial_t \bar{b} = 2A_\infty \bar{b}$$

Thus then zero's of $a(k), \bar{a}(k)$ (evalues) $k_j, \bar{k}_j$ are const in time

For $\rho(k,t) = b(k,t)/a(k,t)$; $\bar{\rho} = \bar{b}(k,t)/\bar{a}(k,t)$:

$$\rho(k,t) = \rho(k,0)e^{-2A_\infty(k)t}, \qquad \bar{\rho}(k,t) = \bar{\rho}(k,0)e^{2A_\infty(k)t}$$

Similarly find:

$$C_j(t) = C_j(0)e^{-2A_\infty(k_j)t}, \qquad \bar{C}_j(t) = \bar{C}_j(0)e^{2A_\infty(\bar{k}_j)t}$$

In inverse problem use time dependence of scattering data ...

Solitons–Reflectionless Potls

Can obtain pure soliton solutions; for genl $q(x,t), r(x,t)$ systems IST with: $\rho = 0, \bar{\rho} = 0$ i.e. reflectionless potls; inverse prob reduces to a linear algebraic system:

$$\bar{N}_l(x,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j(t) e^{2ik_j x} N_j(x,t)}{\bar{k}_l - k_j}$$

$$N_p(x,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{m=1}^{\bar{J}} \frac{\bar{C}_m(t) e^{-2i\bar{k}_m x} \bar{N}_m(x,t)}{k_p - \bar{k}_m},$$

with reconstruction:

$$r(x,t) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j(t) N_j^{(2)}(x,t)$$

$$q(x,t) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j(t) \bar{N}_j^{(1)}(x,t)$$

One Soliton Solution–General Case

Using the time-dependence of $C_1(t)$ and symmetry:
$r(x,t) = -q(x,t)^*$

General one soliton soln:

$$q(x) = 2\eta e^{-2i\xi x + 2i\mathrm{Im}A_\infty(k_1)t - i\psi_0}\,\mathrm{sech}\left[2\left(\eta(x-x_0) + \mathrm{Re}A_\infty(k_1)t\right)\right]$$

where

$$k_1 = \xi + i\eta, \qquad C_1(0) = 2\eta e^{2\eta x_0 + i(\psi_0 + \pi/2)}$$

One Soliton Solutions-Key Eqs

Special one soliton cases:

i) NLS:
$r(x,t) = -q^*(x,t) \in \mathbb{C}, k_1 = \xi + i\eta, \quad A_\infty(k_1) = 2ik_1^2$

$$q(x,t) = 2\eta e^{-2i\xi x + 4i\left(\xi^2 - \eta^2\right)t - i\psi_0} \operatorname{sech}\left[2\eta\left(x - 4\xi t - x_0\right)\right]$$

ii) mKdV:
$r(x,t) = -q(x,t) \in \mathbb{R}, \ k_1 = i\eta, \quad A_\infty(k_1) = -4ik_1^3 = -4\eta^3$

$$q(x,t) = 2\eta \operatorname{sech}\left[2\eta\left(x - 4\eta^2 t - x_0\right)\right]$$

One Soliton Solutions-Key Eqs–con't

Special one soliton cases–con't

iii) SG:
$r(x,t) = -q(x,t) \in \mathbb{R}, \;\; k_1 = i\eta, \quad A_\infty(k_1) = \frac{i}{4k_1} = \frac{1}{4\eta}$

$$q(x,t) = -\frac{u_x}{2} = -2\eta \operatorname{sech}\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right],$$

or in terms of u, a simple 'kink':

$$u(x,t) = 4\,\tan^{-1}\exp\left[2\eta\left(x + \frac{1}{4\eta}t - x_0\right)\right]$$

One Soliton –Nonlocal NLS

Nonlocal NLS: $r(x,t) = -q^*(-x,t)$: $\quad k_1 = i\eta_1, \bar{k}_1 = -i\bar{\eta}_1$

$$C_1(t) \;=\; C_1(0)e^{+4i\eta_1^2 t} \;=\; |c|e^{i(\varphi+\pi/2)}e^{+4i\eta_1^2 t}, \qquad |c| = \eta_1 + \overline{\eta}_1$$

$$\overline{C}_1(t) \;=\; \overline{C}_1(0)e^{-4i\overline{\eta}_1^2 t} \;=\; |\overline{c}|e^{i(\overline{\varphi}+\pi/2)}e^{-4i\overline{\eta}_1^2 t}, \qquad |\overline{c}| = \eta_1 + \overline{\eta}_1$$

Find a two parameter ‘breathing’ one soliton solution

$$q(x,t) = -\frac{2(\eta_1 + \overline{\eta}_1)e^{i\overline{\varphi}}e^{-4i\overline{\eta}_1^2 t}e^{-2\overline{\eta}_1 x}}{1 + e^{i(\varphi+\overline{\varphi})}e^{4i(\eta_1^2 - \overline{\eta}_1^2)t}e^{-2(\eta_1+\overline{\eta}_1)x}}$$

Note $|c| = |\overline{c}| = \eta_1 + \overline{\eta}_1$ eigenvalues and ‘norming’ const related!

1-soliton reduces to NLS 1-soliton when $\eta_1 = \overline{\eta}_1$ and $\varphi + \overline{\varphi} = 0$

One Soliton With Symmetry–con't

Recall: two parameter 'breathing' one soliton solution

$$q(x,t) = -\frac{2(\eta_1 + \overline{\eta}_1)e^{i\overline{\varphi}}e^{-4i\overline{\eta}_1^2 t}e^{-2\overline{\eta}_1 x}}{1 + e^{i(\varphi+\overline{\varphi})}e^{4i(\eta_1^2-\overline{\eta}_1^2)t}e^{-2(\eta_1+\overline{\eta}_1)x}}$$

Note that there are singularities at $x = 0$ with:

$$1 + e^{i(\varphi+\overline{\varphi})}e^{4i(\eta_1^2-\overline{\eta}_1^2)t} = 0 \quad \text{or at}$$

$$t = t_n = \frac{(2n+1)\pi - (\varphi+\overline{\varphi})}{4(\eta_1^2 - \overline{\eta}_1^2)}\ , \qquad n \in \mathbb{Z}$$

Singularity disappears when $\eta_1 = \overline{\eta}_1$ and $\varphi + \overline{\varphi} \neq (2n+1)\pi, n = \mathbb{Z}$

Conserved quantities

$a(k, t)$ is conserved in time; it can be related to the conserved quantities. This follows from the relation

$$a(k, t) = \lim_{x \to +\infty} \phi^{(1)}(x, k; t) e^{ikx}$$

and the large k asymptotic expn for the efcn: $\phi = \left(\phi^{(1)}, \phi^{(2)}\right)^T$
The first few conserved quantities are:

$$\begin{aligned} C_1 &= -\int q(x) r(x) dx, \qquad C_2 = -\int q(x) r_x(x) dx \\ C_3 &= \int \left(q_x(x) r_x(x) + (q(x) r(x))^2 \right) dx \end{aligned}$$

Similar ideas lead to conservation laws

Conserved quantities–con't

For example, with the reductions $r = \mp q^*$ these constants of the motion can be written as

$$
\begin{aligned}
C_1 &= \pm \int |q(x)|^2 \, dx, \qquad C_2 = \pm \int q(x) q_x^*(x) dx \\
C_3 &= \int \left(\mp |q_x(x)|^2 + |q(x)|^4 \right) dx
\end{aligned}
$$

Inverse Pb–Triangular Representations: Towards GLM

For general $q(x), r(x)$:

Assuming triangular representations for $N, \bar{N}$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_x^{+\infty} K(x,s) e^{ik(s-x)} ds, \quad s > x, \quad \mathrm{Im} k \geq 0$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_x^{+\infty} \bar{K}(x,s) e^{-ik(s-x)} ds, \quad s > x, \quad \mathrm{Im} k \leq 0$$

substituting into prior integral eq and taking FTs, GLM eq follow

Inverse Problem–via GLM Eq–con't

For general $q(x), r(x)$ find for $y \geq x$

$$\bar{K}(x,y) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} F(x+y) + \int_x^{+\infty} K(x,s)F(s+y)ds = 0$$

$$K(x,y) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \bar{F}(x+y) + \int_x^{+\infty} \bar{K}(x,s)\bar{F}(s+y)ds = 0$$

where

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \rho(\xi)e^{i\xi x}d\xi - i\sum_{j=1}^{J} C_j e^{ik_j x}$$

$$\bar{F}(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \bar{\rho}(\xi)e^{-i\xi x}d\xi + i\sum_{j=1}^{\bar{J}} \bar{C}_j e^{-i\bar{k}_j x}$$

GLM: Reconstruction – Symmetry

Reconstruction for general $q(x), r(x)$

$$q(x) = -2K^{(1)}(x,x), \qquad r(x) = -2\bar{K}^{(2)}(x,x)$$

Symmetry reduces the GLM eq; with $r(x) = \mp q(x)^*$ find

$$\bar{F}(x) = \mp F^*(x), \quad \bar{K}(x,y) = \begin{pmatrix} K^{(2)}(x,y) \\ \mp K^{(1)}(x,y) \end{pmatrix}^*$$

In this case the GLM eq reduces to

$$K^{(1)}(x,y) = \pm F^*(x+y) \mp \int_x^{+\infty} ds \int_x^{+\infty} ds' K^{(1)}(x,s') F(s+s') F^*(y+s)$$

When $r(x) = \mp q(x) \in \mathbb{R}$ then $F(x)$ and $K(x,y)$ are $\in \mathbb{R}$

Finally: add time dependence of scattering data to GLM eq

Conclusion and Remarks

- Discussed in these lectures:
- Compatible linear systems–Lax Pairs–2×2 systems and extensions
- IST method–nonlinear Fourier transform
- IST associated with KdV
- IST for general q, r: 2×2 systems
- q, r systems with symmetry:
  - $r(x,t) = \mp q^*(x,t)$: NLS
  - $r(x,t) = \mp q(x,t) \in \mathbb{R}$; mKdV, SG
  - $r(x,t) = \mp q^*(-x,t)$: nonlocal NLS
- Not discussed: long time asymptotic analysis where solitons and similarity solns/Painleve fcns (e.g. for KdV/mKdV) play important roles

Conclusion and Remarks

- May also do IST for many other systems, some physically interesting
  - Higher order and more complex $1+1$d PDE evolution systems: N Wave, Boussinesq, vector NLS eq
  - Nonlocal eq such as Benjamin-Ono (BO) and Intermediate Long wave eq
  - Discrete problems: e.g. Toda lattice, discrete ladder systems, integrable discrete NLS
  - $2+1$d systems such as Kadomtsev-Petviashvili (KP), Davey-Stewartson, N Wave systems
  - In $2+1$d there are some important extensions/new ideas needed for IST: notably nonlocal RH and DBAR problems: e.g. KPI and KPII eq

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