Integrability in Discrete Differential Geometry: From DDG to the classification of discrete integrable systems

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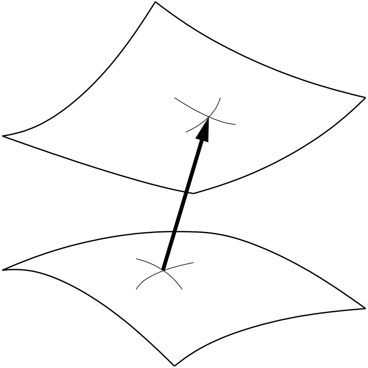
CRC 109 “Discretization in Geometry and Dynamics”

Discrete Differential Geometry

- **Aim:** Development of discrete equivalents of the geometric notions and methods of differential geometry. The latter appears then as a limit of refinements of the discretization.
- **Question:** Which discretization is the best one?
  - (Theory): preserves fundamental properties of the smooth theory
  - (Applications): represent smooth shape by a discrete shape with just few elements; best approximation

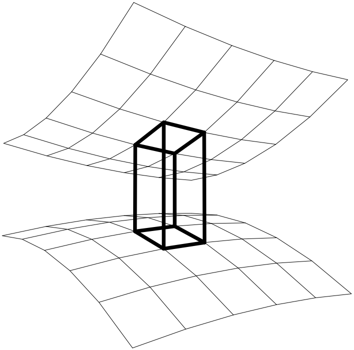
Surfaces and transformations

Classical theory of (special classes of) surfaces (constant curvature, isothermic, etc.)



special transformations (Bianchi, Bäcklund, Darboux)

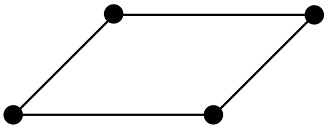
General and special Quad-surfaces



discrete $\rightarrow$ symmetric

Basic idea

Do not distinguish discrete surfaces and their transformations.
Discrete master theory.



Example - planar quadrilaterals as discrete conjugate systems.
Multidimensional Q-nets [Doliwa-Santini '97].

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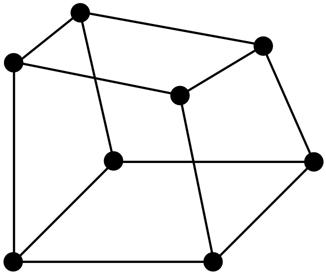
Discrete master theory.



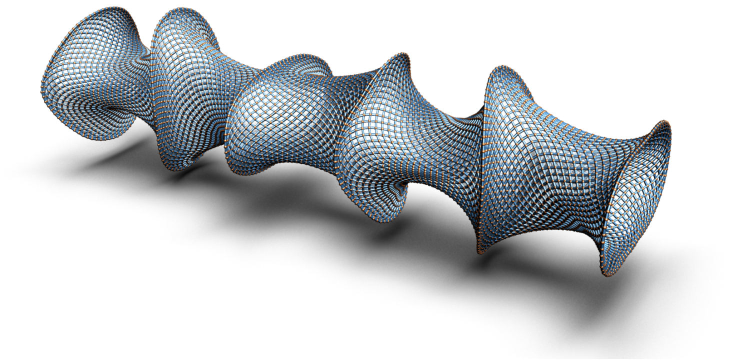
Example - planar quadrilaterals as discrete conjugate systems.
Multidimensional Q-nets [Doliwa-Santini '97].

Discretization Principles

- **Transformation Group Principle.** Smooth geometric objects and their discretizations belong to the same geometry, i.e. are invariant with respect to the same transformation group
  (discrete Klein's Erlangen Program)
- **Consistency Principle.** Discretizations of smooth parametrized geometries can be extended to multidimensional consistent nets
  (Integrability)

Multidimensional Q-nets (projective geometry) can be restricted to an arbitrary quadric ($\Rightarrow$ Discretization of classical geometries) [Doliwa '99]

Differential geometry

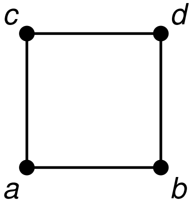


Smooth limit:

- Differential geometry follows from incidence theorems of projective geometry

Integrability as Consistency

- Equation



$$Q(a, b, c, d) = 0$$

- Consistency

Integrability as Consistency

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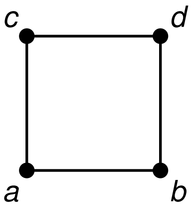

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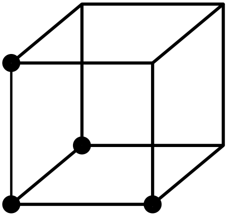

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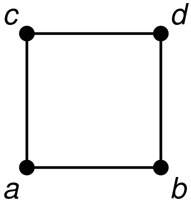

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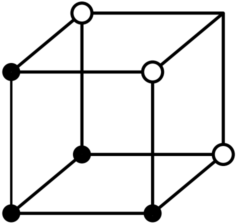

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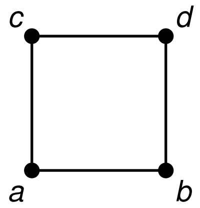

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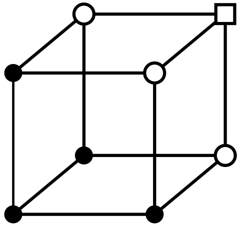


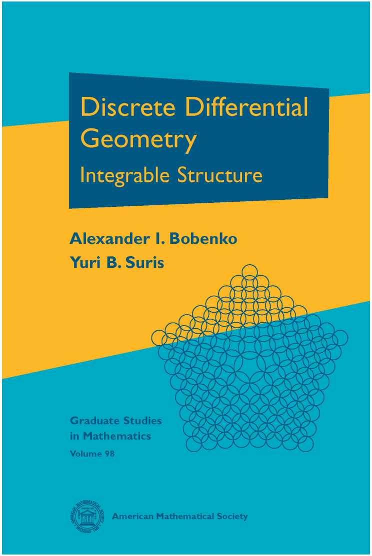
Why integrability?

Can be derived from consistency:

- Lax representation
- Bäcklund-Darboux transformations

Bobenko-Suris ['02], Nijhoff ['02]

Book in DDG

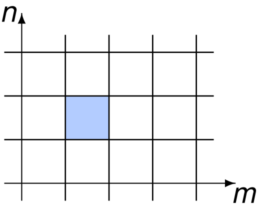


A.I. Bobenko, Y.B. Suris, Discrete differential geometry. Integrable structure. AMS, Graduate Studies in Mathematics, v.98, 2008, xxiv+404 pp.

Classification 2D

- hyperbolic nonlinear equation $Q(a, b, c, d) = 0$
- Q multi-affine (can be resolved with respect to any variable)
- Classification of integrable (i.e. consistent) equations. [Adler, B., Suris '03]

Classification. Method's overview



$$Q_{m,n}(x_{m,n}, x_{m+1,n}, x_{m,n+1}, x_{m+1,n+1}) = 0$$

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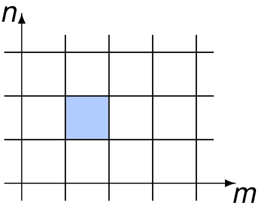
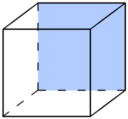


integrability =

3D-consistency



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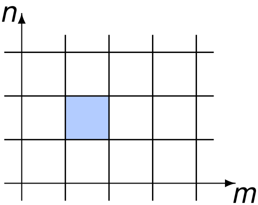

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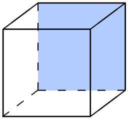

analysis of singular solutions

list of integrable equations

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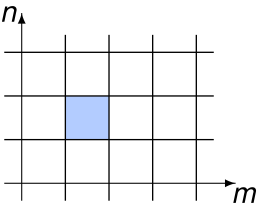

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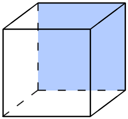

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$$Q_{m,n}(x_{m,n}, x_{m+1,n}, x_{m,n+1}, x_{m+1,n+1}) = 0$$

assumptions:

multi-affine Q

list of integrable equations

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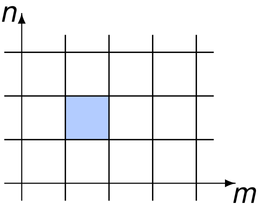

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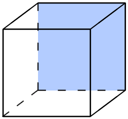

analysis of
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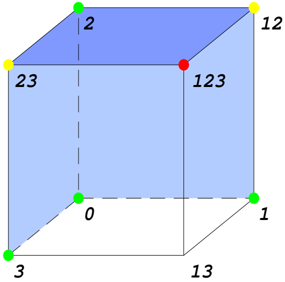
$$Q_{m,n}(x_{m,n}, x_{m+1,n}, x_{m,n+1}, x_{m+1,n+1}) = 0$$

assumptions:

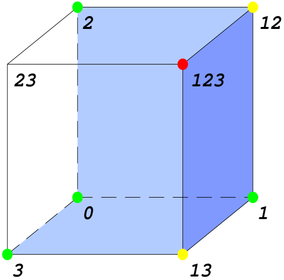
multi-affine Q
+ some nondegeneracy condition

list of integrable equations



3D-consistency: the values of x_{123} computed in 3 possible ways coincide identically on the initial values x, x_1, x_2, x_3.

3D-consistency



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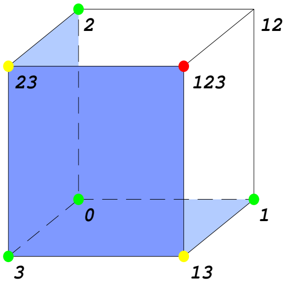


3D-consistency: the values of x_{123} computed in 3 possible ways coincide identically on the initial values x, x_1, x_2, x_3.

Singular solutions

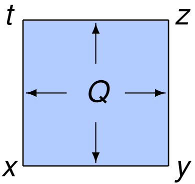
We consider only multi-affine equations (= of the first degree on each unknown):

$$Q(x, y, z, t) = a_1 xyzt + \cdots + a_{16} = 0. \tag{1}$$

The important role play biquadratic curves:

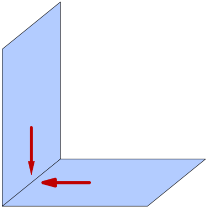
$$h(x, y) = Q_z Q_t - QQ_{zt} = h_1 x^2 y^2 + \cdots + h_9 = 0.$$

We associate such curve to each edge of the square cell



Biquadratics

The key observation is given by the following theorem.



Theorem. Let the equations be 3D-consistent and all involved biquadratics be not degenerate. Then, for each edge of the cube, the equations corresponding to adjacent faces give rise to one and the same biquadratic curve.

The nondegeneracy assumption means that a biquadratic polynomial $h(x, y)$ must be free of the factors of the form $x - \mathrm{const}$ and $y - \mathrm{const}$ $\Longrightarrow$ two types of equations.

Singular solutions

The idea of the proof.

Choose the singular initial data on the face $(1, 2)$. This leads to an undetermined value of x_{123}. However, due to consistency, x_{123} can be found without using this face. Therefore, the initial data on the faces $(1, 3)$ and $(2, 3)$ must be singular as well. Therefore, the singular curves on these faces have the same projections on the common edges.

Möbius transformations

The classification is made modulo $(PSL_2(\mathbb{C}))^8$, that is the variables in all vertices of the cube are subjected to independent Möbius transformations.

It is important that the following commutative diagram is compatible with the action of this group.

Möbius transformations

$$
\begin{array}{ccccc}
r_4(x_4) & \xleftarrow{\delta_3} & h^{34}(x_3,x_4) & \xrightarrow{\delta_4} & r_3(x_3) \\
\delta_1 \big\uparrow & & \big\uparrow \delta_{12} & & \big\uparrow \delta_2 \\
h^{14}(x_1,x_4) & \xleftarrow{\delta_{23}} & Q(x_1,x_2,x_3,x_4) & \xrightarrow{\delta_{14}} & h^{23}(x_2,x_3) \\
\delta_4 \big\downarrow & & \big\downarrow \delta_{34} & & \big\downarrow \delta_3 \\
r_1(x_1) & \xleftarrow{\delta_2} & h^{12}(x_1,x_2) & \xrightarrow{\delta_1} & r_2(x_2)
\end{array}
$$

where

$$\delta_{ij}(Q) = Q_{x_i}Q_{x_j} - QQ_{x_i,x_j}, \qquad \delta_i(h) = h_{x_i}^2 - 2hh_{x_i,x_i}.$$

List of 2D integrable equations

Theorem. Up to Möbius transformations, any 3D-consistent system with nondegenerate biquadratics is one of the following list ($\alpha = \alpha^{(i)}$, $\beta = \alpha^{(j)}$, $\mathrm{sn}(\alpha) = \mathrm{sn}(\alpha; k)$):

$$\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) = \delta\alpha\beta(\beta - \alpha) \qquad (Q_1)$$

$$\begin{aligned} &\alpha(x - x_j)(x_i - x_{ij}) - \beta(x - x_i)(x_j - x_{ij}) \\ &\quad +\alpha\beta(\alpha - \beta)(x + x_i + x_j + x_{ij}) \\ &\quad = \alpha\beta(\alpha - \beta)(\alpha^2 - \alpha\beta + \beta^2) \end{aligned} \qquad (Q_2)$$

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$$\begin{aligned}\Big(\alpha - \frac{1}{\beta}\Big)(xx_i + x_j x_{ij}) - \Big(\beta - \frac{1}{\alpha}\Big)(xx_j + x_i x_{ij}) \\ -\Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big)(xx_{ij} + x_i x_j) \\ -\frac{\delta}{4}\Big(\alpha - \frac{1}{\alpha}\Big)\Big(\beta - \frac{1}{\beta}\Big)\Big(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\Big) = 0,\end{aligned} \qquad (Q_3)$$

$$\begin{aligned}\mathrm{sn}(\alpha)\,\mathrm{sn}(\beta)\,\mathrm{sn}(\alpha-\beta)(k^2 xx_i x_j x_{ij} + 1) + \mathrm{sn}(\alpha)(xx_i + x_j x_{ij}) \\ -\mathrm{sn}(\beta)(xx_j + x_i x_{ij}) - \mathrm{sn}(\alpha-\beta)(xx_{ij} + x_i x_j) = 0.\end{aligned} \qquad (Q_4)$$

List of 2D integrable equations

$Q1$ [Quispel-Nijhoff-Capel-Van der Linden '84]

$Q2$ [Adler-Bobenko-Suris '03]

$Q3_{\delta=0}$ [Quispel-Nijhoff-Capel-Van der Linden '84]

$Q3_{\delta\neq0}$ [Adler-Bobenko-Suris '03]

$Q4$ [Adler '98]

List of 2D integrable equations

Equations with degenerated biquadratics:

$$(x - x_{ij})(x_i - x_j) = \alpha - \beta \qquad (H_1)$$

$$(x - x_{ij})(x_i - x_j) + (\beta - \alpha)(x + x_i + x_j + x_{ij}) = \alpha^2 - \beta^2 \qquad (H_2)$$

$$\alpha(xx_i + x_j x_{ij}) - \beta(xx_j + x_i x_{ij}) = \delta(\beta^2 - \alpha^2) \qquad (H_3)$$