Symmetry approach to classification of integrable systems

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Testing for integrability

- How do we test whether a given system is integrable?
- What are the integrability conditions?
- Can we describe all integrable systems of a certain type (classification problem)?
- Can we give a complete picture of all possible integrable systems of all orders (global classification)?

To answer these challenging questions we ought to decide what integrability is.

In order to classify equations we have to define the equivalence relation and ideally give a method to check whether two given equations are equivalent or not.

Testing for integrability. Various approaches to classification

- 1975 Wahlquist, Estabrook: Pseudo-potentials (a method to find Lax representations) for a given equation
- 1976 Kulish: Perturbative analysis of conservation laws.
- 1977 Ablowitz, Segur: Painlevé test for integrability
- 1979 Shabat, Sokolov, AVM, Yamilov, Svinolupov, Adler: Symmetry approach to classification of integrable PDEs and differential-difference systems
- 1980 Fokas: existence of a higher symmetry as criteria for integrability
- 1987 Hietarinta: classification of bi-linear (Hirota) representations
- 1997 Kodama, AVM: Asymptotic integrability
- 1998 Sanders, Wang: Symbolic method. Global results in classification of integrable equations
- 2002 AVM, Novikov: Perturbative symmetry approach
- 2003 Adler, Bobenko, Suris: classification of 3-D consistent integrable difference equations
- 2009 Ferapontov, Novikov, Roustemoglou, ... :Integrable deformations of hydrodynamic type systems
- 2011 AVM, Wang, Xenitidis, Garifullin, Yamilov, ...: Integrable partial difference equations.
- Algebraic entropy, singularity confinement, numerical simulations, ...

God, Thou great symmetry,
Who put a biting lust in me
From whence my sorrows spring,
For all the frittered days
That I have spent in shapeless ways
Give me one perfect thing.

Anna Wickham, 1921.

Let us consider a partial differential equation with one dependent variable u and two independent variables t, x

$$\Phi(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots) = 0 \tag{1}$$

where the lower indexes denote partial derivatives ($u_x = \partial_x u,\ u_t = \partial_t u,\ u_{tt} = \partial_t^2 u,\ u_{xt} = \partial_x \partial_t u$, etc) and function Φ depends on a finite number of arguments. We shall assume that Φ is polynomial (or, in some cases, a locally holomorphic) function of its arguments. The ring of all polynomial (or locally holomorphic) functions of variables $t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}, \ldots$ we shall denote $\mathcal{R}_0$.

$$\mathcal{R}_0 = [\mathbb{C}; x, t, u; D_x, D_t]$$

Definition

A function $g = g(t, x, u, u_t, u_x, u_{tt}, u_{tx}, \ldots) \in \mathcal{R}_0$ is a *symmetry* (a generator of a symmetry) of equation (1) if for any solution u of (1) function

$$\bar{u} = u + \varepsilon g + \mathcal{O}(\varepsilon^2)$$

satisfies equation

$$\Phi(t, x, \bar{u}, \bar{u}_t, \bar{u}_x, \bar{u}_{tt}, \bar{u}_{tx}, \bar{u}_{xx}, \ldots) = \mathcal{O}(\varepsilon^2).$$

The latter is equivalent to the following equation

$$\frac{\partial \Phi}{\partial u} g + \frac{\partial \Phi}{\partial u_x} D_x(g) + \frac{\partial \Phi}{\partial u_t} D_t(g) + \frac{\partial \Phi}{\partial u_{tt}} D_t^2(g) + \frac{\partial \Phi}{\partial u_{tx}} D_t D_x(g) \cdots = 0$$

or

$$\Phi_*(g) = 0$$

where Φ_* is the Fréchet derivative of Φ. For any $a \in \mathcal{R}_0$ the Fréchet derivative a_* is defined as linear differential operator

$$a_* = \frac{\partial a}{\partial u} + \frac{\partial a}{\partial u_x} D_x + \frac{\partial a}{\partial u_t} D_t + \frac{\partial a}{\partial u_{tt}} D_t^2 + \frac{\partial a}{\partial u_{tx}} D_t D_x \dots \ .$$

In each case the sum is finite since a has a finite number of arguments. Derivations D_x, D_t can be written in the form

$$D_x = \frac{\partial}{\partial x} + \sum_{i,j=0}^{\infty} u_{i+1,j} \frac{\partial}{\partial u_{i,j}}\,, \quad D_t = \frac{\partial}{\partial t} + \sum_{i,j=0}^{\infty} u_{i,j+1} \frac{\partial}{\partial u_{i,j}},$$

where $u_{i,j} = \frac{\partial^{i+j} u}{\partial x^i \partial t^j}$.

- Equation $\Phi_*(g) = 0$ should be satisfied on **all solutions** of $\Phi = 0$, t.e. **modulo** $\Phi = 0$ **and its differential consequences** (such as $D_x\Phi = 0$, $D_t\Phi = 0, \dots$).
- $g \neq 0$ modulo $\Phi = 0$ and its differential consequences.

In other words, we should consider a differential ideal $J_\Phi \subset \mathcal{R}_0$ generated by the element Φ

$$J_\Phi = \{ \sum_{p,q \geq 0}^{m,n} a_{p,q} D_x^p D_t^q(\Phi) \,|\, a_{p,q} \in \mathcal{R}_0,\ \ m,n \in \mathbb{Z}_{\geq 0} \}$$

and the quotient ring $\mathfrak{R}_\Phi = \mathcal{R}_0 / J_\Phi$.

A symmetry is a non-zero element $g \in \mathfrak{R}_\Phi$ such that $\Phi_*(g) = 0$ (in $\mathfrak{R}_\Phi$).

Dynamical variables. Evolutionary PDEs

Let us consider evolutionary PDEs

$$u_t = f(x, u_0, u_1, \dots, u_n). \tag{2}$$

Here we adopt notations $u_k = \partial_x^k u$. As **dynamical variables** we can take the infinite set $x, u_0, u_1, u_2 \dots$.

Any t-derivative can be re-expressed in terms of the dynamical variables and

$$\mathcal{R}_0/\langle u_t - f\rangle \simeq \mathcal{R} = [\mathbb{C}; x, t; u; D_x]$$

where $\langle u_t - f\rangle \subset \mathcal{R}_0$ is a differential ideal and D_x is reduced to

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1}\frac{\partial}{\partial u_i}; \quad D_x t = 0, D_x x = 1, \partial_x u_0 = u_1, D_x u_1 = u_2 \dots \ .$$

In $\mathcal{R}$ the derivation $D_t : \mathcal{R} \mapsto \mathcal{R}$ is reduced to

$$D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D_x^i(f)\frac{\partial}{\partial u_i}; \quad D_t t = 1, D_t x = 0, D_t u_0 = f, D_t u_1 = D_x f \dots \ .$$

Dynamical variables. Evolutionary PDEs

Equation $u_t = f(x, u_0, u_1, \dots, u_n) \iff$ two commuting derivations $[D_x, D_t] = 0 \iff$ two compatible infinite dimensional dynamical systems

$$\begin{array}{ll} D_x u_0 = u_1, & D_t u_0 = f, \\ D_x u_1 = u_2, & D_t u_1 = D_x(f), \\ \dots & \dots \\ D_x u_k = u_{k+1}, & D_t u_k = D_x^k(f), \\ \dots & \dots . \end{array}$$

Example

KdV

$$u_t = u_3 + 6uu_1 \qquad \iff$$

$$\begin{array}{ll} D_x u_0 = u_1, & D_t u_0 = u_3 + 6u_0 u_1, \\ D_x u_1 = u_2, & D_t u_1 = u_4 + 6u_1^2 + 6u_0 u_2, \\ D_x u_2 = u_3, & D_t u_2 = u_5 + 18u_1 u_2 + 6u_0 u_3, \\ \dots & \dots . \end{array}$$

Dynamical variables. Evolutionary PDEs

Definition

A derivation Y of $\mathcal{R}$ (a vector field) is called **evolutionary** if $[D_x, Y] = 0$.

Theorem

Let $Y = \sum_{i=0}^{\infty} Y_i \frac{\partial}{\partial u_i}$, $Y_k \in \mathcal{R}$ be an evolutionary derivation, then $Y_i = D_x^i Y_0$.

Proof: $[D_x, Y](u_k) = D_x(Y_k) - Y(u_{k+1}) = D_x(Y_k) - Y_{k+1} = 0 \Rightarrow Y_k = D_x^k(Y_0)$. □

Thus, an evolutionary derivation can be written as

$$D_G = \sum_{i=0}^{\infty} D_x^i(G) \frac{\partial}{\partial u_i}, \quad G \in \mathcal{R},$$

and G is called the **characteristic** of the evolutionary derivation D_G)

Theorem

Let D_G, D_H be two evolutionary derivations, then the derivation $[D_G, D_H]$ is also evolutionary with the characteristic function $K = D_G(H) - D_H(G) = H_*(G) - G_*(H)$.

With evolutionary derivation D_G we associate the infinite dimensional dynamical system $(u_k)_\tau = D_x^k(G)$ and a PDE $u_\tau = G$.

Dynamical variables. Hyperbolic (elliptic) equations

Let us consider hyperbolic (elliptic) PDEs

$$\Phi = u_{z\bar{z}} - f(z, \bar{z}, u, u_z, u_{\bar{z}}) = 0 \tag{3}$$

where z and $\bar{z}$ are two independent variables (if $z, \bar{z}$ are complex conjugated, then (3) is elliptic, if they are real, then it is a hyperbolic equation).
If u is a solution to the equations, then using the equation we can express any mixed derivative $\partial_z^k \partial_{\bar{z}}^n u$ in terms of $z, \bar{z}, u, u_z, u_{\bar{z}}, u_{zz}, u_{\bar{z}\bar{z}}, \ldots$:

$$u_{z\bar{z}} = f, \;\; u_{zz\bar{z}} = \frac{\partial f}{\partial z} + \frac{\partial f}{\partial u} u_z + \frac{\partial f}{\partial u_z} u_{zz} + \frac{\partial f}{\partial u_{\bar{z}}} f, \ldots \; .$$

Let us introduce more convenient notations

$$u_0 = \bar{u}_0 = u, \quad u_k = \frac{\partial^k u}{\partial z^k}, \quad \bar{u}_k = \frac{\partial^k u}{\partial \bar{z}^k}.$$

Let $\mathcal{R} = (\mathbb{C}; z, \bar{z}, u, u_1, \bar{u}_1, u_2, \bar{u}_2, \ldots)$ denotes a ring of (locally holomorphic) functions. In this notations z and $\bar{z}$ derivatives $D, \bar{D}$ of any $a \in \mathcal{R}$ can be written in the form

$$D(a) = \frac{\partial a}{\partial z} + u_1 \frac{\partial a}{\partial u_0} + u_2 \frac{\partial a}{\partial u_1} + \cdots + f \frac{\partial a}{\partial \bar{u}_1} + \bar{D}(f) \frac{\partial a}{\partial \bar{u}_2} + \cdots$$

$$\bar{D}(a) = \frac{\partial a}{\partial \bar{z}} + \bar{u}_1 \frac{\partial a}{\partial \bar{u}_0} + \bar{u}_2 \frac{\partial a}{\partial \bar{u}_1} + \cdots + f \frac{\partial a}{\partial u_1} + D(f) \frac{\partial a}{\partial u_2} + \cdots$$

Dynamical variables. Hyperbolic (elliptic) equations

Thus we have two derivations $D, \bar{D}$ in $\mathcal{R}$ which can be defined recursively:

$$D = \frac{\partial}{\partial z} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k} + \sum_{k=1}^{\infty} \bar{D}^{k-1}(f) \frac{\partial}{\partial \bar{u}_k}$$

$$\bar{D} = \frac{\partial}{\partial \bar{z}} + \sum_{k=0}^{\infty} \bar{u}_{k+1} \frac{\partial}{\partial \bar{u}_k} + \sum_{k=1}^{\infty} D^{k-1}(f) \frac{\partial}{\partial u_k}$$

Derivations $D, \bar{D}$ commute $[D, \bar{D}] = 0$. They correspond to two compatible infinite dimensional dynamical systems

$$D(u) = u_1,\ D(u_1) = u_2,\ D(\bar{u}_1) = f,\ D(u_2) = u_3,\ D(\bar{u}_2) = \frac{\partial f}{\partial \bar{z}} + \bar{u}_1 \frac{\partial f}{\partial u_0} + f \frac{\partial f}{\partial u_1} + \bar{u}_2 \frac{\partial f}{\partial \bar{u}_1}, \ldots$$

$$\bar{D}(u) = \bar{u}_1,\ \bar{D}(u_1) = f,\ \bar{D}(\bar{u}_1) = \bar{u}_2,\ \bar{D}(u_2) = \frac{\partial f}{\partial z} + u_1 \frac{\partial f}{\partial u_0} + u_2 \frac{\partial f}{\partial u_1} + f \frac{\partial f}{\partial \bar{u}_1},\ \bar{D}(\bar{u}_2) = \bar{u}_3, \ldots$$

Theorem

If a vector field

$$X = G \frac{\partial}{\partial u} + \sum_{k=1}^{\infty} G_k \frac{\partial}{\partial u_k} + \sum_{k=1}^{\infty} \bar{G}_k \frac{\partial}{\partial \bar{u}_k}$$

commutes with D and $\bar{D}$, then

$$G_k = D^k(G), \qquad \bar{G}_k = \bar{D}^k(G). \tag{4}$$

Proof.

Indeed, if $[D,X]=0,\ [\bar D,X]=0$ then

$$(DX-XD)(u_k)=D(G_k)-X(u_{k+1})=D(G_k)-G_{k+1}=0,\Rightarrow\ \ G_k=D^k(G),$$

$$(\bar DX-X\bar D)(\bar u_k)=\bar D(\bar G_k)-X(\bar u_{k+1})=\bar D(\bar G_k)-\bar G_{k+1}=0,\Rightarrow\ \ \bar G_k=\bar D^k(\bar G).$$

□

Conditions (4) are necessary, but not sufficient. We also need to check that $[D,X](\bar u_k)=0,\ [\bar D,X](u_k)=0$. It leads to

$$[D,X](\bar u_1)=D(\bar G_1)-X(f)=D\bar D(G)-f_*(G)=\Phi_*(G)=0.$$

The latter is nothing, but the condition that G is a generator of a symmetry for equation $\Phi=0$ (3). Thus all coefficients $G_k,\hat G_k$ of the derivation X commuting with $D,\bar D$ can be expressed in terms of a **characteristic** function $G\in\mathcal{R}$. It is natural to denote

$$D_G=G\frac{\partial}{\partial u}+\sum_{k=1}^{\infty}D^k(G)\frac{\partial}{\partial u_k}+\sum_{k=1}^{\infty}\bar D^k(G)\frac{\partial}{\partial \bar u_k}. \qquad (5)$$

Having represented a PDE $\Phi = 0$ as a compatible system of two infinite dimensional dynamical systems corresponding to derivations D_x, D_t in a certain set of dynamical variables, a symmetry can be viewed as a third infinite dimensional system, which is compatible with the first two.

Definition

We shall say that a derivation D of the ring $\mathfrak{R}_\Phi$ is a local symmetry of PDE $\Phi = 0$ if

1. $[D_x, D] = [D_t, D] = 0$;
2. $D(x) = D(t) = 0$.

In the evolutionary case $u_t = f$ these conditions mean that derivation is evolutionary $D = D_G$ and its characteristic satisfies the linearised equation

$$(D_t - f_*)G = 0.$$

The characteristic function G and the corresponding PDE $u_\tau = G$ are also often called symmetry of equation $u_t = f$.

We say that a system is **integrable** *if it possesses an infinite algebra of symmetries.*

Dynamical variables and symmetries

Non-evolutionary equations, such as the Boussinesq equation

$$u_{tt} = u_{xxxx} + (u^2)_{xx}$$

can be re-written in the form of a system of evolutionary equations as

$$u_t = v, \qquad v_t = u_{xxxx} + (u^2)_{xx},$$

with dynamical variables $V = \{u, v, u_x, v_x, u_{xx}, v_{xx}, \dots\}$. Or as

$$u_t = w_x, \qquad w_t = u_{xxx} + (u^2)_x,$$

with dynamical variables $W = \{u, w, u_x, w_x, u_{xx}, w_{xx}, \dots\}$. Or as

$$u_t = z_{xx}, \qquad z_t = u_{xx} + u^2,$$

with dynamical variables $Z = \{u, z, u_x, z_x, u_{xx}, z_{xx}, \dots\}$.

It is interesting to note that in the dynamical variables V the corresponding evolutionary system has only a finite number of symmetries, but in the variables W or Z the number of commuting is infinite.

Examples:

1. Heat equation $u_t = u_2$, symmetries

$$u_\tau = 1, \ u_{t_n} = u_n, \ u_\eta = 2tu_2 + xu_1.$$

2. The Hopf equation $u_t = uu_x$:

$$u_{t_k} = u^k u_x, \quad u_\tau = f(u, x + tu)u_1$$

3. Burgers equation $u_t = u_{xx} + 2uu_x$, symmetries

$$u_\tau = 1{+}2tu_1, \ u_\eta = 2tu_2{+}xu_1{-}u, \ u_{t_2} = u_2{+}2uu_1, \ u_{t_3} = u_3{+}3uu_2{+}3u_1^2{+}3u^2 u_1$$

4. The KdV equation $u_t = u_{xxx} + 6uu_x$. A few symmetries

$$u_\tau = 1{+}6tu_1, \ u_\eta = 3t(u_3{+}6uu_1){+}xu_1{-}2u, \ u_{t_5} = u_5{+}10u_3 u{+}20u_1 u_2{+}30u^2 u_1, \ldots$$

5. Sine-Gordon equation $u_{z\bar{z}} = \sin u$:

$$u_{\tau_3} = u_3 + \frac{1}{2}u_1^3, \ u_{\tau_5} = u_5 + \frac{5}{2}u_1^2 u_3 + \frac{5}{2}u_1 u_2^2 + \frac{3}{8}u_1^5, \ldots$$

Symmetry reductions

Having a symmetry we can find symmetry reduction: restrict on invariant solutions.
Let G be a generator of a symmetry, then condition $G = 0$ consistent with the equation and leads to symmetry reduction from PDE to a finite system of ODEs.
Example: KdV $u_t = u_3 + 6uu_1$. Let us take a symmetry with a generator

$$G = au_1 + b(u_3 + 6uu_1) + (u_5 + 10u_3 u + 20u_1 u_2 + 30u^2 u_1), \qquad a, b \in \mathbb{R}.$$

Setting $G = 0$ we get an ODE of 5th order. We can express $u_5, u_6, \dots$ in terms of dynamical variables u, u_1, u_2, u_3, u_4 and reduce the infinite dimensional system

$$\begin{aligned} D_x u_0 &= u_1, & D_t u_0 &= u_3 + 6u_0 u_1, \\ D_x u_1 &= u_2, & D_t u_1 &= u_4 + 6u_1^2 + 6u_0 u_2, \\ D_x u_2 &= u_3, & D_t u_2 &= u_5 + 18u_1 u_2 + 6u_0 u_3, \\ &\cdots & &\cdots . \end{aligned}$$

to two compatible systems of order 5. According S.P.Novikov, its solution can be expressed in theta functions for genus 2 algebraic curve. Degeneration of the curve leads to 2-soliton solutions.

For a PDE $\Phi(x, t, u, u_x, u_t, u_{xx}, \dots) = 0$ a local conservation law is defined as a pair of two functions $\rho(x, t, u, u_x, u_t, \dots), \sigma(x, t, u, u_x, u_t, \dots)$ satisfying equation

$$\partial_t \rho(x, t, u, u_x, u_t, \dots) = \partial_x \sigma(x, t, u, u_x, u_t, \dots)$$

on all solutions of the PDE. Functions ρ and σ are called **density** and **flux** of a local conservation law.

Having a conservation law we can find constant of motion (analogs of first integrals in the ODE case).

Example: The KdV equation $u_t = u_{xxx} + 6uu_x$ is itself a conservation law with $\rho_1 = u$. It is easy to check that $\rho_2 = u^2, \rho_3 = 2u^3 - u_1^2$

$$\frac{\partial}{\partial t} u = \frac{\partial}{\partial x}(u_2 + 3u^2), \qquad \frac{\partial}{\partial t} u^2 = \frac{\partial}{\partial x}(2uu_2 - u_1^2 + 4u^3),$$
$$\frac{\partial}{\partial t}(2u^3 - u_1^2) = \frac{\partial}{\partial x}(9u^4 + 6u^2 u_2 + u_2^2 - 12uu_1^2 - 2u_1 u_3),$$

If, for example vanishes rapidly $u(x, t) \to 0$ as $x \to \pm\infty$, so that $\int_{-\infty}^{\infty} \rho_k \, dx$ converges, then

$$\frac{d}{dt} \int_{-\infty}^{\infty} \rho_k \, dx = 0.$$

In dynamical variables and our derivations D_x, D_t associated with the PDE our (pre)-definition of local conservation laws is a pair $\sigma, \rho \in \mathcal{R}$, such that

$$D_t(\rho) = D_x(\sigma). \tag{6}$$

If we take any element $h \in \mathcal{R}$ and define $\rho = D_x(h)$ then condition (6) will be satisfied in a trivial way with $\sigma = D_t(\rho)$. Such densities we call trivial.

Definition

Considering $\mathcal{R}$ as a linear space over the base field $\mathbb{C}$ we say that

1. Two elements $r_1, r_2 \in \mathcal{R}$ are equivalent $r_1 \sim r_2$ if $r_1 - r_2 \in D_x(\mathcal{R})$.
2. A quotient linear space $\hat{\mathcal{R}} = \mathcal{R}/_\sim$ is called a space of functionals.
3. Elements of $\hat{\mathcal{R}}$ are called densities.
4. A non-zero element $\rho \in \hat{\mathcal{R}}$ is called a density of a local conservation law (or simply a conserved density) if $D_t(\rho) = D_x(\sigma)$ for some $\sigma \in \mathcal{R}$ which is called a flux of the conservation law.

Evolutionary case

In what follows

- we shall consider evolutionary equations $u_t = F(x, u, u_1, \ldots, u_n)$ only,
- we shall assume that all functions do not depend on the variable t explicitly and thus $\mathcal{R} = [\mathbb{C}; x; u; D_x]$,
- the differential field of fractions, corresponding to $\mathcal{R}$ will be denoted as $\mathcal{F}$.

In this case the derivation D_x is quite simple

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}$$

and $\mathrm{Ker}(D_x) = \mathbb{C}$.

For equation $u_t = F(x, u, u_1, \ldots, u_n),\ n > 1$ we will find a sequence of necessary conditions for the existence of local symmetries.

In order to proceed we need some facts from the theory of differential operators and formal series.

Let $\mathcal{F}[D_x]$ be the algebra of differential operators over $\mathcal{F}$. Elements $A \in \mathcal{F}[D_x]$ are of the form

$$A = A_N D_x^N + A_{N-1} D_x^{N-1} + \cdots + A_1 D_x + A_0, \qquad A_N \neq 0, \;\; A_n \in \mathcal{F}, \;\; N \in \mathbb{Z}_{\geq 0}$$

and N is called the order of the operator $N = \mathrm{ord} A$. The coefficient A_N is called the leading coefficient $A_N = \mathrm{Lc}(A)$.

The Fréchet derivative

$$a_* = \sum_{k=0}^{M} \frac{\partial a}{\partial u_k} D_x^k \in \mathcal{F}[D_x], \qquad a = a(x, u, u_1, \ldots, u_M) \in \mathcal{F}, \;\; \partial_{u_M} a \neq 0$$

is an example of a differential operator of order M.

The **order** $|a|$ of an element $a \in \mathcal{F}$ is defined as $|a| = \mathrm{ord} a_*$.

Ring of differential operators.

The multiplication in $\mathcal{F}[D_x]$ is given by the Leibniz rule

$$aD_x^k \circ bD^n = \sum_{s=0}^{k} \left(\begin{array}{c} k \\ s \end{array} \right) aD_x^s(b)D_x^{n+k-s}, \qquad k \in \mathbb{N},$$

$$\left(\begin{array}{c} k \\ s \end{array} \right) = \frac{k(k-1)\cdots(k-s+1)}{s!}$$

This multiplication is **associative**, but **not commutative**.

Let A be a differential operator

$$A = \sum_{k=0}^{N} a_k D_x^k$$

then a **conjugated** operator A^+ is defined as

$$A^+ = \sum_{k=0}^{N} (-1)^k D_x^k \circ a_k = \sum_{k=0}^{N} (-1)^k \sum_{s=0}^{k} \left(\begin{array}{c} k \\ s \end{array} \right) D_x^s(a) D_x^{k-s}.$$

This conjugation is an involution in algebra $\mathcal{F}[D_x]$: $(A \circ B)^+ = B^+ \circ A^+$.

Definition

A **variational derivative** $\delta_u(a),\ a \in \mathcal{F}$ is

$$\delta_u(a) = a_*^+(1) = \sum_{k=0}^{|a|} (-1)^k D_x^k \left(\frac{\partial a}{\partial u_k} \right).$$

The linear operator $\delta_u := E$ is called the **Euler operator**.

Theorem

1. If $a \in D_x\mathcal{F}$ then $\delta_u(a) = 0$.
2. If $a(x, u, u_1, \ldots, u_k) \in \mathcal{F}$ is holomorphic in a neighbourhood of the point $(x, 0, \ldots, 0)$ and $\delta_u(a) = 0$, then there exists $b \in \mathcal{F}$ such that $D_x b = a$.

If a is not holomorphic at $u = 0$, for instance $a = u_1 u^{-1}$, but $\delta_u(a) = 0$, then a solution of equation $D_x b = a$ can be found in an extension of $\mathcal{F}$. In this example $b = \log u$.

Definition

The **order of a conserved density** $\mathrm{ord}\rho$ is defined as $\mathrm{ord}\rho = \deg(\delta_u \rho)_*$.

It is invariant definition of the order of a conserved density in a sense that if $\rho_1 \sim \rho_2$, then $\mathrm{ord}\rho_1 = \mathrm{ord}\rho_2$.

Let us list a few useful identities concerning Fréchet and variational derivatives.

Theorem

Let $a, b \in \mathcal{F}$, then

1. $(ab)_* = ab_* + ba_*$,
2. $(D_x(a))_* = D_x \circ a_* = D_x(a_*) + a_* \circ D_x$,
3. $(D_b(a))_* = (a_*(b))_* = D_b(a_*) + a_* \circ b_*$,
4. $(\delta_u a)_* = (\delta_u a)_*^+$,
5. $\delta_u(D_b(a)) = D_b(\delta_u a) + b_*^+(\delta_u a)$.

Ring of differential operators.

Corollary

Let ρ be a density of a conservation law of an evolutionary equation $u_t = F$, then

$$(D_t + F_*^+)\delta_u \rho = 0\,.$$

Definition

Any non-zero solution $\gamma \in \mathcal{F}$ of equation

$$(D_t + F_*^+)\gamma = 0$$

is called a **co-symmetry** of the equation $u_t = F$.

Theorem

Let $\gamma,\ |\gamma| = n$ be a co-symmetry of equation $u_t = F$ such that $\gamma_* = \gamma_*^+$. Then there exists $\rho \in \mathcal{F}$ such that $\gamma = \delta_u \rho$ and $\delta_u D_t(\rho) = 0$.

Moreover

$$\rho = u \int_0^1 \gamma(x, \xi u, \xi u_1, \ldots, \xi u_n)\, d\xi,$$

if the integral converges.

Skewfield of formal series.

For further consideration we will need formal pseudo-differential series, which for simplicity we shall call formal series (of order $N = \mathrm{ord} A \in \mathbb{Z}$)

$$A = a_N D_x^N + a_{N-1} D_x^{N-1} + \cdots + a_1 D_x + a_0 + a_{-1} D_x^{-1} + \cdots , \qquad a_N \neq 0, \quad a_n \in \mathcal{F}.$$

$$\text{or} \qquad A = \sum_{i=-\infty}^{N} a_i D_x^i \, .$$

The coefficient a_N is called the **leading coefficient**, $\mathrm{Lc}(A) = a_N$.
Multiplication is defined exactly in the same way as for differential operators

$$aD_x^k \circ bD^n = \sum_{s=0}^{\infty} \binom{k}{s} aD_x^s(b) D_x^{n+k-s}, \qquad k \in \mathbb{Z},$$

$$\binom{k}{s} = \frac{k(k-1)\cdots(k-s+1)}{s!},$$

but now we allow index k to be negative. If k is a positive integer, then the sum is finite. For negative k the sum is infinite (formal).
Example:

$$D_x^{-1} \circ a = aD_x^{-1} - D_x(a) D_x^{-2} + D_x^2(a) D_x^{-3} + \cdots .$$

Theorem

A set of all formal series with coefficients in the differential field $\mathcal{F}$

$$\mathcal{F}((D_x)) = \{ \sum_{i=-\infty}^{N} a_i D_x^i \mid a_n \in \mathcal{F}, \; N \in \mathbb{Z} \}$$

form a skewfield.

Proof.

We need to show that any non-zero element of $\mathcal{F}((D_x))$ is invertible. Indeed, if

$$A = \sum_{i=-\infty}^{N} a_i D_x^i = a_N \left(1 + \sum_{n=-\infty}^{-1} a_N^{-1} a_{n+N} D_x^n \right) D_x^N$$

then

$$A^{-1} = D_x^{-N} \circ \sum_{k=0}^{\infty} \left(- \sum_{n=-\infty}^{-1} a_N^{-1} a_{n+N} D_x^n \right)^k \circ a_N^{-1}.$$

Obviously in $A^{-1} = b_{-N} D_x^{-N} + b_{-N} D_x^{-N} + \cdots$ each coefficient b_k is a finite sum. Moreover b_{-N-k} is a differential polynomial of the coefficients $a_N, \ldots, a_{N-k}$. □

In particular any differential operator $A \in \mathcal{F}[D_x]$ is invertible and its inverse $A^{-1} \in \mathcal{F}((D_x))$.

Skewfield of formal series.

Example: For series

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots$$

we can find uniquely the inverse element

$$B = b_{-m} D_x^{-m} + b_{-m-1} D_x^{-m-1} + \cdots , \quad b_k \in \mathcal{F}$$

such that $A \circ B = B \circ A = 1$. Indeed, multiplying A and B and equating the result to 1 we find that $a_m b_{-m} = 1$, i.e. $b_{-m} = 1/a_m$, then at D_x^{-1} we have

$$m a_m D_x(b_{-m}) + a_m b_{-m-1} + a_{m-1} b_{-m} = 0$$

and therefore

$$b_{-m-1} = -\frac{a_{m-1}}{a_m^2} - m D_x\left(\frac{1}{a_m}\right), \quad \text{etc.}$$

First k coefficients of the series B can be uniquely determined in terms of the first k coefficients of A.

Moreover, if $(a_m)^{\frac{1}{m}} \in \mathcal{F}$ we can find the m-th root of the series A, i.e. a series

$$C = c_1 D_x + c_0 + c_{-1} D_x^{-1} + c_{-2} D_x^{-2} + \cdots$$

such that $C^m = A$ and if we know first k coefficients of the series A we can find the first k coefficients of the series C.

Example. Let $A = D_x^2 + u$. Assuming

$$C = c_1 D_x + c_0 + c_{-1} D_x^{-1} + c_{-2} D_x^{-2} + \cdots$$

we compute

$$C^2 = C \circ C = c_1^2 D_x^2 + (c_1 D_x(c_1) + c_1 c_0 + c_0 c_1) D_x +$$

$$c_1 D_x(c_0) + c_0^2 + c_1 c_{-1} + c_{-1} c_1 + \cdots ,$$

and compare the result with A. At D_x^2 we find $c_1^2 = 1$ or $c_1 = \pm 1$. Let $c_1 = 1$. At D_x we get $2c_0 = 0$, i.e. $c_0 = 0$. At D_x^0 we have $2c_{-1} = u$, at D_x^{-1} we find $c_{-2} = -u_1/4$, etc.,

$$C = A^{1/2} = D_x + \frac{u}{2} D_x^{-1} - \frac{u_1}{4} D_x^{-2} + \cdots .$$

We can easily find as many coefficients of C as required.

Definition. The **residue** of a formal series $A = \sum_{k \leq n} a_k D_x^k,\ a_k \in \mathcal{F}$ is by definition the coefficient at D_x^{-1}:

$$\mathrm{res}\,(A) = a_{-1}\,.$$

The **logarithmic residue** of A is defined as

$$\mathrm{res}\log A = \frac{a_{n-1}}{a_n}\,.$$

For a formal series

$$A = a_m D_x^m + a_{m-1} D_x^{m-1} + \cdots + a_0 + a_{-1} D_x^{-1} + \cdots$$

First k residues

$$r_{-1} = \ \mathrm{res}\,A^{-\frac{1}{m}},\ r_0 = \ \mathrm{res}\ \log A,\ r_1 = \ \mathrm{res}\,A^{\frac{1}{m}},\ r_2 = \ \mathrm{res}\,A^{\frac{2}{m}},\ \ldots r_{k-2} = \ \mathrm{res}\,A^{\frac{k-2}{m}}$$

can be expressed in terms of first k coefficients of the series and vise versa.

Theorem

For any two formal series A, B of order n and m, respectively, the logarithmic residue satisfies the following identity:

$$\operatorname{res}\log(A \circ B) = \operatorname{res}\log(A) + \operatorname{res}\log(B) + nD_x(\log(b_m)).$$

For any derivation D_G of the differential field $\mathcal{F}$ and any formal series A we have

$$D_G(\operatorname{res}\log A) = \operatorname{res}(D_G(A) \circ A^{-1}).$$

We will use the following important Adler's

Theorem.(M.Adler) For any two formal series A, B the residue of the commutator belongs to $\operatorname{Im} D_x$:

$$\operatorname{res}[A, B] = D_x(\sigma(A, B)),$$

where

$$\sigma(A, B) = \sum_{p \leq \operatorname{ord}(B),\ q \leq \operatorname{ord}(A)}^{p+q+1>0} \binom{p+q+1}{q} \sum_{s=0}^{p+q} (-1)^s D_x^s(a_q) D_x^{p+q-s}(b_q)\,.$$

Formal recursion operator

Let me recall that according my definition, integrable equations are equations possessing higher local symmetries and/or conservation laws.

We are going to show that for an evolutionary differential equation

$$u_t = F, \qquad F = F(x, u, u_1, \ldots, u_n) \in \mathcal{F}, \ |F| = n > 1, \tag{7}$$

existence of a symmetry $u_\tau = G(x, u, \ldots, u_m)$ of high order $m = |G| > n$ implies existence of an **"approximate"** solution $R \in \mathcal{F}((D_x))$ of the formal operator equation

$$D_F(R) - [F_*, R] = 0, \tag{8}$$

while existence of symmetries of arbitrary high order guarantee existence of a formal series R satisfying equation (8). We shall also show that existence of two high order local conserve densities ρ_1, ρ_2 also implies existence of an approximate solution of equation (8). As well, if equation (7) is linearisable by a differential substitution, then it admits a formal recursion operator.

Thus, conditions of solvability for equation (8) will provide us with necessary integrability conditions for equation (7). These conditions will lead us to a **canonical** sequence of local conserved densities for equation (7), and the first one is of the form

$$\rho_{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-\frac{1}{n}} .$$

Ultimately we aim to answer the questions: **whether a given equation is integrable** and what is a complete list of integrable equations?

The name **formal recursion operator** is motivated by the concept of **recursion operators**, which are pseudo-differential operators R satisfying equation $D_F(R) - [F_*, R] = 0$. Such operators do exist (AKNS, Lenard, Olver):

Burgers eq.	$u_t = u_2 + 2uu_1,$	$R_{Bur} = D_x + u + u_1 D_x^{-1},$
KdV eq.	$u_t = u_3 + 6uu_1,$	$R_{KdV} = D_x^2 + 4u + 2u_1 D_x^{-1},$
Sawada-Kotera eq.	$u_t = u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1,$	$R_{SK} = D_x^6 + 6uD_x^4 +$

$$+9u_1 D_x^3 + (9u^2 + 11u_2)D_x^2 + (10u_3 + 21uu_1)D_x + 5u_4 + 16uu_2 + 6u_1^2 + 4u^3 + \\ +(u_5 + 5uu_3 + 5u_1u_2 + 5u^2u_1)D_x^{-1} + u_1 D_x^{-1} \circ (2u_2 + u^2).$$

Theorem

Let R be a recursion operator for $u_t = F$ and G be a symmetry, such that $R(G) \in \mathcal{F}$. Then $R(G)$ is also a symmetry.

Proof. Indeed, the Lie bracket $[F, R(G)] = D_F(R(G)) - D_{R(G)}(F) =$

$$D_F(R)(G) + RD_F(G) - F_*R(G) = D_F(R)(G) + R([F,G]) + RD_G(F) - F_*R(G) =$$

$$(D_F(R) + RF_* - F_*R)(G) + R([F,G]) = 0. \qquad \square$$

Applying R^k to a seed symmetry G_1 we can, in principle, construct an infinite sequence of symmetries $G_n = R(G_{n-1})$.

Co-recursion operator

Theorem

Let R be a recursion operator for equation $u_t = F$ and γ be its co-symmetry such that $R^+(\gamma) \in \mathcal{F}$, then $R^+\gamma$ is also a co-symmetry of the equation.

In this sense R^+ is a **co-recursion** operator.

Proof. Indeed, $D_F(\gamma) = -F_*^+(\gamma)$ and thus,

$$D_F(R^+\gamma) = D_F(R^+)\gamma + R^+ D_F(\gamma) = -F_*^+ R^+(\gamma) + R^+ F_*^+(\gamma) - R^+ F_*^+(\gamma) = -F_*^+(R^+\gamma). \square$$

Example

For the KdV equation $R^+ = D_x^2 + 4u - 2D_x^{-1} \circ u_1$. Taking $\gamma_0 = \frac{1}{2}$ we get $\gamma_k = (R^+)^k \gamma_0$:

$$\gamma_1 = u,\ \gamma_2 = u_2 + 3u^2,\ \gamma_3 = u_4 + 10u_2 u + 5u_1^2 + 10u^3, \ldots\ .$$

There is the issue of locality, i.e. to show that $R^+(\gamma_k) \in \mathcal{F}$ which in this case is equivalent to $(\gamma_k)_* = (\gamma_k)_*^+$.

Formal recursion operator

Formal recursion operators, i.e. formal series satisfying equation

$$D_F(R) - [F_*, R] = 0.$$

form an algebra, which we will denote $\mathfrak{R}(F)$:

Theorem

Let $R_1, R_2 \in \mathfrak{R}(F)$, $\operatorname{ord}(R_1) = n \neq 0$. Then

- $\alpha_1 R_1 + \alpha_2 R_2 \in \mathfrak{R}(F), \qquad \alpha_1, \alpha_2 \in \mathbb{C}$,
- $R_1 \circ R_2 \in \mathfrak{R}(F)$,
- $R_1^{\frac{k}{n}} \in \mathfrak{R}(F)$, $k \in \mathbb{Z}$.

Moreover,

Theorem

Let $R \in \mathfrak{R}(F)$, $\operatorname{ord} R = m \neq 0$. Then $\mathfrak{R}(F) = \mathbb{C}((R^{-\frac{1}{m}}))$.

Meaning: If a formal series $\hat{R} \in \mathfrak{R}(F)$, $\operatorname{ord} \hat{R} = k$, then

$$\hat{R} = \sum_{i=-\infty}^{k} \alpha_i R^{\frac{i}{m}}, \qquad \alpha_i \in \mathbb{C}.$$

Lemma

Let $u_t = F(x, u, \dots, u_n)$, $|F| = n \geqslant 2$ *and* $R = r_m D_x^m + \dots \in \mathfrak{R}(F)$, $ord R = m$. *Then*

$$r_m = \beta_m \left(\frac{\partial F}{\partial u_n}\right)^{\frac{m}{n}}, \qquad \beta_m \in \mathbb{C}\,.$$

Proof. The leading term in $D_F(R) - [F_*, R]$ is

$$\left(-n\left(\frac{\partial F}{\partial u_n}\right) D_x(r_m) + m r_m D_x\left(\frac{\partial F}{\partial u_n}\right)\right) D_x^{n+m-1}.$$

Here we use the condition that $n \geqslant 2$, in this case $\mathrm{ord}\, D_F(R) \leqslant m < n + m - 1$ and this term does not contribute in the equation for the coefficient at D_x^{n+m-1}. Now it is obvious that

$$-n\left(\frac{\partial F}{\partial u_n}\right) D_x(r_m) + m r_m D_x\left(\frac{\partial F}{\partial u_n}\right) = 0 \Longleftrightarrow r_m = \beta_m \left(\frac{\partial F}{\partial u_n}\right)^{\frac{m}{n}}, \qquad \beta_m \in \mathbb{C}\,.\square$$

Canonical densities

For a formal series $R \in \mathcal{F}((D_x))$, $\operatorname{ord}(R) = m \neq 0$ the sequence of canonical densities $\rho_{-1}, \rho_0, \rho_1, \dots$ is defined as

$$\rho_{-1} = \operatorname{res} R^{-\frac{1}{m}} = r_m^{-\frac{1}{m}},$$
$$\rho_0 = \operatorname{res} \log R = \frac{r_{m-1}}{r_m},$$
$$\rho_k = \operatorname{res} R^{\frac{k}{m}}, \qquad k \in \mathbb{N}.$$

Theorem

If R, $\operatorname{ord}(R) = m \neq 0$ is a formal recursion operator ($R \in \mathfrak{R}(F)$), then canonical densities are densities of local conservation laws for equation $u_t = F$

$$D_F(\rho_i) \in D_x(\mathcal{F}), \qquad i = -1, 0, 1, 2, \dots \ .$$

Proof. It follows from Adler's Theorem that

$$D_F \rho_k = D_F \operatorname{res} R^{\frac{k}{m}} = \operatorname{res} D_F R^{\frac{k}{m}} = \operatorname{res} [F_*, R^{\frac{k}{m}}] = D_x \sigma(F_*, R^{\frac{k}{m}}), \qquad k \neq 0$$

.

$$D_F(\rho_0) = D_F \operatorname{res} \log R = \operatorname{res} (D_F(R) R^{-1}) = \operatorname{res} ([F_* R^{-1}, R]) = D_x \sigma(F_* R^{-1}, R). \ \square$$

Formal recursion operator

If there exists a formal recursion operator for equation $u_t = F(x, u, \dots, u_n)$, $|F| = n \geqslant 2$ then $\rho_{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-\frac{1}{n}}$ must be a density of a conservation law.

Example

It is known that equation $u_t = u^n u_n$ is integrable and possesses a recursion operator for $n = 2, 3$. Does it possess a formal recursion operator for $n > 3$? In this case $\rho_1 = u^{-1}$ and we have to verify that

$$(u^{-1})_t = -u^{n-2} u_n \in D_x(\mathcal{F}).$$

Taking the variational derivative we observe that

$$\delta_u(u^{n-2} u_n) = (-1)^n D_x^n(u^{n-2}) + (n-2) u^{n-3} u_n$$

is zero for $n = 2, 3$ and different from zero for $n > 3$. Conclusion: for $n > 3$ this equation does not possess a formal recursion operator.

Example

- For Burgers equation: $\rho_{-1} = 1, \rho_0 = 0, \rho_k = D_x((D_x + u)^{k-1} u) \in D_x(\mathcal{F})$.
- For the KdV equation: $\rho_{-1} = 1, \rho_0 = 0, \rho_1 = 2u, \rho_2 = 2u_1, \rho_3 = 2u_2 + u^2, \dots$

Now we are going to discuss **approximate** solutions of the equation

$$D_F(R) - [F_*, R] = 0, \tag{9}$$

in terms of formal series ($R \in \mathcal{F}((D_x))$).

Definition. A set of k-approximate solutions of the equation (9) is defined as

$$\mathfrak{R}_k = \{A \in \mathcal{F}((D_x)) \,|\, \mathrm{ord}(D_F(A) - [F_*, A]) \leqslant \mathrm{ord} F_* + \mathrm{ord} A - k\} .$$

It is clear that

$$\mathcal{F}((D_x)) = \mathfrak{R}_1 \supset \mathfrak{R}_2 \supset \mathfrak{R}_3 \supset \cdots \supset \mathfrak{R}_\infty = \mathfrak{R}(F).$$

Lemma

A formal series R, $\ \mathrm{ord} R = n \neq 0$ belongs to $\mathfrak{R}_k$ if and only if $R^{\frac{m}{n}} \in \mathfrak{R}_k,\ \ m \in \mathbb{Z}$.

Theorem

Suppose equation $u_t = F$ has a symmetry $G \in \mathcal{F}$ of order k, then $G_* \in \mathfrak{R}_k$.

Proof. Taking the Fréchet derivative of the Lie bracket $[F, G] = 0$ we get

$$(D_F(G) - D_G(F))_* = D_F(G_*) - [F_*, G_*] - D_G(F_*) = 0$$

and thus

$$\operatorname{ord}(D_F(G_*) - [F_*, G_*]) = \operatorname{ord} D_G(F_*) \leqslant \operatorname{ord} F_* = \operatorname{ord} F_* + \operatorname{ord} G_* - k\,. \quad \square$$

We can take a fraction power $G_*^{\frac{1}{k}}$ in order to obtain an approximate recursion operator of order 1, which also belongs to $\mathfrak{R}_k$ due to the Lemma.

Corollary

If equation $u_t = F$ admits symmetries of arbitrary high order, then there exist a formal recursion operator $R \in \mathcal{F}((D_x))$ of any fixed order N satisfying equation

$$D_F(R) = [F_*, R]\,.$$

Formal recursion operator

For equation $u_t = F$ of order n, since F does not depend on t explicitly, F is a symmetry of order n and thus $F_* \in \mathfrak{R}_n$. Thus we know first $n-1$ coefficients of R and the same number of canonical conserved densities

$$\rho_{-1} = \text{res}\, F_*^{-\frac{1}{n}}, \rho_0 = \text{res}\log F_*, \rho_1 = \text{res}\, F_*^{\frac{1}{n}}, \ \ldots\ , \rho_{n-3} = \text{res}\, F_*^{\frac{n-3}{n}}.$$

Theorem

- *Any equation $u_t = F$ of order $n \geqslant 2$ has an approximate formal recursion operator $R \in \mathfrak{R}_n$.*
- *It has an approximate recursion operator with $k > n$ if and only if*

$$D_F(\rho_i) = D_x(\sigma_i),\ \sigma_i \in \mathcal{F}, \qquad i = -1, 0, \ldots, k-n-2.$$

- *Canonical densities $\rho_i,\ i \geqslant n-2$ can be found explicitly in terms of the coefficients $\frac{\partial F}{\partial u_j},\ j = 0, 1, \ldots, n$ of the Fréchet derivative F_* and $\sigma_{-1}, \ldots \sigma_{i-n+1}$.*

Example

Let $u_t = F(x,u,u_1,u_2)$ then $R = F_* \in \mathfrak{R}_2$. Indeed

$$\mathrm{ord}(D_F(F_*) - [F_*, F_*]) \leqslant 2 = \mathrm{ord}(F_*) + \mathrm{ord}(R) - 2.$$

Let us denote $F_2 = \frac{\partial F}{\partial u_2}, F_1 = \frac{\partial F}{\partial u_1}, F_0 = \frac{\partial F}{\partial u}$. We have $\rho_{-1} = \mathrm{res}\, R^{-\frac{1}{2}} = F_2^{-\frac{1}{2}}$.

In order to find solution in $\mathfrak{R}_3$ we represent $R = F_* + aD_x$ and substitute in

$$\begin{array}{ccc} D_F(\,\mathrm{res}\, R^{-\frac{1}{2}}) & = & \mathrm{res}\,[F_*, R^{-\frac{1}{2}}] \\ \| & & \| \\ D_x(\sigma_{-1}) & = & D_x(-F_2^{-\frac{1}{2}} a) \end{array}$$

therefore

$$\sigma_{-1} = -F_2^{-\frac{1}{2}} a$$

Thus

$$\mathrm{ord}(D_F R^{-\frac{1}{2}} - [F_*, R^{-\frac{1}{2}}]) \leqslant -2 = 2 - 1 - k, \; \Rightarrow k = 3.$$

Thus $R = F_* - \sigma_{-1} F_2^{\frac{1}{2}} D_x = F_2 D_x^2 + (F_1 - \sigma_{-1} F_2^{\frac{1}{2}}) D_x \in \mathfrak{R}_3$ and moreover $\rho_0 = (F_1 - \sigma_{-1} F_2^{\frac{1}{2}}) F_2^{-1}$. Making the next correction to R we would find

$$\rho_1 = \rho_{-1} F_0 - \frac{\rho_0^2}{4\rho_{-1}} + \frac{1}{2}\rho_0\sigma_{-1} - \frac{1}{2}\rho_{-1}\sigma_0.$$

Solved problems of classification of integrable equations

The problem of complete description and classification of all integrable equations of the form

$$
\begin{aligned}
u_{xt} &= f(u), \\
u_t &= f(x, t, u, u_x, u_{xx}) \\
u_t &= u_{xxx} + f(x, u, u_x, u_{xx}), \\
u_t &= a(x, u, u_x, u_{xx})u_{xxx} + f(x, u, u_x, u_{xx}), \\
u_t &= u_{xxxxx} + f(x, u, u_x, u_{xx}, u_{xxx}, u_{xxxx}),
\end{aligned}
$$

have been solved (Shabat, Sokolov, Svinolupov, Meshkov, Heredero, Zhiber). There are plenty results for systems of equations. In particularly all systems of two equations

$$\mathbf{u}_t = A(\mathbf{u})\mathbf{u}_{xx} + \mathbf{F}(\mathbf{u}, \mathbf{u}_x), \quad \mathbf{u} = (u, v)^T, \quad \mathrm{Det}A(\mathbf{u}) \neq 0$$

possessing an infinite hierarchy of conservation laws have been classified (AVM, Shabat, Yamilov).
Differential-difference equations with conservation laws where studied

$$
\begin{aligned}
(u_n)_t &= F(u_{n-1}, u_n, u_{n+1}), \\
(u_n)_t &= F(u_{n-2}, u_{n-1}, u_n, u_{n+1}, u_{n+2}).
\end{aligned}
$$

The first one was completely classified by Yamilov, the problem of classification for the second type of equations was recently (partially) solved by V.Adler.
This method can also be applied to the study of integrable difference equations. Integrability conditions for quadrilateral equations

$$Q(u_{n,m}, u_{n+1,m}, u_{n,m+1}, u_{n+1,m+1}) = 0$$

was found by AVM, Wang and Xenitidis.

I would like to illustrate the method on a simple enough example of the Korteweg-de Vries type equation

$$u_t = u_{xxx} + f(u, u_x)$$

and make a few steps towards the proof of the following statement:

Theorem

Equation

$$u_t = F = u_3 + f(u, u_1) \tag{10}$$

admits an infinite algebra of symmetries if and only if it is one from the following list

$$u_t = u_3 - \frac{\alpha^2}{2} u_1^3 + (\alpha_1 \exp(2\alpha u) + \alpha_2 \exp(-2\alpha u) + \alpha_0)\, u_1, \tag{11}$$

$$u_t = u_3 + \alpha_3 u_1^3 + \alpha_2 u_1^2 + \alpha_1 u_1 + \alpha_0, \tag{12}$$

$$u_t = u_3 + \left(\alpha_2 u^2 + \alpha_1 u + \alpha_0\right) u_1, \tag{13}$$

$$u_t = u_3 + \alpha_2 u_1 + \alpha_1 u + \alpha_0. \tag{14}$$

where $\alpha, \alpha_k \in \mathbb{C}$ are arbitrary constants.

Example of classification

Existence of symmetries **implies** the existence of a formal recursion operator R satisfying the equation

$$D_t(R) - [F_*, R] = 0. \tag{15}$$

In our case

$$F_* = D_x^3 + f_1 D_x + f_0\,, \qquad f_1 = \frac{\partial f}{\partial u_1}, \quad f_0 = \frac{\partial f}{\partial u}$$

has constant leading coefficient and the next coefficient (at D^2) is zero. Thus canonical densities $\rho_{-1} = \operatorname{res} R^{-\frac{1}{3}} = 1$, $\rho_0 = \operatorname{res} \log R = 0$ thus

$$R = F_* + r_{-1} D_x^{-1} + \cdots\ .$$

Taking $R = D_x^3 + f_1 D_x + f_0 + r_{-1} D_x^{-1} + \cdots$ we can easily find that

$$R^{\frac{1}{3}} = D_x + \frac{1}{3} f_1 D_x^{-1} + \frac{1}{3}\left(f_0 - D_x(f_1)\right) D_x^{-2} + \cdots$$
$$R^{\frac{2}{3}} = D_x^2 + \frac{2}{3} f_1 + \frac{1}{3}\left(2f_0 - D_x(f_1)\right) D_x^{-1} + \cdots$$

and thus canonical densities are

$$\rho_1 = \operatorname{res} R^{\frac{1}{3}} = \frac{1}{3} f_1\,, \quad \rho_2 = \operatorname{res} R^{\frac{2}{3}} = \frac{1}{3}\left(2f_0 - D_x(f_1)\right)\,, \quad \rho_3 = \operatorname{res} R = r_{-1}.$$

Example of classification

Thus our conditions are (after obvious re-scaling):

$$\rho_1 = \frac{\partial f}{\partial u_1}, \quad D_t\rho_1 = D_x\sigma_1\,,\ \sigma_1 \in \mathcal{F},$$

$$\rho_2 = \frac{\partial f}{\partial u_0}, \quad D_t\rho_2 \in D_x(\mathcal{F}),$$

$$\rho_3 = \sigma_1, \qquad D_t\rho_3 \in D_x(\mathcal{F}).$$

Applying the Euler operator $\delta/\delta u$ to $D_t\rho_1$ we find an explicit form

$$0 = \frac{\delta}{\delta u} D_t\left(\frac{\partial f}{\partial u_1}\right) = 3u_4\left(u_2\,\frac{\partial^4 f}{\partial u_1^4} + u_1\,\frac{\partial^4 f}{\partial u_1^3\partial u}\right) + \cdots$$

of the first integrability condition. Equation

$$u_2\,\frac{\partial^4 f}{\partial u_1^4} + u_1\,\frac{\partial^4 f}{\partial u_1^3\partial u} = 0$$

gives rise to

$$f(u_1, u) = \lambda u_1^3 + A(u)u_1^2 + B(u)u_1 + C(u),$$

where λ is a constant.
For such f the first condition turns out to be equivalent to

$$\lambda A' = 0, \qquad B''' + 8\lambda B' = 0,$$

$$(B'C)' = 0, \qquad AB' + 6\lambda C' = 0.$$

The second integrability condition has the form

$$D_t\left(\frac{\partial f}{\partial u}\right) = D(\sigma_2).$$

Using this fact we can derive a few more differential relations between $A(u)$, $B(u)$, $C(u)$. Solving them all together we obtain the following list of equations

$$\begin{aligned}
u_t &= u_{xxx} + c_1 u_x + c_2 u + c_3\,, \\
u_t &= u_{xxx} + (c_1 u^2 + c_2 u + c_3) u_x\,, \\
u_t &= u_{xxx} + c_1 u_x^3 + c_2 u_x^2 + c_3 u_x + c_4\,, \\
u_t &= u_{xxx} - \frac{1}{2} u_x^3 + (c_1 e^{2u} + c_2 e^{-2u} + c_3)\, u_x\,,
\end{aligned}$$

where c_1, c_2, c_3, c_4 are arbitrary constants. In the latter equation we normalize λ to $-1/2$ by a scaling. Only these equations have passed through the first two necessary integrability conditions ($D_t(\rho_1), D_t(\rho_2) \in \mathrm{Im}(D)$). Actually all these equations are integrable, i.e. possess infinitely many commuting symmetries, higher conservation laws, have Lax's representations, etc. In this particular case first two integrability conditions proved to be sufficient for the classification.

Example of classification

Integrability conditions for more general equation $u_t = u_{xxx} + f(x, u, u_x, u_{xx})$ are:

$$D_t \left(\frac{\partial f}{\partial u_2} \right) = D_x \sigma_0 \,,$$

$$D_t \left(\frac{\partial f}{\partial u_1} - \frac{1}{3} \left(\frac{\partial f}{\partial u_2} \right)^2 \right) = D_x \sigma_1$$

$$D_t \left(\frac{\partial f}{\partial u} - \frac{1}{3} \left(\frac{\partial f}{\partial u_2} \right) \left(\frac{\partial f}{\partial u_1} \right) + \frac{2}{27} \left(\frac{\partial f}{\partial u_2} \right)^3 + \frac{1}{3} \sigma_0 \right) = D_x \sigma_2$$

$$D_t(\sigma_1) = D_x(\sigma_3).$$

Quadrilateral equations:

$$Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) \in \mathbb{C}[u; \mathcal{S}, \mathcal{T}], \quad J_Q = \langle Q; \mathcal{S}, \mathcal{T} \rangle \subset \mathbb{C}[u; \mathcal{S}, \mathcal{T}],$$

Difference fields:

$$\mathcal{F}_Q = \mathrm{Frac}(\mathbb{C}[u; \mathcal{S}, \mathcal{T}]/J_Q), \quad \mathcal{F}_{\mathsf{s}} = \mathrm{Frac}(\mathbb{C}[u; \mathcal{S}]), \quad \mathcal{F}_{\mathsf{t}} = \mathrm{Frac}(\mathbb{C}[u; \mathcal{T}]), \quad .$$

Definition

An element $K \in \mathcal{F}_Q$ is called a symmetry of the equation $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0$ if in $\mathcal{F}_Q$

$$Q_*(K) = 0, \quad Q_* = \sum_{i,j \in \{0,1\}} \frac{\partial Q}{\partial u_{i,j}} \mathcal{S}^i \mathcal{T}^j.$$

Definition

Let A be a formal series of order N

$$A = a_N \mathcal{S}^N + a_{N-1}\mathcal{S}^{N-1} + \cdots + a_1 \mathcal{S} + a_0 + a_{-1}\mathcal{S}^{-1} + \cdots\,, \quad a_k \in \mathcal{F}_Q.$$

The **residue** $\mathrm{res}(A)$ and **logarithmic residue** $\mathrm{res}\ln(A)$ are defined as

$$\mathrm{res}(A) = a_0, \qquad \mathrm{res}\ln(A) = \ln(a_N)\,.$$

Theorem

Let $A = a_N \mathcal{S}^N + a_{N-1}\mathcal{S}^{N-1}\cdots$ *and* $B = b_M \mathcal{S}^M + b_{M-1}\mathcal{S}^{M-1}\cdots$ *be two Laurent formal series of order* N *and* M *respectively. Then*

$$\mathrm{res}[A, B] = (\mathcal{S} - \mathbf{1})(\sigma(A, B)),$$

where $\sigma(A, B) \in \mathcal{F}_Q$

$$\sigma(A, B) = \sum_{n=1}^{N}\sum_{k=1}^{n} \mathcal{S}^{-k}(a_{-n})\mathcal{S}^{n-k}(b_n) - \sum_{n=1}^{M}\sum_{k=1}^{n} \mathcal{S}^{-k}(b_{-n})\mathcal{S}^{n-k}(a_n).$$

Theorem

If a quadrilateral difference equation possess an infinite sequence of symmetries $K_n \in \mathcal{F}_\mathbf{s}$ of increasing order $0 < ord_+(K_{p+1}) - ord_+(K_p) = N$ then it has a formal recursion operator $\mathfrak{R}$ of order N.

Theorem

Let $Q(u_{0,0}, u_{1,0}, u_{0,1}, u_{1,1}) = 0$ be a quadrilateral difference equation.
(i) *If there exist two* **s***–pseudo-difference operators $\mathfrak{R}$ and $\mathfrak{P}$ such that*

$$Q_* \circ \mathfrak{R} = \mathfrak{P} \circ Q_*,$$

then $\mathfrak{R}$ is a recursion operator of the difference equation.
(ii) *The above relation is valid if and only if*

$$\mathcal{T}(\mathfrak{R}) - \mathfrak{R} = [\Phi \circ \mathfrak{R}, \Phi^{-1}] \ ,$$

where $\Phi = (Q_{u_{1,1}}\mathcal{S} + Q_{u_{0,1}})^{-1} \circ (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}})$, and

$$\mathfrak{P} = (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}}) \circ \mathfrak{R} \circ (Q_{u_{1,0}}\mathcal{S} + Q_{u_{0,0}})^{-1}.$$

Theorem

If a (formal) recursion operator $\mathfrak{R}$ is represented by a first order formal series $\mathfrak{R} = r_1\mathcal{S} + r_0 + r_{-1}\mathcal{S}^{-1} + \cdots$, *then*

$$\text{(i)} \qquad (\mathcal{T} - \mathbf{1})(\ln r_1) = (\mathcal{S} - \mathbf{1})\mathcal{S}^{-1}\left(\ln \frac{Q_{u_{1,1}}}{Q_{u_{1,0}}}\right),$$

$$\text{(ii)} \qquad (\mathcal{T} - \mathbf{1})(r_0) = (\mathcal{S} - \mathbf{1})\mathcal{S}^{-1}(r_1 F),$$

$$\text{(iii)} \qquad (\mathcal{T} - \mathbf{1})(r_{-1}\mathcal{S}^{-1}(r_1) + r_0^2 + r_1\mathcal{S}(r_{-1})) = (\mathcal{S} - \mathbf{1})(\sigma_2),$$

where

$$\sigma_2 = \mathcal{S}^{-1}(r_1 F)\left\{\mathcal{S}^{-1}(r_0) + r_0 - \mathcal{S}^{-2}(r_1 F)\right\} - \\ -(1 + \mathcal{S}^{-1})\left(r_1\, G\, \mathcal{S}^{-1}(r_1 F)\right),$$

and F, G *denote*

$$F = \frac{Q_{u_{0,1}}\mathcal{S}^{-1}(Q_{u_{1,0}}) - Q_{u_{0,0}}\mathcal{S}^{-1}(Q_{u_{1,1}})}{Q_{u_{1,0}}\mathcal{S}^{-1}(Q_{u_{1,1}})}, \qquad G = \frac{Q_{u_{0,0}}}{Q_{u_{1,0}}}.$$

Problem 1: *Find conditions on $a \in \mathcal{F}_Q$, such that the difference equation $a = \mathcal{T}(b) - b$ is solvable in $\mathcal{F}_Q$, and if so find $b \in \mathcal{F}_Q$ (same for $a = \mathcal{S}(b) - b$).*

Problem 2: *Determine whether the kernel spaces $\mathrm{Ker}(\mathcal{T} - \mathbf{1})$ and $\mathrm{Ker}(\mathcal{S} - \mathbf{1})$ are trivial: $\mathrm{Ker}(\mathcal{T} - \mathbf{1}) = \mathrm{Ker}(\mathcal{S} - \mathbf{1}) = \mathbb{C}$? If not, give a description of these spaces.*

Kernel spaces $\mathrm{Ker}(\mathcal{T} - \mathbf{1})$ and $\mathrm{Ker}(\mathcal{S} - \mathbf{1})$ can be nontrivial
It depends on the choice of Q. For example, if

$$Q = uu_{11} - (u_{10} - 1)(u_{01} - 1)$$

then

$$\left(\frac{u_{20}}{u_{10} - 1}\right)\left(\frac{u - 1}{u_{10}}\right) \in \mathrm{Ker}(\mathcal{T} - \mathbf{1})$$

and $\mathcal{F}_Q$ has a nontrivial subfield of $\mathcal{T}$–constants.

If $a \in \mathcal{F}_{\mathbf{t}}$ the answer is well known:

$$a \in \mathrm{Im}(\mathcal{T} - \mathbf{1}) + \mathbb{C} \ \Leftrightarrow \ \sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\partial a}{\partial u_{0n}} = 0,$$

element $b \in \mathcal{F}_{\mathbf{t}}$ can be easily found and $\mathrm{Ker}(\mathcal{T} - \mathbf{1}) = \mathbb{C}$.

$$u_{n,m}u_{n+1,m} + u_{n,m+1}u_{n+1,m+1} + u_{n+1,m}u_{n+1,m+1} = 0$$

Canonical conservation laws ($\Delta_m = \mathcal{T} - 1,\ \Delta_n = \mathcal{S} - 1$):

$$\Delta_m \left(\log \frac{u_{n,m}u_{n-1,m}}{u_{n+1,m}^2} \right) = \Delta_n \left(\log \frac{u_{n,m}u_{n-1,m}}{u_{n,m+1}(u_{n-1,m} + u_{n,m+1})} \right)$$

$$\Delta_m (\mathcal{S} + 1) \frac{u_{n-2,m}}{u_{n,m}} = \Delta_n \left(\frac{u_{n,m+1}u_{n-2,m}}{u_{n,m}(u_{n-1,m} + u_{n,m+1})} - \frac{u_{n-1,m}}{u_{n,m+1}} - 1 \right)$$

$$\Delta_m \left\{ (\mathcal{S} + 1) \left(\frac{u_{n-2,m}^2}{u_{n,m}^2} + \frac{u_{n-3,m}u_{n-2,m}}{u_{n-1,m}u_{n,m}} \right) + 2 \frac{u_{n-2,m}u_{n-1,m}}{u_{n,m}u_{n+1,m}} \right\} = \Delta_n \sigma$$

Coefficients of the formal recursion operator

$$r_1 = \frac{u_{n,m}u_{n-1,m}}{u_{n+1,m}^2}$$

$$r_0 = (\mathcal{S} + 1) \frac{u_{n-2,m}}{u_{n,m}}$$

$$r_{-1} = \frac{u_{n-3,m}u_{n,m}}{u_{n-1,m}^2} + \frac{u_{n,m}^2}{u_{n-2,m}u_{n-1,m}}$$

First symmetry

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}t_1} = \frac{u_{n,m}u_{n-1,m}}{u_{n+1,m}}$$

Second symmetry

$$\frac{\mathrm{d}u_{n,m}}{\mathrm{d}t_2} = \frac{u_{n,m}u_{n-1,m}^2}{u_{n+1,m}^2} + \frac{u_{n-2,m}u_{n-1,m}}{u_{n+1,m}} + \frac{u_{n-1,m}u_{n,m}^2}{u_{n+1,m}u_{n+2,m}}$$

Open problems:

- Foundation of the theory (differential-difference algebra).
- Connection of symmetry approach and Lax-Darboux structure.
- Classification of Lax structures.
- Classification of the corresponding elementary Darboux maps.
- Lenard's scheme for DΔEs and PΔEs.
- Integrability conditions for non-quadrilateral equations.
- Integrability conditions for systems of DΔEs and PΔEs.
- Classification of integrable DΔEs of order higher than $(-1, +1)$.
- Classification of integrable PΔEs and system of PΔEs.
- Non-local extensions and non-evolutionary equations.
- Integrability conditions for multi-dimensional equations.
- Differential and difference equations in "non-commutative" cases.
- Theory of normal forms for approximately integrable systems.
- Connection with differential and difference Galois theory.