Geometric and algebraic aspects of integrability
Durham, 25 July - 04 August, 2016

The inverse spectral transform for integrable dispersionless PDEs: Cauchy problem, longtime behavior and wave breaking; with applications to nonlinear physics

(seminar dedicated to Manakov's memory)

with S. V. Manakov: formal IST and all its consequences
with P. G. Grinevich and D. Wu: rigorous aspects of the IST
Also: Santucci and G. Yi (former students)

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Outline

Soliton PDEs vs Integrable dispersionless PDEs

Some basic examples of dPDEs

IST for vector fields and dPDEs

Analytic description of wave breaking

Solvable nonlinear RH problems and exact implicit solutions (NO TIME)

Rigorous aspects of the IST theory

How do we deal we the nonlocality of the PDEs?

Two Cauchy problems in Nature in which dKP is relevant (NO TIME?)

INTEGRABLE SOLITON PDEs

Waves propagating in weakly nonlinear and dispersive media are well described by integrable soliton equations: KdV, NLS, ... 1) The Inverse Spectral Transform (IST) is the spectral method allowing one to solve the Cauchy problem for such PDEs, predicting that a localized disturbance evolves into a number of soliton pulses + radiation. Soliton = balance between nonlinearity and dispersion. 2) Soliton PDEs arise in hierarchies of commuting flows, sharing similar behavior. 3) Soliton PDEs are in low dimensions.

INTEGRABLE DISPERSIONLESS PDEs

1) Lax pair of integrable dPDEs is made of vector fields $\Rightarrow$ can be in arbitrary dimensions; 2) Due to the lack of dispersion, dPDEs may exhibit wave breaking at finite time. 3) A novel IST for vector fields has been recently developed, to solve the Cauchy problem, allowing, in particular, to establish if, due to the lack of dispersion, the nonlinearity of the PDE is “strong enough” to cause the gradient catastrophe of localized multidimensional disturbances and to study the analytic details of such a wave breaking. 4)dPDs are intimately related to Twistor theory.

Commuting vector fields generate integrable PDEs in arbirary dimensions [Zakharov Shabat '79]

EXAMPLES

The commutation $[\hat{L}_1, \hat{L}_2] = 0$ of the vector fields:

$$\hat{L}_j \equiv \partial_{t_j} + \lambda \partial_{z_j} + \vec{u}_{z_j} \cdot \nabla_{\vec{x}}, \quad j = 1, 2 \tag{1}$$

is equivalent to the nonlinear vector PDE in $N + 4$ dimensions [Manakov-PMS 06]:

$$\vec{u}_{t_1 z_2} - \vec{u}_{t_2 z_1} + \left(\vec{u}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{u}_{z_2} - \left(\vec{u}_{z_2} \cdot \nabla_{\vec{x}}\right) \vec{u}_{z_1} = \vec{0}, \tag{2}$$

and its divergenceless reduction $\nabla_{\vec{x}} \cdot \vec{u} = 0$.

1) Its deepest scalar Hamiltonian reduction in $2M + 4$ dimensions:

$$\theta_{t_2 z_1} - \theta_{t_1 z_2} + \{\theta_{z_1}, \theta_{z_2}\} = c(t_1, t_2, z_1, z_2)$$

$$\hat{L}_j = \partial_{t_j} + \lambda \partial_{z_j} + \{\theta_{z_j}, \cdot\}$$

$$\{f, g\} = \sum_{k=1}^{M} (f_{x_k} g_{x_{M+k}} - f_{x_{M+k}} g_{x_k})$$

i) the first heavenly equation

$N = 2$; $\partial_{t_1}, \partial_{t_2} = 0$, $\Rightarrow$ $\{\theta_{x_1}, \theta_{x_2}\}_{z_1, z_2} = c(z_1, z_2)$ ii) the second heavenly equation (anti-self-duality + Einstein equations):

$N = 2$; $z_1 = x_1$, $z_2 = x_2$ $\Rightarrow$ $\theta_{t_2 x_1} - \theta_{t_1 x_2} + \theta_{x_1 x_1} \theta_{x_2 x_2} - \theta^2_{x_1 x_2} = c(t_1, t_2)$

Intimately related to SDYM via $U \quad \to \quad \vec{u} \cdot \nabla_{\vec{x}}$

$$L_i = \lambda \partial_{z_i} + \partial_{t_i} + U_{z_i}, \ i = 1, 2, \quad [L_1, L_2] = 0 \quad \Rightarrow$$

$$U_{t_1 z_2} - U_{t_2 z_1} + [U_{z_1}, U_{z_2}] = 0$$

Recursion operator [Marvan, Sergyeyev 2012] and bi-Hamiltonian structures follow directly from those of the SDYM [Bruschi, Levi, Ragnisco 1981]:

$$SDYM: \quad \Theta_1 \equiv \partial_{z_1}, \ \Theta_2 \equiv \partial_{t_1} + ad(U_{z_1}), \quad \Phi \equiv \Theta_2 (\Theta_1)^{-1}$$

$$\Rightarrow \quad R = \Theta_2 (\Theta_1)^{-1}, \ \ \Theta_2 \vec{f} = \vec{f}_{t_1} + \left(\vec{u}_{z_1} \cdot \nabla_{\vec{x}}\right) \vec{f} - \left(\vec{f} \cdot \nabla_{\vec{x}}\right) \vec{u}_{z_1}$$

2) The system of two nonlinear PDEs in $2+1$ dimensions [Manakov and PMS '06]:

$$\begin{aligned} &u_{xt} + u_{yy} + (uu_x)_x + v_x u_{xy} - v_y u_{xx} = 0, \\ &v_{xt} + v_{yy} + uv_{xx} + v_x v_{xy} - v_y v_{xx} = 0, \end{aligned} \tag{3}$$

commutation condition $[\tilde{L}_1, \tilde{L}_2] = 0 \ \forall \lambda$, involving also ∂_λ :

$$\begin{aligned} &\tilde{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x \partial_\lambda, \\ &\tilde{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \end{aligned} \tag{4}$$

it describes the most general integrable Einstein-Weyl metric structure [Dunajski '08; Dunajski,Ferapontov,Kruglikov '14]
1a) The $v = 0$ reduction of (3), the celebrated dKP equation

$$(u_t + uu_x)_x + u_{yy} = 0, \quad u = u(x, y, t) \in \mathbb{R}, \qquad x, y, t \in \mathbb{R}, \tag{5}$$

describing the evolution of small amplitude, nearly one-dimensional waves when dispersion and dissipation are negligeable. It is a basic prototype model in the description of multidimensional wave breaking [Manakov, PMS 2008].

Commutation condition for a pair of Hamiltonian 2D vector fields:

$$
\begin{aligned}
&\hat{L}_1 \equiv \partial_y + \lambda\partial_x - u_x\partial_\lambda = \partial_y + \{H_1, \cdot\}_{(\lambda,x)}, \\
&\hat{L}_2 \equiv \partial_t + (\lambda^2 + u)\partial_x + (-\lambda u_x + u_y)\partial_\lambda = \partial_t + \{H_2, \cdot\}_{(\lambda,x)}, \\
&H_1 = \tfrac{\lambda^2}{2} + u(x,y), \qquad H_2 = \tfrac{\lambda^3}{3} + \lambda u - \partial_x^{-1} u_y,
\end{aligned}
\tag{6}
$$

dKP hierarchy:

$$
H_{n t_m} - H_{m t_n} + \{H_m, H_n\}_{(\lambda,x)} = 0, \quad H_n \equiv \frac{1}{n}\,(f^n)_{\geq 0}\,, \tag{7}
$$

where f is the eigenfunction analytic in a neighborough of $\lambda = \infty$, with the expansion:

$$
f = \lambda + u\lambda^{-1} - \partial_x^{-1}(u_y)\lambda^{-2} + \sum_{j\geq 3, j\in\mathbb{Z}} q_j \lambda^{-j}, \tag{8}
$$

1b. The $u = 0$ reduction of (3), the Pavlov equation

$$
v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{xy}, \quad v = v(x,y,t) \in \mathbb{R}, \quad x,y,t \in \mathbb{R}, \tag{9}
$$

associated with the non-Hamiltonian one-dimensional vector fields

$$
\begin{aligned}
&\hat{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x, \\
&\hat{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x - v_y)\partial_x.
\end{aligned}
\tag{10}
$$

IST for VECTOR FIELDS [Manakov and PMS '05-06]
Basic example: the dKP system

$$\begin{aligned} & u_{xt} + u_{yy} = -(uu_x)_x - v_x u_{xy} + v_y u_{xx}, \; u, v \in \mathbb{R}, \;\; x, y, t \in \mathbb{R}, \\ & v_{xt} + v_{yy} = -uv_{xx} - v_x v_{xy} + v_y v_{xx} \end{aligned} \tag{11}$$

describing the most general Einstein-Weyl metric, and its Lax pair formulation $\hat{L}_1\psi = 0, \;\; \hat{L}_2\psi = 0, \;\; \Leftrightarrow \;\; [\hat{L}_1, \hat{L}_2] = 0$

$$\begin{aligned} & \tilde{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x\partial_\lambda, \\ & \tilde{L}_2 \equiv \partial_t + (\lambda^2 + \lambda v_x + u - v_y)\partial_x + (-\lambda u_x + u_y)\partial_\lambda, \end{aligned} \tag{12}$$

Novel features of the IST for vector fields
Since the Lax pair is made of vector fields (Hamiltonian in the dKP ($v = 0$ reductions):

1) The space of eigenfunctions is a ring: if $f_1, \; f_2$ are two eigenf.s, then an arbitrary differentiable function $F(f_1, f_2)$ of them is also an eigenf.

2) In the Hamiltonian reductions, the space of eigenfunctions is also a Lie algebra, whose Lie bracket is the natural Poisson bracket: if $f_1, \; f_2$ are two eigenf.s, then their Poisson bracket $\{f_1, f_2\}$ is also an eigenf.

Cauchy problem for rapidly decreasing real 2D waves

$\hat{L}_1 \equiv \partial_y + (\lambda + v_x)\partial_x - u_x \partial_\lambda$

If f is a solution of $\hat{L}_1 f = 0$, then

$$\begin{aligned} & f(x,y,\lambda) \to f_\pm(\xi,\lambda), \quad y \to \pm\infty, \\ & \xi := x - \lambda y; \end{aligned} \tag{13}$$

i.e., asymptotically, f is an arbitrary function of $\xi = (x - \lambda y)$, and λ.

Real (Jost) eigenfunctions $\vec{\phi}_\pm(x,y,\lambda)$:

$$\vec{\phi}_\pm \equiv \begin{pmatrix} \phi_{1\pm}(x,y,\lambda) \\ \phi_{2\pm}(x,y,\lambda) \end{pmatrix} \to \begin{pmatrix} \lambda \\ -\lambda y + x \end{pmatrix} \equiv \vec{\xi}, \quad y \to \pm\infty. \tag{14}$$

intimately related to the system of real ODEs

$$\frac{dx}{dy} = \lambda + v_x(x,y), \quad \frac{d\lambda}{dy} = -u_x(x,y) \tag{15}$$

defining the characteristics of $\hat{L}_1$.

If the potentials (u, v) are sufficiently regular, the solution $(x(y), \lambda(y))$ of the ODE (15) exists unique globally in the (time) variable y, with the following free particle asymptotic behavior

$$x(y) \to \lambda_{\pm} y + x_{\pm}, \ \ \lambda(y) \to \lambda_{\pm}, \ \ y \to \pm\infty, \tag{16}$$

reducing to the asymptotics

$$x(y) \to \lambda y + x_{\pm}, \ \ \lambda(y) = \lambda = \text{constant}, \ \ y \to \pm\infty, \tag{17}$$

in the Pavlov reduction $u = 0$. Once the asymptotics $\lambda_{\pm}, x_{\pm}$ are constructed in terms of the initial data $x_0 = x(y_0), \lambda_0 = \lambda(y_0)$ of the ODE: $\lambda_{\pm}(x_0, y_0, \lambda_0), x_{\pm}(x_0, y_0, \lambda_0)$, the real eigenfunctions $\vec{\phi}_{\pm}$, that are particular constants of motion of the ODE, are given by

$$\vec{\phi}_{\pm}(x_0, y_0, \lambda_0) = (x_{\pm}(x_0, y_0, \lambda_0), \lambda_{\pm}(x_0, y_0, \lambda_0)). \tag{18}$$

Another important ingredient of the formalism is given by the complex eigenfunction $\vec{\psi}$, defined by the asymptotics

$$\vec{\psi}(y,\vec{x},\lambda) \sim \vec{\xi}, \quad x^2+y^2 \to \infty, \quad \lambda \notin \mathbb{R}, \tag{19}$$

analytic for $\lambda \notin \mathbb{R}$, having continuous boundary values $\vec{\psi}^{\pm}(x,y,\lambda), \ \lambda \in \mathbb{R}$ from above and below the real λ axis, with the following asymptotics for large complex λ:

$$\begin{array}{l} \vec{\psi}^{\pm}(x,y,\lambda) = \vec{\xi} + \frac{1}{\lambda}\vec{U}(x,y) + \vec{O}\left(\frac{1}{\lambda^2}\right), \quad |\lambda| >> 1, \\ \vec{U}(x,y) \equiv \left(\begin{array}{c} -yu(x,y) - v(x,y) \\ u(x,y) \end{array} \right). \end{array} \tag{20}$$

Scattering and spectral data. The $y=+\infty$ limit of $\vec{\phi}_-$ defines the natural (y - time) *scattering vector* $\vec{\sigma}$ for $\hat{L}_1$:

$$\lim_{y\to+\infty} \vec{\phi}_-(x,y,\lambda) \equiv \vec{\mathcal{S}}(\vec{\xi}) = \vec{\xi} + \vec{\sigma}(\vec{\xi}). \tag{21}$$

Since the space of eigenfunctions is a ring, the eigenfunctions $\vec{\psi}^{\pm}$ for $\lambda \in \mathbb{R}$ can be expressed in terms of the real eigenfunctions $\vec{\phi}_{\pm}$, and this expression defines the *spectral data* $\vec{\chi}^{\pm}_{\beta}(\xi,\lambda)$:

$$\begin{aligned} &\vec{\psi}^{\pm}(x,y,\lambda) = \vec{\mathcal{K}}^{\pm}_{-}(\vec{\phi}_-(x,y,\lambda)) = \vec{\mathcal{K}}^{\mp}_{+}(\vec{\phi}_+(x,y,\lambda)), \quad \lambda\in\mathbb{R}, \\ &\vec{\mathcal{K}}^{\pm}_{\beta}(\vec{\xi}) \equiv \vec{\xi} + \vec{\chi}^{\pm}_{\beta}(\vec{\xi}), \quad \vec{\xi} = (\xi,\lambda), \end{aligned} \tag{22}$$

where $\vec{\chi}^{+}_{\beta}(\vec{\xi})$ and $\vec{\chi}^{-}_{\beta}(\vec{\xi})$ are analytic wrt the first argument ξ respectively in the upper and lower halves of the complex ξ - plane, as a consequence of the analyticity properties of $\vec{\psi}^{\pm}$.

Evaluating

$$\begin{aligned}
&\vec{\mathcal{K}}^{+}_{-}(\vec{\phi}_{-}(x,y,\lambda)) = \vec{\mathcal{K}}^{-}_{+}(\vec{\phi}_{+}(x,y,\lambda)), \quad \lambda \in \mathbb{R}, \\
&\vec{\mathcal{K}}^{\pm}_{\beta}(\vec{\xi}) \equiv \vec{\xi} + \vec{\chi}^{\pm}_{\beta}(\vec{\xi}), \quad \vec{\xi} = (\xi, \lambda),
\end{aligned} \tag{23}$$

at $y = +\infty$, one obtains the following linear Riemann - Hilbert (RH) problem with a shift:

$$\begin{aligned}
&\vec{\sigma}(\xi,\lambda) + \vec{\chi}^{+}_{-}(\vec{\xi} + \vec{\sigma}(\xi,\lambda)) - \vec{\chi}^{-}_{+}(\xi,\lambda) = \vec{0}, \\
&|\vec{\chi}^{\pm}_{\beta}(\xi,\lambda)| = O(\xi^{-1}), \quad \xi \sim \infty
\end{aligned} \tag{24}$$

equivalent to a linear Fredholm integral equation, allowing one to uniquely construct the spectral data $\vec{\chi}^{+}_{-}$ and $\vec{\chi}^{-}_{+}$ from the scattering data $\vec{\sigma}$, under the hypothesis that the mapping $\xi \to \xi + \sigma_1(\xi,\lambda)$ be invertible.

Reality conditions (from reality of the potentials):

$$(u,v) \in \mathbb{R}^2 \;\Rightarrow\; \vec{\phi}_{\pm} \in \mathbb{R}^2,\; \vec{\psi}^{-} = \overline{\vec{\psi}^{+}}, \;\; \vec{\sigma} \in \mathbb{R}^2,\; \vec{\chi}^{-}_{\alpha} = \overline{\vec{\chi}^{+}_{\alpha}}, \;\; \lambda \in \mathbb{R}. \tag{25}$$

Two inverse problems

The first inversion (the reconstruction of $\vec{\phi}_-$ from the spectral data $\vec{\chi}_-^+$) is provided by the nonlinear integral equation

$$\vec{\phi}_-(x,y,\lambda) + H_\lambda \vec{\chi}^+_{-I}(\vec{\phi}_-(x,y,\lambda) + \vec{\chi}^+_{-R}(\vec{\phi}_-(x,y,\lambda)) = \vec{\xi}, \tag{26}$$

where $\vec{\chi}^+_{-R}$ and $\vec{\chi}^+_{-I}$ are the real and imaginary parts of $\vec{\chi}_-^+$, and H_λ is the Hilbert transform operator wrt λ

$$H_\lambda f(\lambda) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda - \lambda'} d\lambda'. \tag{27}$$

Equation (26) expresses the fact that the RHS of (22) for $\vec{\psi}^+$ is the boundary value of a function analytic in the upper half λ plane. Once $\vec{\phi}_-$ is reconstructed from $\vec{\chi}_-^+$ solving the nonlinear integral equation (26), equations (22) give $\vec{\psi}^\pm$, and (u, v) is finally reconstructed from

$$\begin{aligned} u(x,y) &= \lim_{\lambda\to\infty} \left(\lambda(\psi_2^-(x,y,\lambda) - \lambda\right), \\ v(x,y) &= -yu - \lim_{\lambda\to\infty} \left(\lambda(\psi_1^-(x,y,\lambda) - (x - \lambda y)\right), \end{aligned} \tag{28}$$

A second inverse problem can be obtained eliminating the real eigenfunctions from the first of equations (22) for $\vec{\psi}^{\pm}$, obtaing a 2 vector nonlinear RH (NRH) problem on the real line:

$$\begin{array}{l}
\psi_1^+(\lambda) = \mathcal{R}_1\left(\psi_1^-(\lambda), \psi_2^-(\lambda)\right), \quad \lambda \in \mathbb{R}, \\
\psi_2^+(\lambda) = \mathcal{R}_2\left(\psi_1^-(\lambda), \psi_2^-(\lambda)\right), \\
\psi_1^+(\lambda) = -y\lambda + x + O(\lambda^{-1}), \quad \psi_2^+(\lambda) = \lambda + O(\lambda^{-1}), \quad \lambda \sim \infty.
\end{array} \tag{29}$$

for the RH data $\vec{\mathcal{R}}$, constructed, via algebraic manipulation, from the spectral data. Once the analytic eigenfunctions are reconstructed through the solution of the NRH problem (29), the solution of the nonlinear PDE (3) is obtained from (28). We remark that, in the two basic reductions, the RH data are constrained as follows:

$$\begin{array}{l}
\mathcal{R}_2(\zeta_1, \zeta_2) = \zeta_2, \quad \text{Pavlov reduction}, \\
\{\mathcal{R}_1, \mathcal{R}_2\}_{\zeta_1, \zeta_2} = 1, \quad \text{dKP reduction}.
\end{array} \tag{30}$$

Evolution of the spectral data. The evolution of the scattering, spectral, and RH data is described by the following simple formula [Manakov,PMS 2006,07]:

$$\Sigma_1(\xi,\lambda,t) = \Sigma_1(\xi - \lambda^2 t, \lambda, 0) \tag{31}$$

for the Pavlov equation, and

$$\begin{aligned} &\Sigma_1(\xi,\lambda,t) = t\left(\Sigma_2(\xi - \lambda^2 t, \lambda, 0)\right)^2 + \Sigma_1(\xi - \lambda^2 t, \lambda, 0), \\ &\Sigma_2(\xi,\lambda,t) = \Sigma_2(\xi - \lambda^2 t, \lambda, 0) \end{aligned} \tag{32}$$

for the dKP equation.

We remark that, from the eigenfunctions $\vec{\phi}_\pm, \vec{\psi}^\pm$ of $\hat{L}_1$, one can constructs the common eigenfunctions $\vec{\Phi}_\pm, \vec{\Psi}^\pm$ of $\hat{L}_1$ and $\hat{L}_2$ through the formulae

$$\begin{aligned} &\Phi_{\pm 1} = \phi_{\pm 1} - t\left(\phi_{\pm 2}\right)^2, \quad \Phi_{\pm 2} = \phi_{\pm 2}, \\ &\Psi_1^\pm = \psi_1^\pm - t\left(\psi_2^\pm\right)^2, \quad \Psi_2^\pm = \psi_2^\pm, \end{aligned} \tag{33}$$

Nonlinear Riemann - Hilbert dressing. Let $\vec{\Psi}^{\pm}(\lambda)$ be the solutions of the following 2 vector NRH problem on the line

$$\vec{\Psi}^{+}(\lambda) = \vec{\mathcal{R}}\left(\vec{\Psi}^{-}(\lambda)\right), \quad \lambda \in \mathbb{R}, \tag{34}$$

with the normalization

$$\vec{\Psi}^{\pm}(\lambda) = \begin{pmatrix} -t\lambda^2 - y\lambda + x - 2ut \\ \lambda \end{pmatrix} + \vec{O}(\lambda^{-1}), \quad \lambda \sim \infty, \tag{35}$$

for the RH data $\vec{\mathcal{R}}(\vec{\zeta}) = (\mathcal{R}_1(\vec{\zeta}), \mathcal{R}_2(\vec{\zeta})), \ \vec{\zeta} \in \mathbb{C}^2$. Then $\vec{\Psi}^{\pm}(\lambda)$ are eigenfunctions of $\hat{L}_j$ $j = 1, 2$: $\hat{L}_j \vec{\Psi}^{\pm} = \vec{0}, \ j = 1, 2$, and one obtains the following spectral characterization of the solution u:

$$u = F(x - 2ut, y, t) \in \mathbb{R}, \tag{36}$$

where the spectral function F, defined by

$$F(\xi, y, t) = -\int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R_2\left(\Psi_1^{-}(\lambda; \xi, y, t), \Psi_2^{-}(\lambda; \xi, y, t)\right), \tag{37}$$

is connected to the initial data via the direct problem [**?**].

The longtime behavior of dKP solutions

Let $t >> 1$ and

$$\begin{aligned} & x = \xi + v_1 t, \quad y = v_2 t, \\ & \xi - 2ut, v_1, v_2 = O(1), \quad v_2 \neq 0, \quad t >> 1. \end{aligned} \tag{38}$$

On the parabola

$$x + \frac{y^2}{4t} = \xi \quad (v_1 = -\frac{v_2^2}{4}), \tag{39}$$

the longtime behaviour of the solutions of the dKP equation is given by

$$\begin{aligned} & u = \frac{1}{\sqrt{t}} G\left(x + \frac{y^2}{4t} - 2ut, \frac{y}{2t}\right)(1 + o(1)), \\ & G(\xi, \eta) = -\frac{1}{2\pi i} \int_{\mathbb{R}} d\mu R_2\left(\xi + \mu^2 + a_1(\mu; \xi, \eta), \eta + a_2(\mu; \xi, \eta)\right), \end{aligned} \tag{40}$$

where $a_j(\mu : \xi, \eta)$ solve "asymptotic" RH problem on the μ real axes:

$$\begin{aligned} & \vec{A}^+(\mu; \xi, \eta) = \vec{A}^-(\mu; \xi, \eta) + \vec{R}(\vec{A}^-(\mu; \xi, \eta)), \quad \mu \in \mathbb{R}, \\ & \vec{A}^\pm(\mu; \xi, \eta) = \begin{pmatrix} \xi + \mu^2 \\ \eta \end{pmatrix} + \vec{a}(\mu; \xi, \eta). \end{aligned} \tag{41}$$

Small initial data start evolving according to $u_{tx} + u_{yy} = 0$. Only in the longtime regime the nonlinear term becomes relevant, causing the breaking of the small localized initial wave in a point of the parabola.

NO breaking mechanism instead for the the Pavlov equation
[Manakov and PMS 09]
RH INVERSE PROBLEM (Pavlov): $\mathcal{R}(\zeta_1, \zeta_2) \quad \Rightarrow \quad v(x, y, t)$
In the Pavlov reduction:

$$\vec{\mathcal{R}}(\overline{\vec{\mathcal{R}}(\bar{\vec{\zeta}})}) = \vec{\zeta}, \quad \forall \vec{\zeta} \in \mathbb{C}^2,, \\ \mathcal{R}_1(\vec{\zeta}) = \zeta_1 \quad \Rightarrow \quad \psi_1^+(\lambda) = \psi_1^-(\lambda) = \lambda \quad \Rightarrow \quad u = 0 \tag{42}$$

the nonlinear RH problem becomes scalar for $\psi_2^{\pm}(\lambda)$:

$$\Phi^+(\lambda) = \mathcal{R}(\lambda, \psi^-(\lambda)), \quad \lambda \in \mathbb{R} \tag{43}$$

with normalization:

$$\Phi^{\pm}(\lambda) = -\lambda^2 t - \lambda y + x + O(\tfrac{1}{\lambda}), \tag{44}$$

Then

$$v(x, y, t) = \int_{\mathbb{R}} \frac{d\lambda}{2\pi i} R\Big(\lambda, \Phi^-(\lambda; x, y, t)\Big). \tag{45}$$

The IST allows one to show that solutions $u(x, y, t)$ of dKP depend on x through the combination $x - 2ut$; i.e., these solutions can be written in the characteristic form

$$
\begin{aligned}
&u = F(\zeta, y, t), \;\; \zeta = x - 2F(\zeta, y, t)t,, \\
&F(x, y, 0) = u(x, y, 0),
\end{aligned} \tag{46}
$$

in analogy with the case of the Riemann equation $u_t + u^m u_x = 0$, for which the dependence of the solution $u(x, t)$ on x is through the combination $x - u^m t$. For this reason, the IST for dKP can be viewed as a generalization of the method of characteristics. The formulation (46) becomes explicit in the small field limit

$$
\begin{aligned}
&u \sim \tilde{u}(x - 2ut, y, t), \\
&\tilde{u}_{xt} + \tilde{u}_{yy} = 0, \quad \tilde{u}(x, y, 0) = u(x, y, 0).
\end{aligned} \tag{47}
$$

The formulation (46) has allowed one to study in an analytically explicit way the interesting features of the gradient catastrophe of two dimensional waves at finite time and in the longtime regime in terms of the initial data [Manakov,PMS 2008,2011,2012].

Given F from the inverse problem, we solve (46b) wrt ζ : $\zeta(x, y, t)$ and we replace it in (46a), obtaining the solution $u = F(\zeta(x, y, t), y, t)$ of the Cauchy problem for dKP. Therefore the Singularity Manifold (SM) of dKP is the two - dimensional manifold characterized by the equation

$$\mathcal{S}(\zeta, y, t) \equiv 1 + 2F_\zeta(\zeta, y, t)t = 0; \tag{48}$$

Since

$$\nabla_{(x,y)} u = \frac{\nabla_{(\zeta,y)} F(\zeta, y, t)}{1 + 2F_\zeta(\zeta, y, t)t}, \tag{49}$$

the gradient of the wave becomes infinity on the SM, and the wave "breaks".

The first time t_b at which $\mathcal{S} = 0$ in a point $\vec{\zeta}_b = (\zeta_b, y_b)$ of the (ζ, y) - plane:

$$t_b \equiv \text{global min}\ \check{t}(\zeta, y) = \check{t}(\zeta_b, y_b), \quad \Rightarrow \quad 1 + 2F_\zeta(\vec{\zeta}_b, t_b)t_b = 0; \tag{50}$$

conditions characterizing the breaking point $(\vec{\zeta}_b, t_b)$:

$$
\begin{aligned}
&1 + 2t_b F_\zeta(\vec{\zeta}_b, t_b) = 0\\
&F_\zeta(\vec{\zeta}_b, t_b) < 0, \quad F_\zeta(\vec{\zeta}_b, t_b) + t_b F_{\zeta t}(\vec{\zeta}_b, t_b) < 0,\\
&F_{\zeta\zeta}(\vec{\zeta}_b, t_b) = F_{\zeta y}(\vec{\zeta}_b, t_b) = 0,\\
&F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b) > 0, \quad \beta \equiv F_{\zeta\zeta\zeta}(\vec{\zeta}_b, t_b) F_{\zeta yy}(\vec{\zeta}_b, t_b) - F^2_{\zeta\zeta y}(\vec{\zeta}_b, t_b) > 0.
\end{aligned}
\tag{51}
$$

At $t = t_b$ the wave breaks in the point $\vec{x}_b = (x_b, y_b)$ of the (x, y) - plane defined by

$$
x_b = \zeta_b + 2F(\vec{\zeta}_b, t_b) t_b. \tag{52}
$$

Therefore, generically, the solution breaks at the finite point (x_b, y_b, t_b) of space-time; in addition, due to (49), all derivatives of u blow up at (x_b, y_b, t_b), except the derivative along the "transversal line of breaking", characterized by the vector field $\hat{V} = 2F_y t\partial_x + \partial_y$, for which

$$
\hat{V} u = F_y. \tag{53}
$$

Now we study the analytic behavior of the dKP solution near breaking, evaluating the characteristic equations $\zeta = x - 2F(\zeta, y, t)t$ in the regime:

$$x = x_b + x', \;\; y = y_b + y', \;\; t = t_b + t', \;\; \zeta = \zeta_b + \zeta', \tag{54}$$

where x', y', t', ζ' are small, obtaining, at the leading order, the cubic

$$Cubic(\zeta'; x', y', t') \equiv {\zeta'}^3 + a(y'){\zeta'}^2 + b(y', t')\zeta' - \gamma X(x', y', t') = 0, \tag{55}$$

where

$$\begin{array}{l}
a(y') = \frac{3F_{\zeta\zeta y}}{F_{\zeta\zeta\zeta}} y', \qquad b(y', t') = \gamma[2(F_\zeta + t_b F_{\zeta t})t' + F_{\zeta yy} t_b {y'}^2], \\
X(x', y', t') = x' - 2F(\zeta_b, y, t)t' - 2\,[F(\zeta_b, y, t) - F]\, t_b \sim \\
x' + \frac{F_y}{F_\zeta} y' - 2(F + t_b F_t)t' - \frac{F_{yy}}{2|F_\zeta|} {y'}^2 - 2(F_y + t_b F_{yt})y' t' - \frac{F_{yyy}}{6|F_\zeta|} {y'}^3, \\
\gamma = \frac{6|F_\zeta|}{F_{\zeta\zeta\zeta}},
\end{array} \tag{56}$$

corresponding to the maximal balance

$$|\zeta'|, |y'| = O(|t'|^{1/2}), \quad X = O(|t'|^{3/2}). \tag{57}$$

The Function $\mathcal{S}$ reads, at the leading order,

$$\mathcal{S} = 2(F_\zeta + t_b F_{\zeta t})t' + \left(F_{\zeta\zeta\zeta}{\zeta'}^2 + 2F_{\zeta\zeta y} y'\zeta' + F_{\zeta yy}{y'}^2\right) t_b. \tag{58}$$

The solution is therefore described, around the breaking point, by

$$\begin{aligned} & u = F(\zeta_b + \zeta', y_b + y', t_b + t') \sim F + F_\zeta \zeta' + F_y y' + O(t'), \\ & Cubic\,(\zeta'; x', y', t') = 0. \end{aligned} \tag{59}$$

[Manakov,PMS 2008,2011,2012]

Equivalently, as in 1+1, one eliminates ζ', obtaining a cubic for u:

$$Cubic\left(\frac{u - F - F_y y'}{F_\zeta}; x', y', t'\right) = 0. \tag{60}$$

After breaking. If $t > t_b$ ($t' > 0$), the SM in space-time coordinates is given by the zero discriminant condition $\Delta(x', y', t') = 0$, delimiting a compact region of the (x,y) plane with two cusp points $(x_C^{\pm}(t'), y_C^{\pm}(t'))$:

$$y_C^{\pm}(t') \equiv \pm|F_\zeta|\sqrt{\frac{2F_{\zeta\zeta\zeta}}{\beta}}\sqrt{t'}, \tag{61}$$

$$\begin{aligned} x_C^{\pm}(t') \equiv & -\tfrac{F_y}{F_\zeta}y_C^{\pm}(t') + \left(F + \left(\tfrac{F_y}{F_\zeta}\right)^2\right)t' + \tfrac{F_{yy}}{2|F_\zeta|}y_C^{\pm}(t')^2 + 2(F_y + t_bF_{yt})y_C^{\pm}(t')t' \\ & + \tfrac{F_{yyy}}{6|F_\zeta|}y_C^{\pm}(t')^3 - \tfrac{2}{\gamma}\tfrac{a^3(y_C^{\pm}(t'))}{27} = \pm F_y\sqrt{\tfrac{2F_{\zeta\zeta\zeta}}{\beta}}\sqrt{t'} + O(t'), \end{aligned} \tag{62}$$

Summarizing:

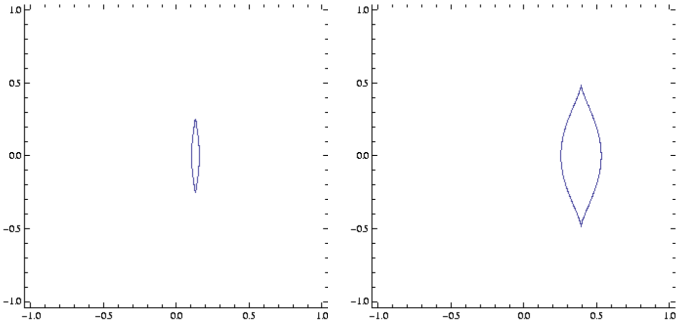
$$\begin{array}{ll} \Delta(x', y', t') < 0 & \text{if } y_C^{-}(t') < y' < y_C^{+}(t'),\ x_B^{-}(y', t') < x' < x_B^{+}(y', t'), \\ \Delta(x', y', t') = 0 & \text{if } y_C^{-}(t') \le y' \le y_C^{+}(t'),\ x' = x_B^{\pm}(y', t'), \\ \Delta(x', y', t') > 0 & \text{otherwise}, \end{array} \tag{63}$$

$$\begin{aligned} x_B^{\pm}(y', t') \equiv & -\tfrac{F_y}{F_\zeta}y' + \left(F + \left(\tfrac{F_y}{F_\zeta}\right)^2\right)t' + \tfrac{F_{yy}}{2|F_\zeta|}{y'}^2 + 2(F_y + t_bF_{yt})y't' \\ & + \tfrac{F_{yyy}}{6|F_\zeta|}{y'}^3 + \tfrac{2}{\gamma}\left(-\tfrac{a(y')b(y',t')}{18} - \tfrac{a(y')}{3}Q(y', t') \pm |Q(y', t')|^{3/2}\right). \end{aligned} \tag{64}$$

We end this section remarking that, since

$$x_B^+(y', t') - x_B^-(y', t') = O(t'^{3/2}), \tag{65}$$

it follows that the transversal width of the three valued region, estimated by the distance of the two cusps, is $O(t'^{1/2})$, while the longitudinal width is $O(t'^{3/2})$. Consequently, the three valued region develops, from the breaking point (x_b, y_b), with infinite speed in the transversal direction, and with zero speed in the longitudinal one. Typical shapes of the multivalued region for y-symmetric data:



Rigorous aspects of the IST for the Pavlov equation [Grinevich, PMS, Wu '14]

In analogy with KP1, f.i., a basic role is played by the real non analytic eigenfunctions and by the complex analytic ones. But the analogy ends here.

Indeed the construction and the proof of existence and uniqueness of KP1 (and of ALL soliton PDEs) eigenfunctions follows from the standard construction of the corresponding Green's functions and of the associated Fredholm integral equations. Since our Lax operators are vector fields, bounded (Fredholm) operators cannot be constructed and completely different strategies must be invented to characterize the eigenfunctions associated with vector fields and prove their existence and uniqueness.

To avoid extra technicalities, we assume that $v_0(x,y) = v(x,y,0) \in \mathbb{R}$ is smooth and has compact support:

$$v_0(x,y) = 0 \ \text{ outside the area } \ -D_x \le x \le D_x, -D_y \le y \le D_y.$$

Step 1: We construct the Jost functions and the classical scattering data.

By definition, the Jost functions are solutions of:

$$L\varphi_{\pm}(x,y,\lambda) = 0, \ \ L = \partial_y + (\lambda + v_x)\partial_x,$$

such that

$$\varphi_{\pm}(x,y,\lambda) \to x - \lambda y \ \text{ as } \ y \to \pm\infty.$$

These eigenfunctions are constant on the characteristics, i.e. are constant on the solutions of the corresponding ODE:

$$\frac{dx}{dy} = \lambda + v_x(x,y).$$

Consider the solutions of the Cauchy problem: $x(y) = x_0$ at $y = y_0$;
We have the following asymptotic:

$$x(y) \to \lambda y + x_{\pm}(x_0, y_0, \lambda), \ \ y \to \pm\infty.$$

It is easy to see that

$$x_{\pm}(x_0, y_0, \lambda) \to x_0 - \lambda y_0 \ \text{ as } \ y_0 \to \pm\infty;$$

therefore

$$\varphi_{\pm}(x_0, y_0, \lambda) = x_{\pm}(x_0, y_0, \lambda).$$

Since $\varphi_{\pm}$ enumerate the trajectories of L, they are a good basis for any eigenfunction of L, for $\lambda \in \mathbb{R}$.
The classical scattering amplitude $\sigma(\xi, \lambda)$ is defined $\xi \in \mathbb{R}$, $\lambda \in \mathbb{R}$ as the function connecting the asymptotics at $y \to +\infty$ and $y \to -\infty$:

$$x_+(x_0, y_0, \lambda) = x_-(x_0, y_0, \lambda) + \sigma(x_-(x_0, y_0, \lambda), \lambda).$$

Therefore:

$$\varphi_+(x, y, \lambda) \to x - \lambda y + \sigma(x - \lambda y, \lambda) \ \text{ as } \ y \to -\infty.$$

The regularity properties of $\sigma(\xi, \lambda)$ follow from the standard ODE theory.

Step 2: We construct the eigenfunction, analytic in the spectral parameter.
For complex λ, let us introduce the following complex notations:

$$z = x - \lambda y, \quad \bar{z} = x - \bar{\lambda} y$$

Equation on the wave function takes the form:

$$L\Phi^{\pm}(x, y, \lambda) = 0, \quad L = \partial_y + (\lambda + v_x)\partial_x.$$

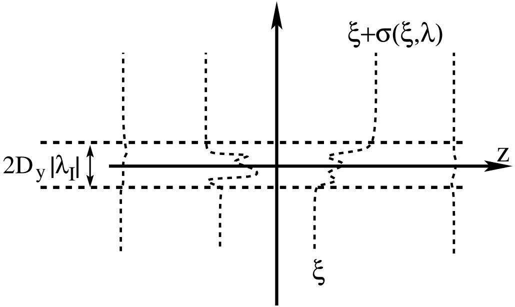
and can be written as a Beltrami equation:

$$\begin{aligned} &[\partial_{\bar{z}} + b(z, \bar{z}, \lambda)\partial_z]\,\Phi(z, \bar{z}, \lambda) = 0, \\ &b(z, \bar{z}, \lambda) = \tfrac{v_x(z,\bar{z})}{2i\lambda_I + v_x(z,\bar{z})}. \end{aligned} \tag{66}$$

Since $v \in \mathbb{R}$, , $|b| < 1$ and the Beltrami equ is uniquely solvable without the small norm assumption; in addition the solution is holomorphic in λ for $\mathrm{Im}\,\lambda \neq 0$.
What happens if $|\,\mathrm{Im}\,\lambda| \to 0$?. If the limiting values $\Phi^{\pm}(x, y, \lambda) = \Phi(x, y, \lambda \pm \epsilon),\ \lambda \in \mathbb{R}$ exist, they can be represented as functions of $\varphi_{\pm}$:

$$\begin{aligned} &\Phi^{-}(x, y, \lambda) = \varphi_{-}(x, y, \lambda) + \chi_{-}(\varphi_{-}(x, y, \lambda), \lambda) \\ &= \varphi_{+}(x, y, \lambda) + \chi_{+}(\varphi_{+}(x, y, \lambda), \lambda) \end{aligned} \tag{67}$$

defining the spectral data $\chi_{\pm}(\xi, \lambda)$.



If $\mathrm{Im}\,\lambda \ll 1$, $\mathrm{Im}\,\lambda < 0$, outside a small neighborhood of the real line in the z-plane (the support of v), $\Phi^-_{\bar{z}} = 0 \;\Rightarrow\; \Phi^-$ is holomorphic in z: $\Phi^-(x, y, \lambda) = \hat{\Phi}(z, \lambda)$, and almost constant on the characteristics:

$$\hat{\Phi}(\xi - i\epsilon, \lambda) \sim \hat{\Phi}(\xi + \sigma(\xi, \lambda) + i\epsilon, \lambda) \tag{68}$$

In addition, from (67):

$$\begin{aligned} \hat{\Phi}(\xi - i\epsilon, \lambda) &= \Phi^-(\xi, \lambda) = \xi + \chi_-(\xi, \lambda), \quad y < -D_y \\ \hat{\Phi}(\xi + i\epsilon, \lambda) &= \Phi^-(\xi, \lambda) = \xi + \chi_+(\xi, \lambda), \quad y > D_y \end{aligned} \tag{69}$$

Combining these equations, one obtains:

a linear Riemann problem with a shift for the spectral data $\chi_{\pm}(\xi,\lambda)$:

$$\sigma(\xi,\lambda)+\chi_{+}(\xi+\sigma(\xi,\lambda),\lambda)-\chi_{-}(\xi,\lambda)=0, \quad \xi\in\mathbb{R},$$

where the functions $\chi_{\pm}(\xi,\lambda)$ are analytic in ξ in the upper half-plane and in the lower half-plane respectively, with $\chi_{\pm}(\xi,\lambda)\to 0,\ |\xi|\to\infty$.

The solution of such a linear RH problem with a shift exists unique without a small norm assumption

The trivial t-dependence of the scattering and spectral data:

$$\begin{aligned}\sigma(\xi,\lambda,t)&=\sigma(\xi-\lambda^2 t,\lambda,0),\\ \chi_{\pm}(\xi,\lambda,t)&=\chi_{\pm}(\xi-\lambda^2 t,\lambda,0),\end{aligned} \tag{70}$$

is what justifies the IST method

The inverse spectral problem

The equation of the inverse spectral problem is the nonlinear integral equation:

$$\psi_-(x,y,t,\lambda) - H_\lambda \chi_{-I}\big(\psi_-(x,y,t,\lambda),\lambda\big) + \chi_{-R}\big(\psi_-(x,y,t,\lambda),\lambda\big) = x - \lambda y - \lambda^2 t,$$

where χ_{-R} and χ_{-I} denote the real and imaginary parts of χ_- respectively, and H_λ – denotes the Hilbert transform in λ: $H_\lambda f(\lambda) = \frac{1}{\pi} PV \int\limits_{-\infty}^{\infty} \frac{f(\lambda')}{\lambda-\lambda'} d\lambda'$. In terms of the Hilbert transform analyticity of $\chi_-(\xi,\lambda)$ ξ in the lower half-plane is equivalent to: $\chi_{-R} - H_\xi \chi_{-I} = 0$, and the solution of the Pavlov equation is reconstructed via

$$v(x,y,t) = -\frac{1}{\pi}\int_{\mathbb{R}} \chi_{-I}\left(\psi_-(x,y,t,\lambda),\lambda\right) d\lambda. \tag{71}$$

Unique solvability of the integral equation of the inverse problem

Theorem

Let the spectral data $\chi_-(\xi,\lambda)$ satisfy the following constraints:

1. $\chi_-(\xi,\lambda)$, $\partial_\xi\chi_-(\xi,\lambda)$ *are differentiable*
2. $$|\partial_\xi\chi_{-R}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right), \quad |\partial_\xi\chi_{-I}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right).$$
3. *For some $C > 0$*
$$|\chi_-(\xi,\lambda)| \le \frac{C}{1+|\lambda|}$$

Then for all $x, y, t \in \mathbb{R}$, $t \ge 0$, the inverse problem integral equation is uniquely solvable (the integral operator is a contraction) and $\psi(x,y,t,\lambda) = x - \lambda y - \lambda^2 t + \omega(x,y,t,\lambda)$, where $\omega(x,y,t,\lambda) \in L^2(d\lambda) \cap L^\infty(d\lambda)$.

The inverse problem solves the direct:

Theorem

Assume, that we have the following constraints on the inverse spectral data $\chi_{\pm}(\xi,\lambda)$:

1. $|\partial_\xi \chi_{-R}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right), \quad |\partial_\xi \chi_{-I}(\xi,\lambda)| \le \frac{1}{4}\tan\left(\frac{\pi}{8}\right).$
2. $|\partial_\xi^n \chi_{-}(\xi,\lambda)| \le \frac{C}{1+|\lambda|^{2+n}}$, $n = 0, 1, 2, 3.$
3. $|\partial_\xi^n \partial_\lambda \chi_{-}(\xi,\lambda)| \le \frac{C}{1+|\lambda|^{3+n}}$, $n = 0, 1,$

and denote by $\omega(x,y,t,\lambda) = \psi(x,y,t,\lambda) - (x - \lambda y - \lambda^2 t)$. Then

- *Then $\omega_x, \omega_y, \omega_t \in L^2(d\lambda) \cap L^4(d\lambda)$, $\omega_{xx}, \omega_{xy}, \omega_{xt}, \omega_{yy} \in L^2(d\lambda)$, they depend continuously on x, y, t, they are uniformly bounded in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$, and $\psi(x,y,t,\lambda)$ satisfies the Lax pair for the Pavlov equation.*

At last:

Theorem

Suppose $v_0(x, y)$ is a smooth, compact support, and small norm initial condition. Then above IST provides a real solution of the Cauchy problem for the Pavlov equation

$$\begin{aligned} &v_{xt} + v_{yy} = v_y v_{xx} - v_x v_{yy}, \\ &v(x, y, 0) = v_0(x, y) \end{aligned} \tag{72}$$

such that $v, v_x, v_y, v_{xx}, v_{xy}, v_{yy}, v_{xt} \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+) \cap L^\infty(\mathbb{R} \times \mathbb{R} \times \mathbb{R}^+)$.

What do we mean by “small data” in our problem?

Let us associate the following constants with the Cauchy data

$$B_0 = \int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_x(x,y)| \right] dy,$$

$$B_1 = \exp\left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xx}(x,y)| \right] dy \right] - 1,$$

$$B_2 = \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] (1+B_1)^3,$$

$$\begin{aligned} B_3 \quad &= \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxx}(x,y)| \right] dy \right] 3(1+B_1)^2 B_2 + \\ &+ \left[\int_{-\infty}^{+\infty} \left[\max_{x\in\mathbb{R}} |v_{xxxx}(x,y)| \right] dy \right] (1+B_1)^4, \end{aligned} \tag{73}$$

$$\hat{B}_0 = \left[\int_{-\infty}^{+\infty} \left(\sqrt{\int_{-\infty}^{+\infty} |v_x(x,y)|^2 dx} \right) dy \right] \cdot \frac{1}{\sqrt{1-B_1}}, \tag{74}$$

$$\hat{B}_1 = \left[\int_{-\infty}^{+\infty} \left(\sqrt{\int_{-\infty}^{+\infty} |v_{xx}(x,y)|^2 dx} \right) dy \right] \cdot \frac{1+B_1}{\sqrt{1-B_1}}. \tag{75}$$

Theorem

Assume that

1. $v(x,y) = 0$ *outside the area* $-D_x \leq x \leq D_x, \ -D_y \leq x \leq D_y$*.*
2. $B_0 \leq \frac{1}{4}$,
3. $B_1 \leq \frac{1}{2}$,
4. $8B_0 + 8B_2 + 2\sqrt{2}\hat{B}_0 < \pi$,
5. $2B_1 + \frac{\sqrt{2}}{\pi}(32B_1 + 16\hat{B}_0) + \frac{1}{\pi}(8B_3 + 16B_2^2 + 56B_1 + 16B_1^2)\left(B_0 + \frac{2}{\pi}[2B_0 + \hat{B}_0]\right) < \tan\left(\frac{\pi}{8}\right)$.

Then the unique solubility conditions for the inverse problem are fulfilled.

What happens, if we consider the inverse problem

$$\psi_-(x,y,t,\lambda)-H_\lambda\chi_{-I}\big(\psi_-(x,y,t,\lambda),\lambda\big)+\chi_{-R}\big(\psi_-(x,y,t,\lambda),\lambda\big)=x-\lambda y-\lambda^2 t,$$

with inverse data such that $\chi_{-R}-H_\xi\chi_{-I}\neq 0$?
It can be shown that we obtain the same solutions of the Pavlov equation, but the normalization of the wave function will be different from the Jost one at $y\to-\infty$.

HOW DO WE DEAL WITH NONLOCALITY OF dPDEs? [Grinevich, PMS '15]

$$(u_t + uu_x)_x + u_{yy} = 0, \text{ dKP}$$
$$v_{xt} + v_{yy} + v_x v_{xy} - v_y v_{xx} = 0, \text{ Pavlov}$$

nonlocal evolutionary forms and Cauchy problems:

$$u_t + uu_x = -\partial_x^{-1} u_{yy}, \quad u \in \mathbb{R}, \quad x, y, \in \mathbb{R}, \quad t > 0, \qquad (76)$$
$$u(x, y, 0) \ \text{given}$$

$$v_t = v_x v_y - \partial_x^{-1} [v_y + v_x^2]_y, \quad v \in \mathbb{R}, \quad x, y \in \mathbb{R}, \quad t > 0,$$
$$v(x, y, 0) \ \text{given}$$

∂_x^{-1} is the formal inverse of ∂_x, defined up to an arbitrary integration constant depending on y and t. On the other hand, the IST for integrable dispersionless PDEs provides us with a unique solution of the Cauchy problem in which the functions $u(x, y, 0), v(x, y, 0)$ are assigned, corresponding to a specific choice of such integration constant.

Result for dKP. 1) The IST formalism corresponds to the following evolutionary form of the Pavlov equation, for $t \geq 0$:

$$u_t(x, y, t) + u(x, y, t)u_x(x, y, t) = \int_x^{+\infty} u_{yy}(x', y, t)\, dx'. \tag{77}$$

2) In addition, for any smooth compact support initial condition and any $t > 0$, the solution develops the constraint

$$\partial_y^2 \mathcal{M}(y, t) \equiv 0, \text{ where } \mathcal{M}(y, t) = \int_{-\infty}^{+\infty} u(x, y, t)\, dx \tag{78}$$

identically in y and t.
This is the first of the so-called Manakov constraints and, to make the dynamics smoother initially, one can choose the initial condition in order to satisfy such constraint (it can be easily satisfied by lots of smooth initial data ..). BUT the more Manakov constraints one satisfies initially, the less relevant is dKP in applications ..

Result for Pavlov. 1) The IST formalism corresponds to the following evolutionary form of dKP, for $t \geq 0$:

$$v_t(x,y,t) = v_x(x,y,t)\, v_y(x,y,t) + \int_x^{+\infty} [v_y(x',y,t) + (v_{x'}(x',y,t))^2]_y \, dx'. \tag{79}$$

2) In addition, for any smooth compact support initial condition and any $t > 0$, the solution develops the constraint

$$\partial_y \mathcal{M}(y,t) \equiv 0, \text{ where } \mathcal{M}(y,t) = \int_{-\infty}^{+\infty} \left[v_y(x,y,t) + (v_x(x,y,t))^2 \right] \, dx, \tag{80}$$

identically in y and t, but, unlike the Manakov constraints for KP and for dKP, no rapidly decaying smooth initial data can satisfy this condition at $t = 0$. Indeed, if we have well-localized Cauchy data, then $\mathcal{M}(y,0) = \text{const}$, and $\mathcal{M}(y,0) \to 0$ for $|y| \to \infty$; therefore $\mathcal{M}(y,0) \equiv 0$. On the other hand,

$$\int\limits_{-\infty}^{+\infty} \mathcal{M}(y,0)\, dy = \int\limits_{-\infty}^{+\infty} \int\limits_{-\infty}^{+\infty} (v_x(x,y,0))^2 \, dxdy > 0,$$

unless $v_x(x,y,0) \equiv 0$.

Scketch of the proof (in the Pavlov case).

$$
\begin{aligned}
&v(x,y,t) = -\frac{1}{\pi}\int\limits_{\mathbb{R}} \chi_{-l}(\psi_{-}(x,y,t,\lambda),\lambda)d\lambda. \\
&v_t(x,y,t) = -\frac{1}{\pi}\int_{\mathbb{R}} \partial_\tau \chi_{-l}(x-\lambda y-\lambda^2 t+\omega(x,y,t,\lambda),\lambda)\psi_{-t}d\lambda, \quad t>0
\end{aligned}
\tag{81}
$$

For $t=0$ the integral repr. of v_t diverges and the calculation of v_t requires an additional investigation.

Our strategy is the following:
1) We calculate the t-derivative of $v(x,y,t)$ for $t \geq 0$ in what we call the “leading order approximation”;
2) We show that the correction to the leading order approximation vanishes for $x \to \pm\infty$.

The leading order approximation:

1) We replace $\sigma(\tau,\lambda)$ by the leading term $\sigma_L(\tau,\lambda)$ of the $\frac{1}{\lambda}$ expansion, for $\lambda \to \pm\infty$.

$$\sigma_L(\tau,\lambda) = \frac{sign(\lambda)}{\lambda^2} V_2\left(-\frac{\tau}{\lambda}\right). \tag{82}$$

where

$$V_2(y) = \int\limits_{-\infty}^{\infty} (-v_y - v_x^2)(x,y)dx,$$

2) The shifted RH problem is replaced by the standard one

$$\chi_{L-}(\tau,\lambda) - \chi_{L+}(\tau,\lambda) = \sigma_L(\tau,\lambda), \quad \tau \in \mathbb{R}, \tag{83}$$

where $\chi_{L\pm}(\tau,\lambda)$ are analytic in τ in the upper and lower half-planes $\mathbb{C}^\pm$ respectively, whose solution reads

$$\chi_{L-l}(\tau,\lambda) = -\frac{\chi_{2-l}\left(-\frac{\tau}{\lambda}\right)}{\lambda^2}, \tag{84}$$

where

$$\chi_{2-}(\zeta) - \chi_{2+}(\zeta) = V_2(\zeta).$$

3) We replace the eigenfunction by its normalization in the representation of v:

$$v_L(x,y,t) = -\frac{1}{\pi}\int\limits_{\mathbb{R}} \chi_{L-l}(x-\lambda y-\lambda^2 t,\lambda)d\lambda, \tag{85}$$

obtaining: $\partial_t v_L(x,y,t)=0, \quad t\geq 0, \quad x>0.$
For $x<0$:

$$v_L(x,y,t) = \frac{1}{|x|\pi}\int\limits_{2\sqrt{t|x|}}^{\infty} \frac{\chi_{2-l}(y-z)+\chi_{2-l}(y+z)}{\sqrt{1+\frac{4tx}{z^2}}}\,dz,$$

$$\partial_t v_L(x,y,t) = \frac{2}{\pi}\int\limits_{\sqrt{4t|x|}}^{\infty} \frac{[\chi'_{2-l}(y+z)-\chi'_{2-l}(y-z)]dz}{\sqrt{z^2-4t|x|}}.$$

Then

$$\partial_t v_L(x,y,t)\big|_{t=0+} = \int_{-\infty}^{+\infty}[v_y(x',y,0)+(v_{x'}(x',y,0))^2]_y\,dx', \quad x<0. \tag{86}$$

We see that, for $t=0+$ and $x<0$, the function $\partial_t v_L(x,y,t)$ does not depend on x. On the contrary, if $t>0$, then this function decays at $x\to-\infty$ as $O\left(\frac{1}{(t|x|)^{3/2}}\right)$.

One can show that, for $|x| \to \infty$, the exact formulas are well approximated by the leading order formulas, for fixed y, $t \geq 0$ and $|x| \to \infty$:

$$\partial_t v(x,y,t) - \partial_t v_L(x,y,t) = O\left(\frac{1}{\sqrt{|x|}}\right). \tag{87}$$

From the Pavlov equation in the non-evolutionary form (76), we see that $\partial_t v(x,y,t)\big|_{t=0+}$ is constant in x in both intervals $x < -D_x$ and $x > D_x$ outside the support of the Cauchy data. Taking $|x| \to \infty$, and using the above equs, we immediately obtain that:

$$\partial_t v(x,y,t)\big|_{t=0+} = \begin{cases} \int_{-\infty}^{+\infty} [v_y(x',y,0) + (v_{x'}(x',y,0))^2]_y \, dx', & x < -D_x, \\ 0, & x > D_x, \end{cases}$$

recall:

$$v_t = v_x v_y - \partial_x^{-1}[v_y + (v_{x'})^2]_y \tag{88}$$

which is consistent only with the choice $\partial_x^{-1} = -\int_x^{\infty} dx'$. We also obtain that

$$\partial_t v(x,y,t) \to 0, \quad \text{for} \quad x \to \pm\infty, \quad t > 0,$$

and, together with the fact that both $\partial_x v(x,y,t), \partial_x v(x,y,t) \to 0$ for $x \to \pm\infty$, $t > 0$, the evolutionary Pavlov, evaluated at $x \to -\infty$, implies the constraint $\mathcal{M}(y,t) \equiv 0$. We remark that, once this constraint is satisfied, for $t > 0$, all possible choices of ∂_x^{-1} become

Applications to wave breaking phenomena in Nature. Let us pretend that the family of nonlinear wave equations [Santucci, PMS 2012] describes some Physics:

$$(f(W))_{TT} = \triangle W, \qquad \triangle = \sum_{i=1}^{n} \partial_{X_i}^2, \qquad W = W(\vec{X}, T) \tag{89}$$

For **small amplitudes**: $W = \epsilon w$, $0 < \epsilon \ll 1$:

$$f(W) = f(0) + f'(0)\epsilon w + \frac{1}{2} f''(0)\epsilon^2 w^2 + \cdots \tag{90}$$

at $O(\epsilon)$, w satisfies the linear wave equation

$$w_{TT} = c^2 \triangle w, \qquad c = 1/\sqrt{f'(0)}, \text{ if } f'(0) > 0. \tag{91}$$

If the waves are **quasi one-dimensional** and we choose X_1 as the direction of propagation, the wave lengths in the trasversal directions are small: $\vec{k}_\perp = \epsilon^\alpha \vec{\kappa}_\perp$. Then the dispersion relation becomes

$$\omega = c\sqrt{k_1^2 + \vec{k}_\perp^2} = ck_1 \sqrt{1 + \epsilon^{2\alpha} \frac{\vec{\kappa}_\perp^2}{k_1^2}} \simeq ck_1 \left(1 + \epsilon^{2\alpha} \frac{\vec{\kappa}_\perp^2}{2k_1^2}\right), \tag{92}$$

and the phase of a monochromatic wave reads

$$\vec{k} \cdot \vec{X} - \omega T = k_1 (X_1 - cT) + \epsilon^\alpha \vec{\kappa}_\perp \cdot \vec{X}_\perp - \frac{c\epsilon^{2\alpha}}{2} \frac{\vec{\kappa}_\perp^2}{k_1^2} T, \tag{93}$$

$$\begin{cases} x &= X_1 - cT, \\ \vec{y} &= \epsilon^{\alpha}\vec{X}_{\perp}, \quad y_i = X_{i+1}, \; i = 1, \ldots, n-1, \\ t &= \epsilon^{2\alpha}\frac{c}{2}T. \end{cases} \quad (94)$$

In the new variables and imposing $\alpha = 1/2$ to get the maximal balance, one obtains, at $O(\epsilon^2)$, the *dKP* equation in $n+1$ dims

$$\begin{aligned} &(u_t + uu_x)_x + \triangle_{\perp} u = 0, \\ &u = -\frac{c^2}{2} f''(0) w. \end{aligned} \tag{95}$$

If the term $f''(0)$ vanishes, the maximal balance involves the cubic term and is achieved for $\alpha = 1$, obtaining the modified dKP:

$$\left(u_t + u^2 u_x\right)_x + \triangle_{\perp} u = 0, \tag{96}$$

and so on. The relation between the solution W of the nonlinear wave equation and the dKP solution u:

$$W(X_1, \vec{X}_{\perp}, T) = \epsilon u\left(X_1 - cT, \epsilon^{\frac{m}{2}}\vec{X}_{\perp}, \epsilon^{m}\frac{c}{2}T\right) = \epsilon u\left(x, \vec{y}, t\right) \tag{97}$$

The most relevant examples are:
1) Riemann equation for $n=1$:

$$u_t + u^m u_x = 0 \tag{98}$$

2) the integrable dKP equation for $(m,n) = (1,2)$:

$$(u_t + uu_x)_x + u_{yy} = 0 \tag{99}$$

3) the nonintegrable Khokhlov - Zobolotskaya (KZ) equation for $(m,n) = (1,3)$

$$(u_t + uu_x)_x + u_{y_1 y_1} + u_{y_2 y_2} = 0 \tag{100}$$

4) the nonintegrable modified dKP (mdKP) equation for $(m,n) = (2,2)$:

$$(u_t + u^2 u_x)_x + u_{yy} = 0 \tag{101}$$

Are dKP equations good models?

1) Information travels with finite speed for the wave equation, and with infinite speed for dKP.
2) The wave equation is a second order local PDE, whose solution is prescribed imposing two initial data: the initial position and momentum; dKP is a first order nonlocal PDE:

$$u_t + uu_x - \int_x^{\infty} u_{yy} dx' = 0, \tag{102}$$

whose solution is prescribed by the initial condition for u;
3) Unlike the wave equation, dKP (as KP) is affected by a non smooth behavior at $t = 0^+$; f.i., for any smooth initial condition,

$$\int_{\mathbb{R}} u_{yy}(x', y, t) dx' = 0, \quad t > 0. \tag{103}$$

These pathologies can be cured only imposing an infinite set of (the so-called Manakov) constraints; but the more constraints we impose, the less interesting is the dKP dynamics from the point of view of applications ..

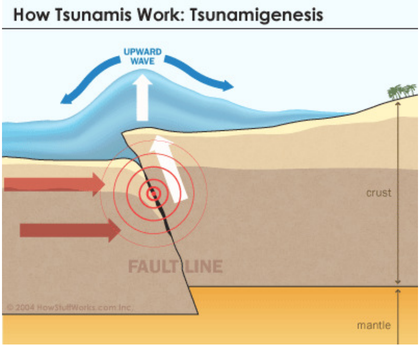
Two Cauchy problems for nonlinear waves in which dKP is relevant
(PMS'16, unpublished)

Weak nonlinearity is often achieved in natural phenomena on Earth;

Quasi-one dimensionality can be achieved essentially in two ways:

1) The initial data of the wave equ. are localized in both direction: $W(X, Y, 0) = \epsilon f(X, Y)$, but the wave front is studied far away from the source. In this case the X direction is the one defined by the positions of the source and of the observer.
From $W(X, Y, 0) = \epsilon u\,(x, \sqrt{\epsilon}y, 0)$, it corresponds to a Cauchy problem of dKP for a fastly varying initial condition in the y direction: $u(x, y, 0) = f(x, \frac{y}{\sqrt{\epsilon}})$

2) The initial data for the wave equation are slowly varying in the tranversal direction: $W(X, Y, 0) = \epsilon f(X, \sqrt{\epsilon} Y)$. In this case, Y is the transversal direction and X is the direction of propagation (f.i., a plate pushes itself under another, creating a bump on the water surface, and the fraction is extended for many Kilometers in the Y direction). It corresponds to a Cauchy problem of dKP for an initial condition localized in both directions: $u(x, y, 0) = f(x, y)$, and its solution requires the full dKP machinery (IST,..).



In the first Cauchy problem for the dKP(m,n) equs

$$(u_t + u^m u_x)_x + \sum_{j=1}^{n-1} u_{y_j y_j} = 0, \qquad \text{dKP(m,n)} \tag{104}$$

a relevant role is played by a family of exact solutions of dKP(m,n) equs associated with their invariance under transformations on the paraboloid $x + \sum_{j=1}^{n-1} \frac{y_j^2}{4t} = const$ (a Lie symmetry) [Manakov, PMS 2011; Santucci, PMS 2012].

Look for sol.s in the form

$$u = v(\xi, t), \quad \xi = x + \frac{1}{4t} \sum_{i=1}^{n-1} y_i^2, \tag{105}$$

reducing $dKP(m,n)$ to a PDE in $1+1$ dim.s

$$v_t + v^m v_\xi + \frac{n-1}{2t} v = 0. \tag{106}$$

Further change of variables

$$\begin{aligned} & v(\xi,t) = t^{-\frac{n-1}{2}} q(\xi, \tau(t)), \\ & \tau(t) = \begin{cases} \frac{1}{c_{m,n}} \; t^{c_{m,n}} + \alpha, & \text{if } c_{m,n} \neq 0 \\ \ln t + \beta, & \text{if } c_{m,n} = 0 \end{cases}, \end{aligned} \tag{107}$$

reduces it to the Riemann equation in the variables (ξ, τ)

$$q_\tau + q^m q_\xi = 0. \tag{108}$$

where

$$c_{mn} = 1 - \frac{m(n-1)}{2}. \tag{109}$$

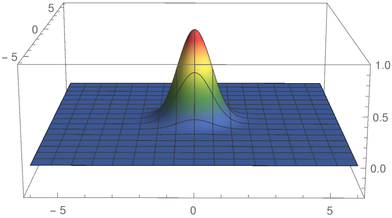
Therefore $dKP(m, n)$ possesses the family of exact implicit solutions

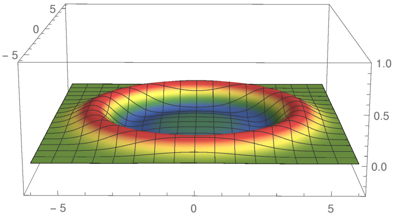
$$\begin{aligned} u &= \frac{1}{t^{\frac{n-1}{2}}} F\Big(x + \sum_{j=1}^{n-1} \frac{y_j^2}{4t} - \frac{1}{c_{mn}} u^m t\Big), \quad c_{mn} \neq 0, \\ u &= \frac{1}{t^{\frac{n-1}{2}}} F\Big(x + \sum_{j=1}^{n-1} \frac{y_j^2}{4t} - u^m t \log t\Big), \quad c_{mn} = 0, \end{aligned} \tag{110}$$

describing wave breaking on parabolic wave fronts; after breaking these solutions are either multivalued or they describe single value discontinuous shocks of dissipative nature.

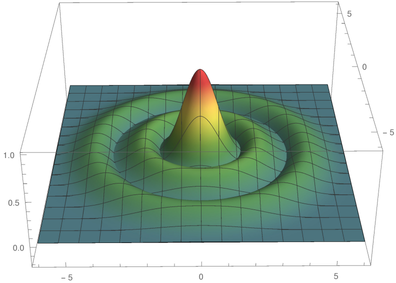
For dKP: $c_{12} = 1/2$; for mdKP: in $2+1$ dims $c_{22} = 0$; for KZ: $c_{13} = 0$.

1^{st} Cauchy problem: the initial data are generic localized single humps; f.i., if the initial momentum is zero and $w(X, Y, 0) = e^{-X^2-Y^2}$

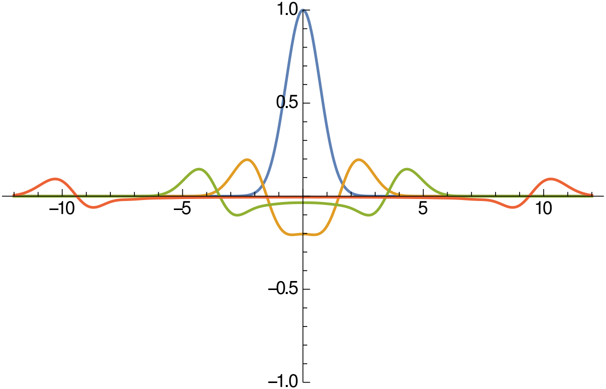




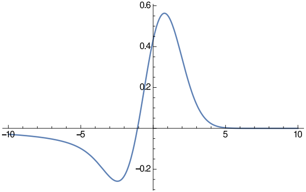
snapshots at different times:



snapshots at different times of the $Y = 0$ section of the wave:



detail of the asymptotic wave front:



$m = 1,\ n = 2$; the initial data of the wave equation are localized in both directions.
For finite T and, in general, $T \ll O(\epsilon^{-1})$, we are in the linear regime

$$\begin{array}{l} w_{TT} = w_{XX} + w_{YY}, \quad w(X, Y, T),\ X, Y \in \mathbb{R},\ T \geq 0, \\ w(X, Y, 0) = A(X, Y), \quad w_T(X, Y, 0) = B(X, Y) \end{array} \tag{111}$$

where the initial profile $A(X, Y)$ and the initial momentum $B(X, Y)$ are smooth and localized, having in mind the physically relevant case of generic convex single humps belonging to the Schwarz space.
It is easy to verify that the total mass of (111) grows linearly in time:

$$\begin{array}{l} M(t) \equiv \int_{\mathbb{R}^2} w(X, Y, T) dX dY = M_1 t + M_0, \\ M_0 \equiv \int_{\mathbb{R}^2} A(X, Y) dX dY, \quad M_1 \equiv \int_{\mathbb{R}^2} B(X, Y) dX dY, \end{array} \tag{112}$$

while the total energy is a constant of motion:

$$E(T) \equiv \frac{1}{2} \int_{\mathbb{R}^2} \left(w_T^2(X, Y, T) + w_X^2(X, Y, T) + w_Y^2(X, Y, T) \right) dX dY = E(0). \tag{113}$$

For the X-mass (the mass of the $Y = 0$ section)

$$\tilde{M}_{lin\ wave}(Y, T) \equiv \int_{-\infty}^{\infty} w(X, Y, T) dX \tag{114}$$

it is easy to prove the following result:

$$\tilde{M}_{lin\ wave}(Y, T) \to \frac{M_1}{2}, \ \text{ as } T \gg 1 \ \text{ and } \ \frac{|Y|}{T} \to 0 \tag{115}$$

faster than any power. In particular, if the initial momentum is zero, then $M_1 = 0$ and $\tilde{M}_{lin\ wave}(Y, T) \to 0$ faster than any power, as $t \to \infty$ and $\frac{|Y|}{T} \to 0$.

We are now ready to show that the pathologies of linearized dKP are actually miracles of the asymptotic model. Since

$$\begin{aligned} &w(X, Y, T) \sim u(X - T, \sqrt{\epsilon} Y, \epsilon \tfrac{T}{2}), \\ &T = O(\epsilon^{-1}), \ X - T = O(1), \ Y = O(\epsilon^{-\frac{1}{2}}), \end{aligned} \tag{116}$$

where $u(x, y, t)$ satisfies the linearized dKP equation

$$u_{xt} + u_{yy} = 0 \tag{117}$$

If w solves the Cauchy problem (111) with $B(X, Y) = 0$, then $\tilde{M}(Y, T) \to 0$ as $t \to \infty$ faster than any power; correspondingly, as far as lin dKP is concerned, the x-mass reads

$$\tilde{M}_{lin\ dKP}(y, t) \equiv \int_{-\infty}^{\infty} u(x, y, t)dx \tag{118}$$

therefore we have the relation

$$\begin{aligned} &\tilde{M}_{lin\ wave}(Y, T) \sim \tilde{M}_{lin\ dKP}(y, t), \\ &y = \sqrt{\epsilon}Y, \quad t = \epsilon\frac{T}{2} \end{aligned} \tag{119}$$

from which we infer that, since any $t > 0$ corresponds to $T \gg 1$, the discontinuity at initial time of the x-mass of linearized dKP is easily explained by the asymptotic property (115) of (111).

$$\begin{aligned} &\tilde{M}_{lin\ dKP}(y, 0) \sim \tilde{M}_{lin\ wave}(Y, 0) \neq 0, \\ &\tilde{M}_{lin\ dKP}(y, t) \sim \tilde{M}_{lin\ wave}(Y, \tfrac{2t}{\epsilon}) \sim 0, \quad t > 0. \end{aligned} \tag{120}$$

If $1 \ll T \ll O(\epsilon^{-1})$; we are still in the linear regime. Applying the stationary phase method to the Fourier representation of the solution:

$$w(X,Y,T) = \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^2} dK_1 dK_2 \hat{A}(K_1,K_2) e^{i(K_1 X + K_2 Y)} \cos(\sqrt{K_1^2 + K_2^2}\, T) + \frac{1}{4\pi^2} \int\limits_{\mathbb{R}^2} \frac{dK_1 dK_2}{\sqrt{K_1^2+K_2^2}} \hat{B}(K_1,K_2) e^{i(K_1 X + K_2 Y)} \sin(\sqrt{K_1^2 + K_2^2}\, T), \tag{121}$$

where $\hat{A}$ and $\hat{B}$ are the Fourier transforms of A and B:

$$\begin{aligned} \hat{A}(K_1,K_2) &= \int_{\mathbb{R}^2} A(X,Y) e^{-i(K_1 X + K_2 Y)} dX dY, \\ \hat{B}(K_1,K_2) &= \int_{\mathbb{R}^2} B(X,Y) e^{-i(K_1 X + K_2 Y)} dX dY, \end{aligned} \tag{122}$$

we obtain the asymptotic formula

$$\begin{aligned} & w \sim \frac{1}{\sqrt{T}} G(R - T, \alpha), \quad T \gg 1, \\ & X = R\cos\alpha, \quad Y = R\sin\alpha, \quad R = \sqrt{X^2+Y^2} = O(T), \end{aligned} \tag{123}$$

where

$$G(\xi,\alpha) \equiv \sqrt{2\pi} \int\limits_0^\infty dk \left[\sqrt{k}\, \mathrm{Re} \left(\hat{A}(k\cos\alpha, k\sin\alpha) e^{ik\xi - i\frac{\pi}{4}} \right) + \frac{1}{2\sqrt{k}} \mathrm{Re} \left(\hat{B}(k\cos\alpha, k\sin\alpha) e^{ik\xi + i\frac{\pi}{4}} \right) \right], \tag{124}$$

Choose, for instance, the gaussian initial condition:

$$A(x,y) = e^{-(a^2x^2+b^2y^2)}, \quad B(x,y) = 0, \tag{125}$$

where a, b are positive $O(1)$ constants; then

$$\hat{A}(k_1,k_2) = \frac{\pi}{ab} e^{-\left(\frac{k_1}{2a}\right)^2 - \left(\frac{k_2}{2b}\right)^2}, \quad \hat{B}(k_1,k_2) = 0, \tag{126}$$

$$G(\xi,\alpha) = \frac{\pi^{3/2}}{2ab\sqrt[4]{C^3(\alpha)}}\left[\Gamma\left(\tfrac{3}{4}\right)\ {}_1F_1\left(\tfrac{3}{4},\tfrac{1}{2},-\tfrac{Z^2}{4}\right) + \right. \\ \left. \tfrac{Z}{2}\Gamma\left(\tfrac{5}{4}\right)\ {}_1F_1\left(\tfrac{5}{4},\tfrac{3}{2},-\tfrac{Z^2}{4}\right)\right] \tag{127}$$

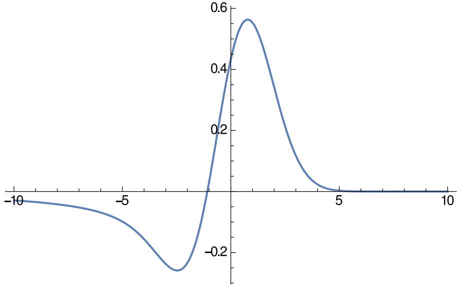
where Γ is the Gamma function, ${}_1F_1$ is the Kummer confluent hypergeometric function:

$$\begin{gathered} {}_1F_1(a;b;z) \equiv \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(b)_n n!}, \\ (a)_n \equiv a(a+1)(a+2)\dots(a+n-1). \end{gathered} \tag{128}$$

and

$$Z \equiv \frac{R-T}{C(\alpha)}, \quad C(\alpha) \equiv \left(\frac{\cos\alpha}{2a}\right)^2 + \left(\frac{\sin\alpha}{2b}\right)^2. \tag{129}$$

We remark that, if an initial condition is radially symmetric, then the solution w is radially symmetric $\forall T$, since the Laplacian is rotation invariant. In our explicit example (125), the initial condition is radially symmetric iff $a = b$. In this case, the coefficient $C = 1/(2a^2) = 1$ (if $a = 1/\sqrt{2}$) does not depend on α, and the asymptotic solution is radially symmetric too.



Asymptotic wave front generated by a Gaussian initial condition is expressed in terms of Kummer special functions.

Going to dKP variables, the circular wave front is approximated by its osculating parabola with vertex on the X axes:

$$R - T \sim x + \frac{y^2}{4t}; \tag{130}$$

in addition,

$$\tan\alpha = \frac{Y}{X} \sim \sqrt{\epsilon}\frac{y}{2t} \sim \alpha. \tag{131}$$

It follows that (123) takes the form of asymptotic solution of linearized dKP:

$$u(x, y, t) \sim \frac{\sqrt{\epsilon}}{\sqrt{t}} G(x + \tfrac{y^2}{4t}, \sqrt{\epsilon}\tfrac{y}{2t}). \tag{132}$$

Radially symmetric initial conditions lead therefore to the exact solution

$$u(x, y, t) \sim \frac{\sqrt{\epsilon}}{\sqrt{t}} G\left(x + \tfrac{y^2}{4t}\right), \tag{133}$$

of lin dKP, constant on the parabola $x + \frac{y^2}{4t} = const$.

When $T = O(\epsilon^{-1})$ we enter the nonlinear regime, and the solution is well described matching the linear asymptotic formula

$$u(x, y, t) \sim \frac{\sqrt{\epsilon}}{\sqrt{t}} G(x + \frac{y^2}{4t}, \sqrt{\epsilon}\frac{y}{2t}) \tag{134}$$

with the dKP exact solution

$$u = \frac{\sqrt{\epsilon}}{\sqrt{t}} F(x + \frac{y^2}{4t} - 2ut), \tag{135}$$

Obtaining

$$u \sim \frac{\sqrt{\epsilon}}{\sqrt{t}} G(x + \frac{y^2}{4t} - 2ut, \sqrt{\epsilon}\frac{y}{2t}). \tag{136}$$

Radially symmetric initial data of lead therefore to the exact dKP solution

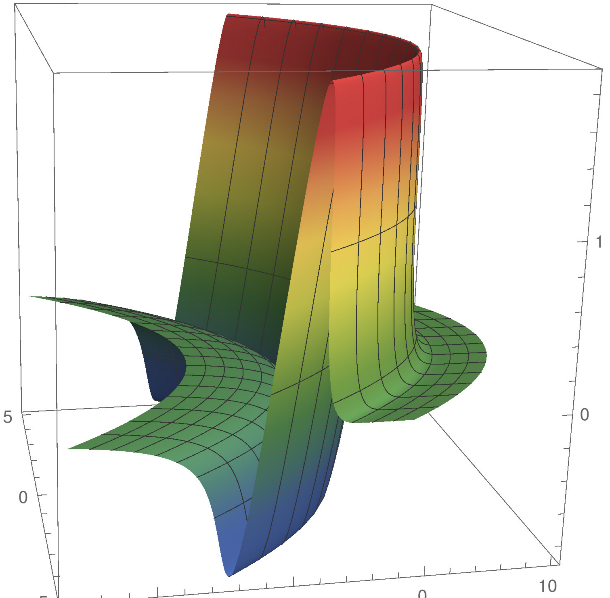
$$u(x, y, t) \sim \frac{\sqrt{\epsilon}}{\sqrt{t}} G\left(x + \frac{y^2}{4t} - 2ut\right), \tag{137}$$

constant on the parabola $x + \frac{y^2}{4t} = const$. Its characteristic form reads

$$\begin{aligned} &q = \sqrt{\epsilon} G(\zeta), \;\; \zeta = \xi - \sqrt{\epsilon} G(\zeta)\tau, \\ &\xi = x + \frac{y^2}{4t}, \;\; \tau = 2\sqrt{t}, \;\; q = \sqrt{t}u \end{aligned} \tag{138}$$

and the solution breaks at $\zeta_b = 1.96, \; \tau_b = 1.49, \; \xi_b = 3.27$; i.e., at $t_b = 0.56$, on the parabola $x + \frac{y^2}{4t_b} = \xi_b$.

The wave front at breaking, corresponding to a Gaussian initial condition for the nonlinear wave equation, is described in terms of confluent Kummer functions:



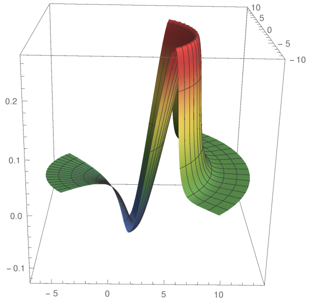
Going back to physical variables, we finally obtain the asymptotic solution of the nonlinear wave equation

$$W \sim \epsilon\sqrt{\frac{2}{T}}G(R-T-WT,\alpha), \tag{139}$$

where

$$G(\xi,\alpha) = \frac{\pi^{3/2}}{2ab\sqrt[4]{C^3(\alpha)}}\left[\Gamma\left(\tfrac{3}{4}\right)\ {}_1F_1\left(\tfrac{3}{4},\tfrac{1}{2},-\tfrac{Y^2}{4}\right)+\right.$$
$$\left.\tfrac{Y}{2}\Gamma\left(\tfrac{5}{4}\right)\ {}_1F_1\left(\tfrac{5}{4},\tfrac{3}{2},-\tfrac{Y^2}{4}\right)\right] \tag{140}$$

For radially symmetric initial data, the dependence on the angle is absent and breaking takes place at $T_b = 2t_b/ = 11.15$ on the circle of radius $R_b = T_b + \xi_b = 14.42$.



Analogously, for mdKP and KZ, for which $c_{mn} = 1 - m(n-1)/2 = 0$:
1) for mdKP $(u_t + u^2 u_x)_x + u_{yy} = 0$ the exact solution reads

$$u = \frac{1}{\sqrt{t}} F\left(x + \frac{y^2}{4t} - u^2 t \log t\right) \tag{141}$$

and the wave breaking in the physical Cauchy problem is decribed by

$$W \sim \frac{\epsilon}{\sqrt{T}} G\left(R - T - \frac{1}{2} W^2 T \log\left(\frac{\epsilon^2 T}{2}\right), \alpha\right) \tag{142}$$

2) for KZ $(u_t + u u_x)_x + u_{yy} + u_{zz} = 0$ the exact solution reads

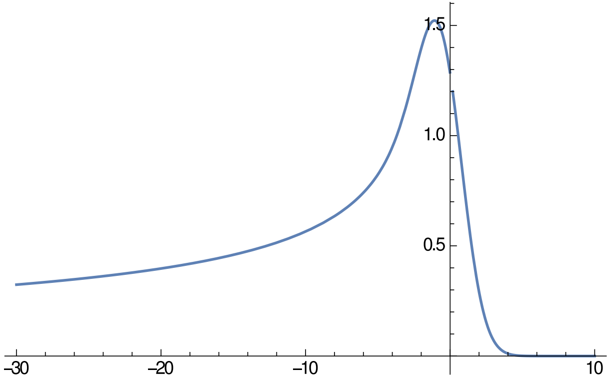
$$u = \frac{1}{t} F\left(x + \frac{y^2}{4t} + \frac{z^2}{4t} - u t \log t\right) \tag{143}$$

and the wave breaking in the physical Cauchy problem is decribed by

$$W \sim \frac{\epsilon}{T} G\left(R - T - \frac{1}{2} W T \log\left(\frac{\epsilon T}{2}\right), \alpha, \beta\right), \tag{144}$$

Also the asymptotic wave front generated by zero initial position and gaussian initial momentum is expressed in term of special functions (Bessel functions):

$$\int_0^\infty \frac{dk}{\sqrt{k}} e^{-k^2} \cos(|X|k + \mathrm{sign}(X)\frac{\pi}{4}) = \frac{1}{\sqrt{2}} e^{-\frac{X^2}{8}} \pi \sqrt{|X|} \times \\ (I_{-\frac{1}{4}}(\frac{X^2}{8}) \cos(\frac{\pi}{4}\,\mathrm{sign}(X)) - I_{\frac{1}{4}}(\frac{X^2}{8}) \sin(\frac{\pi}{4}\,\mathrm{sign}(X)))) \tag{145}$$



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SOLVABLE NONLINEAR RH PROBLEMS [Manakov and PMS '08, '09, '11]

Example 1 The invariant product:

$$\psi_1^+ = \psi_1^- e^{if(\psi_1^- \psi_2^-)}, \quad \psi_2^+ = \psi_1^- e^{-if(\psi_1^- \psi_2^-)}, \quad \lambda \in \mathbb{R}, \tag{146}$$

satisfying the symplectic and reality constraints. Then $\psi_1^+\psi_2^+$ is an invariant of the RH problem:

$$\psi_1^+\psi_2^+ = \psi_1^-\psi_2^- = -t\lambda^3 - y\lambda^2 + (x - 3ut)\lambda - 2yu + 3t\,\partial_x^{-1}u_y \equiv W(\lambda), \tag{147}$$

and the NRH problem linearizes and decouples:

$$\psi_1^+ = \psi_1^- e^{if(W(\lambda))}, \quad \psi_2^+ = \psi_2^- e^{-if(W(\lambda))}. \tag{148}$$

Separating the (+) and (-) parts:

$$\psi_j^+ e^{i(-)^j f^+(\lambda)} = \psi_j^- e^{i(-)^j f^-(\lambda)} = A_j(\lambda), \quad j = 1, 2, \tag{149}$$

where the analytic functions $f^\pm(\lambda)$, defined by

$$f^\pm(\lambda) \equiv \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{d\lambda'}{\lambda' - (\lambda \pm i0)} f(W(\lambda')), \tag{150}$$

exhibit the following asymptotics

$$f^\pm(\lambda) \sim i\sum_{n\geq 1}\langle\lambda^{n-1}f\rangle\lambda^{-n}, \quad \langle\lambda^n f\rangle \equiv \frac{1}{2\pi}\int_{\mathbb{R}}\lambda^n(W(\lambda))d\lambda, \tag{151}$$

if f decays faster than any power

Explicit solution of the NRH problem:

$$\psi_j^{\pm} = A_j(\lambda) e^{i(-)^{j+1} f^{+}(\lambda)}, \;\; j = 1, 2,$$
$$A_1(\lambda) \equiv -t\lambda^2 - (y + t < f >)\lambda + x - 2ut - y < f > -$$
$$t(< \lambda f > + \tfrac{1}{2} < f >^2), \;\; A_2(\lambda) \equiv \lambda - < f >, \tag{152}$$

NB: W depends on the unknowns u and $\partial_x^{-1} u_y$; the $1/\lambda$ terms of the expansions of equations (149) yield the algebraic system

$$q_1^{(1)} = 2t\partial_x^{-1} u_y - yu = -(x - 2ut) < f > + y(< f >^2 /2 + < \lambda f > +$$
$$t(< \lambda^2 f > + < f >< \lambda f > + < f >^3 /6)),$$
$$q_2^{(1)} = u = < \lambda f > - < f >^2 /2. \tag{153}$$

for such unknowns u and $\partial_x^{-1} u_y$. They correspond to the following differential constraint (Lie point symmetry) of dKP
$3tu_t + xu_x + 2yu_y + 2u = 0$. Substituting its general solution

$$u = t^{-2/3} B(x', y'), \;\; x' = \frac{x}{t^{1/3}}, \;\; y' = \frac{y}{t^{2/3}}, \tag{154}$$

into dKP, one obtains the similarity reduction of dKP:

$$B_{x'} + \frac{x'}{3} B_{x'x'} + \frac{2}{3} y' B_{x'y'} - B_{y'y'} - (BB_{x'})_{x'} = 0. \tag{155}$$

whose solution is characterized by the algebraic system (153).

Example 2 The invariant sum

$$\begin{aligned} \psi_1^+ &= \psi_1^- + iaf(\psi_1^- + a\psi_2^-), \\ \psi_2^+ &= \psi_2^- - if(\psi_1^- + a\psi_2^-), \quad \lambda \in \mathbb{R}, \end{aligned} \tag{156}$$

$$\psi_1^+ + a\psi_2^+ = \psi_1^- + a\psi_2^- = -t\lambda^2 - (y-a)\lambda + x - 2ut \equiv W(\lambda), \tag{157}$$

the NRH problem linearizes and decouples:

$$\psi_1^+ = \psi_1^- + iaf(W), \quad \psi_2^+ = \psi_2^- - if(W), \quad \lambda \in \mathbb{R}, \tag{158}$$

its solution:

$$\begin{aligned} \psi_1^\pm &= -t\lambda^2 - y\lambda + x - 2ut + iaf^\pm(\lambda), \\ \psi_2^\pm &= \lambda - if^\pm(\lambda). \end{aligned} \tag{159}$$

Expanding the above equ.s for large λ, it is possible to express the coefficients $q_{1,2}^{(n)}$ of the asymptotic expansions of $\psi_\pm^\pm$ in terms of the spectral function $f(W)$ in the following way:

$$\begin{aligned} q_1^{(n)} &= -a < \lambda^{n-1} f >, \quad n \geq 1, \\ q_2^{(n)} &= < \lambda^{n-1} f >, \quad n \geq 1. \end{aligned} \tag{160}$$

Since W in (157) is function of $q_2^{(1)} = u$ only, equation (160b) for $n = 1$

$$\begin{aligned} u &= \frac{1}{\sqrt{t}} F\left(x + \frac{(y-a)^2}{4t} - 2ut\right). \\ F(z) &= \frac{1}{2\pi} \int f(-\mu^2 + z) d\mu. \end{aligned} \tag{161}$$

From the particular examples to the general method

The goal is to construct RH probl.s i) possessing an invariant; ii) satisfying the reality and symplectic constraints.

Solvable nonlinear RH problems Consider an autonomous Hamiltonian two-dimensional dynamical system with Hamiltonian

$$H(\underline{x}) = \mathcal{H}(E(\underline{x})) \tag{163}$$

where $\underline{x} \equiv (q, p)$ are canonically conjugated coordinates, $E(\underline{x})$ is a polynomial function of the coordinates and $\mathcal{H}(\cdot)$ is an arbitrary function of a single argument, corresponding to the equations of motion

$$\frac{d\underline{x}}{d\tau} = \mathcal{H}'(E)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \nabla_{\underline{x}} E(\underline{x}). \tag{164}$$

Introducing action - angle variables in the usual way:

$$\begin{aligned} &J \equiv \tfrac{1}{2\pi}\oint p(q,H)dq, \quad \Rightarrow \quad H = H(J), \\ &\theta - \theta_0 \equiv \omega(J)\mathcal{H}'(E)(\tau - \tau_0) = \mathcal{H}'(E)\int_{q_0}^{q} \tfrac{\partial p(q',H(J))}{\partial J} dq', \\ &\omega(J) \equiv \tfrac{\partial H(J)}{\partial J}, \end{aligned} \tag{165}$$

the solution can be found inverting the quadrature (165):

$$\begin{aligned} &\vec{x} = \vec{\mathcal{D}}(\theta - \theta_0; \vec{x}_0, J), \\ &\{\mathcal{D}_1, \mathcal{D}_2\}_{(q_0,p_0)} = 1 \text{ symplectic} \end{aligned} \tag{166}$$

Identifying $\vec{x}(\tau) \to \vec{\psi}^{+}(\lambda),\ \vec{x}(\tau_0) \to \vec{\psi}^{-}(\lambda)$, equation (166) becomes the two-dimensional vector NRH problem

$$\vec{\psi}^{+} = \vec{\mathcal{D}}\left(\omega(J(\vec{\psi}^{-}))\mathcal{H}'(E(\vec{\psi}^{-})); \vec{\psi}^{-}, J(\vec{\psi}^{-})\right) \equiv \vec{\mathcal{R}}\left(\vec{\psi}^{-}\right) \tag{167}$$

connecting the $(-)$ and $(+)$ vector functions through the canonical transformation. $E(\vec{x}_0) = E(\vec{x}) \ \to \ E(\vec{\psi}^{-}) = E(\vec{\psi}^{+})$, "invariant" of the NRH problem (167). Since $E(\vec{\psi})$ is a polynomial function of its arguments, equation $E(\vec{\psi}^{-}) = E(\vec{\psi}^{+})$ define a polynomial in λ:

$$E(\vec{\psi}^{-}(\lambda)) = E(\vec{\psi}^{+}(\lambda)) \equiv W(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}), \tag{168}$$

given by the polynomial part of the asymptotic expansion of $E(\vec{\psi}^{\pm})$ for large λ, depending on a finite number of coefficients $\vec{q}_1^{N_1} = (q_1^1, \ldots, q_1^{N_1})$, $\vec{q}_2^{N_2} = (q_2^1, \ldots, q_2^{N_2})$ of the expansion. Since E is a real function of its arguments $\Rightarrow \overline{W(\bar{\lambda})} = W(\lambda) \Rightarrow$

$$\mathcal{H}'(\cdot) = if(\cdot). \tag{169}$$

Define

$$\theta^{\pm}(\lambda) \equiv \frac{1}{2\pi i}\int\limits_{\mathbb{R}} \frac{d\lambda'}{\lambda'-(\lambda \pm i0)} \left(i\omega(J(\vec{\psi}^{-}(\lambda')))f(W(\lambda'))\right)(\lambda'),$$
$$\theta^{\pm}(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}) \equiv -\sum_{n\geq 1} \frac{\langle \lambda^{n-1}\omega f\rangle}{\lambda^n}, \quad |\lambda| >> 1, \tag{170}$$
$$\langle \lambda^n g\rangle \equiv \frac{1}{2\pi}\int_{\mathbb{R}} \lambda^n g(\lambda) d\lambda$$

so that $i\omega f = \theta^+ - \theta^-$, the RH problem becomes

$$\vec{\mathcal{D}}(-\theta^+; \psi^+, J(\psi^+)) = \vec{\mathcal{D}}(-\theta^-; \psi^-, J(\psi^-)), \tag{171}$$

and provides the solution of the problem if $\vec{\mathcal{D}}$ is formally expandible, for large λ, in Laurent series with a finite number of positive powers:

$$\vec{\mathcal{D}}(-\theta^{\pm}(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}); \vec{\psi}^{\pm}, J(\vec{\psi}^{\pm})) = \vec{A}(\lambda)$$
$$\vec{A}(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}) = \left(\vec{\mathcal{D}}(-\theta^{\pm}(\lambda; \vec{q}_1^{N_1}, \vec{q}_2^{N_2}); \vec{\psi}^{\pm}, J(\vec{\psi}^{\pm}))\right)_+ \quad \text{polynomial} \tag{172}$$

Since the negative power part of such expansion is absent, the corresponding coefficients are zero; the first $N_1 + N_2$ of such equations for the first and second component of $\vec{\mathcal{D}}$ define a closed system of algebraic equations for the unknown fields $(\vec{q}_1^{N_1}, \vec{q}_2^{N_2})$, providing the wanted integration of the target nonlinear PDE.

Increasing the richness of the solution space

Let $\psi^{\pm}_{1,2}$ be the solutions of the above solvable NRH problem, satisfying the usual asymptotics:

$$\begin{aligned} &\psi^{\pm}_1(\lambda) = -\lambda^2 t - \lambda y + x - 2tq^{(1)}_2 + \sum_{n\geq 1} \frac{q^{(n)}_1}{\lambda^n}, \\ &\psi^{\pm}_2(\lambda) = \lambda + \frac{q^{(1)}_2}{\lambda} + \sum_{n\geq 2} \frac{q^{(n)}_2}{\lambda^n},\ q^{(1)}_2 = u. \end{aligned} \tag{173}$$

then arbitrary differentiable functions of $(\psi^{\pm}_1, \psi^{\pm}_2)$ are also solutions.

The first transformation: $\psi^{(1)\pm}_1 \equiv \psi^{\pm}_1 + a_1 f^{(1)}\left(\psi^{\pm}_2\right), \quad \psi^{(1)\pm}_2 \equiv \psi^{\pm}_2$

where $a_1 \in \mathbb{R}$ and $f^{(1)}$ is an arbitrary real function of one variable, is triangular, invertible, symplectic and preserves the reality constraint $\Rightarrow\ \psi^{(1)\pm}_{1,2}$ are also canonically conjugated solutions satisfying the reality constraint.

The second transformation:

$\psi^{(2)\pm}_1 \equiv \psi^{(1)\pm}_1, \quad \psi^{(2)\pm}_2 \equiv \psi^{(1)\pm}_2 + a_2 f^{(2)}\left(\psi^{(1)\pm}_1\right)$

has the same properties. Alternating the two transformations, at the m^{th} step, one constructs canonically conjugated solutions $\psi^{(m)\pm}_{1,2}$ satisfying the reality constraint, and parametrized by $(m+1)$ arbitrary real functions $f, f^{(1)}, \dots, f^{(m)}$ of a single argument.

Example: the invariant $\psi_1^+ + a(\psi_2^+)^n,\ n \in \mathbb{N}^+$
example of solvable volume preserving RH data:

$$\mathcal{R}_1(\vec{\zeta}) = \zeta_1 + if(\zeta_2),\ \ \mathcal{R}_2(\vec{\zeta}) = \zeta_2 \tag{174}$$

Elementary transformation:

$$\psi_1^{(1)\pm} \equiv \psi_1^{\pm},\ \ \psi_2^{(1)\pm} \equiv \psi_2^{\pm} + a(\psi_1^{\pm})^N,\ \ (f^{(1)}(\zeta) = \zeta^N) \tag{175}$$

Then:

$$\psi_1^+ = \psi_1^- - if(\psi_2^- + a{\psi_1^-}^n),\ \ \lambda \in \mathbb{R}, \psi_2^+ + a{\psi_1^-}^n = \psi_2^- + a{\psi_1^-}^n \tag{176}$$

Invariance equation $\psi_2^+ + a{\psi_1^+}^n = \psi_2^- + a{\psi_1^-}^n =$
$-t\lambda^2 - y\lambda + x - 2ut + a\left((\psi_1^-)^n\right)_+ \equiv W(\lambda)$,

$$\begin{aligned}
&\psi_1^{\pm} = \lambda - if^{\pm}(\lambda),\ \ \psi_2^{\pm} = W(\lambda) - a{\psi_1^{\pm}}^n,\\
&f^{\pm}(\lambda) \equiv \tfrac{1}{2\pi i}\int_{\mathbb{R}} \tfrac{d\lambda'}{\lambda' - (\lambda \pm i0)} f(W(\lambda')),\\
&f^{\pm}(\lambda) \sim i\sum_{n\geq 1}\langle \lambda^{n-1} f\rangle \lambda^{-n},\quad \langle \lambda^n f\rangle \equiv \tfrac{1}{2\pi}\int_{\mathbb{R}} \lambda^n f(W(\lambda))d\lambda,
\end{aligned} \tag{177}$$

Equation (177a) for $|\lambda| >> 1$ yields $q_1^{(n)} =< \lambda^{n-1} f >,\ \ n \geq 1$.
Since W depends on the $(n-1)$ unknowns $u, q_2^{(n)},\ n = 2, \dots, n-1$, it is an algebraic system of $(n-1)$ equ.s characterizing a family of implicit solutions of dKP parametrized by the arbitrary real function $f(\cdot)$.

The corresponding differential constraint:

$$\left(\left(\psi_2^{\pm} + a{\psi_1^{\pm}}^{n}\right)_{-1}\right)_x = yu_x - 2tu_y + anu_{t_n} = 0$$

where u_{t_n} is the n^{th} flow of the dKP hierarchy.

If $n = 1, 2$, one gets the known solution:

$$\begin{aligned} u = q_2^{(1)} = \tfrac{1}{\sqrt{t}} F\left(x + \tfrac{(y-a)^2}{4t} - 2ut\right), \\ F(z) = \tfrac{1}{2\pi} \int_{\mathbb{R}} f(-\mu^2 + z) d\mu \end{aligned} \qquad (178)$$

If $n=3$, in the longtime regime and for $|a| \ll 1$:

$$\begin{array}{l} u \sim \frac{1}{\sqrt{\tau_3}} F(\xi - a\eta^3 - 2u\tau_3), \\ \tau_3 = t + 3a\eta, \quad \eta = y/2t, \\ \eta = O(1), \quad \xi - a\eta^3 - 2u\tau_3 = O(1), \quad t \gg 1, \quad |a| \ll 1, \end{array} \tag{179}$$

Known the first breaking time τ_b from the well known formula

$$\tau_b = \frac{1}{4F'(\zeta_b)^2} = \min_{\zeta \in \mathbb{R}} \frac{1}{4F'(\zeta)^2}, \quad F'(\zeta_b) < 0, \tag{180}$$

If $a > 0$, the first breaking takes place when $t_b \sim -\infty$ at $y_b \sim -\infty$, outside the asymptotic region of validity of our approximation, travelling towards the inner region along the wave front. Now let t be close to τ_b; then:

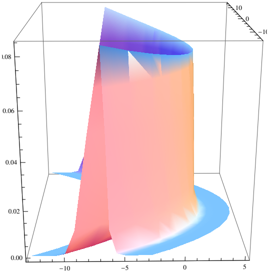
$$y = \frac{2}{3a} t(\tau_b - t) \sim \frac{2}{3a} \tau_b(\tau_b - t), \tag{181}$$

implying that, in the asymptotic region (179), the breaking point moves approximately with the constant speed $2\tau_b/(3a)$ along the wave front.

If $n = 4$, then

$$\begin{aligned} & u \sim \frac{1}{\sqrt{\tau_4}} F(\xi + a\eta^4 - 2u\tau_4), \\ & \tau_4 = t - 6a\eta^2, \\ & \eta = O(1), \ \ \xi + a\eta^4 - 2u\tau_4 = O(1), \ \ t \gg 1, \ \ 0 < a \ll 1. \end{aligned} \tag{182}$$

If the graph of $F(z)$ is a single positive hump, it describes, before breaking, a saddle wave front with saddle point $(\zeta_0 + 2F(\zeta_0)/\sqrt{t}, 0)$, where ζ_0 is the maximum of the hump: $F'(\zeta_0) = 0$.
Known the first breaking time τ_b as before, the first (physical) breaking time t_b is achieved at $y_b = 0$ $(\eta_b = y_b/2t_b = 0)$ and coincides with τ_b, while x_b follows from $x_b = \zeta_b + 2F(\zeta_b)\sqrt{t_b}$.



The basic examples generated for $N = 2$ have natural generalizations in higher dimensions. For example, the following volume preserving NRH problems are solvable:

$$\begin{array}{l} \psi_j^+ = \psi_j^- e^{if_j(\prod_k \psi_k^-)}, \quad j = 1, \dots, N, \\ \sum_j f_j = 0 \;\Rightarrow\; \prod_k \psi_k^- = \prod_k \psi_k^+ \text{ is invariant} \end{array} \tag{183}$$

$$\begin{array}{l} \psi_j^+ = \psi_j^- + if_j(\sum_k \psi_k^-), \quad j = 1, \dots, N, \\ \sum_j f_j = 0 \;\Rightarrow\; \sum_k \psi_k^- = \sum_k \psi_k^+ \text{ is invariant} \end{array} \tag{184}$$

Once we have constructed a solvable volume preserving NRH problem, it is possible to introduce a systematic procedure to increase the richness of the solution space of the integrable PDE, generalizing the procedure presented in [Manakov and PMS 2011]. Consider the triangular transformation

$$\begin{array}{l} \tilde{\psi}_1^{\pm} = \psi_1^{\pm}, \\ \tilde{\psi}_2^{\pm} = \psi_2^{\pm} + f_2^{(1)}(\psi_1^{\pm}), \\ \tilde{\psi}_3^{\pm} = \psi_3^{\pm} + f_3^{(1)}(\psi_1^{\pm}, \psi_2^{\pm}), \\ \vdots, \\ \tilde{\psi}_N^{\pm} = \psi_N^{\pm} + f_N^{(1)}(\psi_1^{\pm}, \dots, \psi_{N-1}^{\pm}) \end{array} \tag{185}$$

where the functions $f_j^{(1)},\ j = 2, \ldots, N$ are entire (polynomial). Then:
i) the transformation is invertible, and the inverse is also entire (polynomial);
ii) the Jacobian of the transformation is 1.
This tranformation can be combined with other triangular transformations of the same type, like, for instance:

$$
\begin{aligned}
&\tilde{\tilde{\psi}}_2^{\pm} = \tilde{\psi}_2^{\pm}, \\
&\tilde{\tilde{\psi}}_3^{\pm} = \psi_3^{\pm} + f_3^{(2)}(\tilde{\psi}_2^{\pm}), \\
&\tilde{\tilde{\psi}}_4^{\pm} = \psi_4^{\pm} + f_4^{(2)}(\tilde{\psi}_2^{\pm}, \tilde{\psi}_3^{\pm}), \\
&\vdots, \\
&\tilde{\tilde{\psi}}_1^{\pm} = \tilde{\psi}_1^{\pm} + f_1^{(2)}(\tilde{\psi}_2^{\pm}, \ldots, \psi_N^{\pm})
\end{aligned}
\tag{186}
$$

Let $\vec{\mathcal{T}}(\psi^{\pm})$ be an arbitrary combination of triangular, invertible and volume preserving transformations of this type, depending on an arbitrary functions of one, two, .., $N-1$ arguments. The normalization of $\vec{\mathcal{T}}(\psi^{-})$:

$$
\vec{\mathcal{T}}(\psi^{-}) = \vec{\mu}(\lambda) + O(\lambda^{-1}), \quad \lambda \sim \infty \tag{187}
$$

in particular, follows from the definition of $\vec{\mathcal{T}}(\psi^{\pm})$. Let $\vec{\mathcal{G}}$ be a solvable volume preserving NRH problem associated with a set of invariants $\{I(\vec{\psi})\}$.

Then the NRH problem

$$\begin{array}{l} \vec{\mathcal{T}}(\psi^{+}) = \vec{\mathcal{G}}(\vec{\mathcal{T}}(\psi^{-})), \\ \vec{\mathcal{T}}(\psi^{-}) = \vec{\mu}(\lambda) + (\lambda^{-1}), \quad \lambda \sim \infty \end{array} \tag{188}$$

can also be solved in terms of the invariants $\{I(\vec{\mathcal{T}}(\vec{\psi}))\}$. Then $\vec{\psi}^{\pm}$, solution of the volume preserving NRH problem

$$\vec{\psi}^{+} = \left(\vec{\mathcal{T}}^{-1} \circ \vec{\mathcal{G}} \circ \vec{\mathcal{T}}\right) \vec{\psi}^{-}, \tag{189}$$

can be directly constructed from $\vec{\mathcal{T}}(\psi^{\pm})$ inverting such a transformation. As a result of this procedure, the solution of the nonlinear PDE depend on a large number of arbitrary functions of several variables. The same procedure applies also to the case of solvable generalized volume preserving NRH problems.