Twistor Theory and Topology of Ward Solitons

LMS - EPSRC Durham Symposium
Geometric and Algebraic Aspects of Integrability
28 July 2016

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2007, Proc. R. Soc. A (M. Dunajski, PP) arXiv:hep-th/0605185
2016, J. Geom. Phys. (PP) arXiv:1504.06038[hep-th]

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Introduction

- Ward chiral model (1988, J.Math.Phys.):

$$\partial_t(J^{-1}\partial_t J)-\partial_x(J^{-1}\partial_x J)-\partial_y(J^{-1}\partial_y J)-[J^{-1}\partial_t J, J^{-1}\partial_y J]=0,$$

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- Ioannidou & Manton (2005): total energy of a time-dependent $SU(2)$ 2-uniton is quantised. (units of 8π)

Uniton solutions

- n - uniton

$$J(x,y,t) = M_1 M_2 \dots M_n, \quad \text{where} \quad M_k = \mathbf{1} - \left(1 - \frac{\mu}{\bar{\mu}}\right) R_k$$

$$\mu \in \mathbb{C} \backslash \mathbb{R}, \quad R_k = \frac{q_k^* \otimes q_k}{||q_k||^2}, \quad q_k \in \mathbb{C}^N$$

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$$q_1 = (1, f), \qquad q_2 = (1 + |f|^2)(1, f) - 2i(tf' + h)(\bar{f}, -1)$$

$$f(z) = z, \quad h(z) = z^2, \quad z = x + iy$$

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- Topology and energy of time-dependent unitons
  1. Energy of uniton is proportional to third homotopy class of extended solution.
  2. Third homotopy class corresponds to second Chern number of the twistor bundle.

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$$\begin{aligned} L_0\Psi &:= (D_y + D_t - \lambda(D_x + \Phi))\Psi = 0 \\ L_1\Psi &:= (D_x - \Phi - \lambda(D_t - D_y))\Psi = 0 \end{aligned}$$

$$\Psi(x^\mu, \lambda) \in GL(N, \mathbb{C}), \quad \lambda \in \mathbb{CP}^1, \quad D_\mu\Psi = \partial_\mu\Psi + A_\mu\Psi$$

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- $J(x^\mu) = \Psi^{-1}(x^\mu, \lambda = 0)$.

Topological Degree

- Restricted extended solution $\Psi(x, y, t, \lambda)$

$$\psi(x, y, \theta) := \Psi(x, y, 0, -\cot\frac{\theta}{2})$$

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$(S^2 \times [0, 1])/((S^2 \times \{0\}) \cup (S^2 \times \{1\}) \cup (\{x_0\} \times [0, 1]))$

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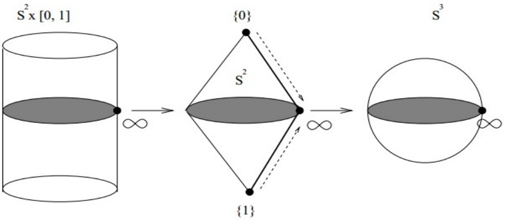

Freedom: $\psi(x, y, 0) = \mathbf{1}$

- $\psi : S^3 \to U(N)$, third homotopy class $\pi_3(U(N)) = \mathbb{Z}$

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Theorem Total energy $E = -\frac{1}{2}\int_{\mathbb{R}^2} \mathrm{Tr}((J^{-1}J_t)^2 + (J^{-1}J_x)^2 + (J^{-1}J_y)^2)\, dxdy$ of an n-uniton solution with $\mu = me^{i\phi}$ is quantised and equal to

$$E_{(n)} = 4\pi \left(\frac{1 + m^2}{m}\right) |\sin(\phi)|\ [\psi]$$

Proof: Exploit the Bäcklund relation (Dai & Terng 2007).

Twistor Theory

- Twistor Correspondence (Ward 1977)

  Solutions of SDYM on spacetime $\longleftrightarrow$ *Holomorphic vector bundles on twistor space*

- Integrability: existence of twistor construction

- Most (low dim) integrable systems can be realised as symmetry reductions of SDYM eq in 4 dim.

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  - $\mathbb{R}^3$: Euclidean monopole eq.

    (Ward 1981, Hitchin 1982)

  - $\mathbb{R}^{2,1}$: Yang-Mills-Higgs system/ Ward chiral model

    (Ward 1989)

Ward Chiral model

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$\mathbb{R}^{2,1}$		$T\mathbb{P}^1$
points p	$\longleftrightarrow$	"*real*" *sections* $\hat{p}$
real null planes	$\longleftrightarrow$	*real points*
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- J *satisfying trivial scattering BC* $\lim_{r\to\infty} \psi(x, y, \theta) = \mathbf{1}$ $\longleftrightarrow$ *Hol. vector bundles over compactified minitwistor space,* $E \to \overline{T\mathbb{P}^1}$

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- What is the spacetime interpretation of $c_2(E)$?

- Correspondence space

$$F = \mathbb{C}^3 \times \mathbb{CP}^1$$

$$\swarrow \qquad \searrow$$

$$M_{\mathbb{C}} = \mathbb{C}^3 \qquad\qquad T\mathbb{P}^1$$

$$F := \{(p, z) \in \mathbb{C}^3 \times T\mathbb{P}^1 \ : \omega = 2x\lambda + y(\lambda^2 - 1) + t(1 + \lambda^2))\}$$

  - $\Psi(x, y, t, \lambda)$ on $\mathbb{R}^{2,1} \times \mathbb{CP}^1$
  - $\psi(x, y, \theta)$ on $\mathbb{R}^2 \times S^1$

Compactified twistor fibration $(Z_0, Z_1, Z_2, Z_3) \in \mathbb{C}^4 - \{0\}$

- $T\mathbb{P}^1$ as a cone minus the vertex $\mathcal{C} - \{\mathbf{z_0}\} \subset \mathbb{CP}^3$

$$Z_1^2 + Z_2^2 - Z_3^2 = 0, \quad \mathbf{z_0} = [Z_0 \neq 0, 0, 0, 0]$$

- null planes in $\mathbb{C}^3$ $\longleftrightarrow$ points of $\mathcal{C} - \{\mathbf{z_0}\}$

$$\mathbb{CP}^2_\infty \quad \longleftrightarrow \quad \mathbf{z_0}$$

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- $\overline{T\mathbb{P}^1}$: fibrewise compactification of $T\mathbb{P}^1$
  - $\mathbb{C}$ - fibre $\longrightarrow \mathbb{CP}^1$ - fibre (add L_∞)
  - Blow-up $\tilde{\mathcal{C}}$ of $\mathcal{C}$ $\quad \{\mathbf{z_0}\} \to L_\infty$

Restricted correspondence space

- $\hat{\mathcal{F}}$: Blow-up of $\hat{f}$ along $\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\}$
  - $\mathbb{CP}^2_\infty \times \{\mathbf{z_0}\} \longrightarrow \mathbb{CP}^2_\infty \times L_\infty$

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  $\tau_\mathbb{R} := \mathbb{RP}^2$ compactification of $t = 0$, $\mathbb{R}^2$ - plane

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Proposition

$q : \mathcal{F} \longrightarrow \overline{T\mathbb{P}^1}$ is onto.

Proof Every null plane in $\mathbb{CP}^3$ intersects $\tau_\mathbb{R}$.

Second Chern number

- $c_2(E) = -\frac{1}{8\pi^2}\int_{\mathcal{F}} \mathrm{Tr}(F \wedge F)$
- Let $\mathcal{F}_+ := \{(p, z) \in \mathcal{F} : \mathrm{Im}(\lambda) > -\epsilon\}$

  $\mathcal{F}_- := \{(p, z) \in \mathcal{F} : \mathrm{Im}(\lambda) < \epsilon\}$
- Choose a connection A that is flat over $\mathcal{F}_+$
- Stokes' Theorem

  $c_2(E) = \frac{1}{24\pi^2}\int_{\partial\mathcal{F}_-} \mathrm{Tr}\left((\psi^{-1}d\psi)^3\right) = [\psi]$

Conclusions

- Total energy of time-dependent Ward unitons is proportional to third homotopy class of extended soln.
- Construct a compactified twistor fibration which fibre over space-like initial surface and $\overline{T\mathbb{P}^1}$.
- Second Chern number of the twistor bundle corresponds to third homotopy class of extended soln.
  - Interpretation as total energy.
- Quantisation of energy can be generalised to energy-momentum quantisation - $SO(1,1)$ invariance.