SDYM equations on the self-dual background

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Theorem (Dunajski, Ferapontov and Kruglikov (2014))

There exist local coordinates (z, w, x, y) *such that any ASD conformal structure in signature* $(2,2)$ *is locally represented by a metric*

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2, \qquad (1)$$

where the functions $F,\ G : M^4 \to \mathbb{R}$ *satisfy a coupled system of third-order PDEs,*

$$\begin{aligned} &\partial_x(Q(F)) + \partial_y(Q(G)) = 0, \\ &(\partial_w + F_y\partial_x + G_y\partial_y)Q(G) + (\partial_z + F_x\partial_x + G_x\partial_y)Q(F) = 0, \qquad (2) \end{aligned}$$

where

$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

System (2) arises as $[X_1, X_2] = 0$ from the dispersionless Lax pair

$$X_1 = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda,$$
$$X_2 = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda.$$

Due to compatibility conditions, f_1 and f_2 can be expressed through F and G,

$$f_1 = -Q(G), \quad f_2 = Q(F),$$
$$Q = \partial_w\partial_x - \partial_z\partial_y + F_y{\partial_x}^2 - G_x{\partial_y}^2 - (F_x - G_y)\partial_x\partial_y.$$

Correspondence between ASD conformal structures and integrable system defined by generic commuting vector fields.
Real case with the signature (2,2) or, generally, complex analytic case may be considered.
Reductions:
Dunajski system - null Kähler case, divergence free vector fields
$f_1, f_2 = 0$ (no ∂_λ in the vector fields), divergence free - *Plebanski's second heavenly equation* (ASD, Ricci flat)

Integrability properties of this Lax pair

The hierarchy, Lax-Sato equations, the dressing scheme - Bogdanov, Dryuma and Manakov (2007)
The structure of the hierarchy in terms of vector fields

$$X_1^n = \partial_{z^n} - \lambda^n\partial_x + F_1^n(\lambda)\partial_x + G_1^n(\lambda)\partial_y + f_1^n(\lambda)\partial_\lambda,$$
$$X_2^n = \partial_{w^n} - \lambda^n\partial_y + F_2^n(\lambda)\partial_x + G_2^n(\lambda)\partial_y + f_2^n(\lambda)\partial_\lambda,$$

where we have two infinite sets of times z^n, w^n and two 'basic' variables x, y, the coefficients of vector fields are polynomials in λ of the order $n-1$.
Multidimensional version contains N infinite sets of times and N 'basic' variables.

Extension of the Lax pair

Consider a gauge field **A** in some (matrix) Lie algebra and 'covariant vector fields' X_1, X_2

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + F_x\partial_x + G_x\partial_y + f_1\partial_\lambda + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + F_y\partial_x + G_y\partial_y + f_2\partial_\lambda + A_2$$

(here A_1, A_2 do not depend on λ). Lax pairs of this structure were already present in Zakharov and Shabat (1979).

The commutator of two covariant vector fields contains vector field part and Lie algebraic part,

$$[\nabla_{X_1}, \nabla_{X_2}] = [X_1, X_2] + X_1 A_2 - X_2 A_1 + [A_1, A_2]$$

Demanding both parts to be equal to zero, from the first part we get the system describing conformally ASD metric, and the second part gives the system for A_1, A_2

$$\partial_x A_2 = \partial_y A_1,$$
$$(\partial_z + F_x\partial_x + G_x\partial_y)A_2 - (\partial_w + F_y\partial_x + G_y\partial_y)A_1 + [A_1, A_2] = 0.$$

ASDYM case

For $F = G = 0$ we have

$$X_1 = \partial_z - \lambda\partial_x,$$
$$X_2 = \partial_w - \lambda\partial_y,$$

$$\frac{1}{2}g = dwdx - dzdy.$$

The extended Lax pair takes the form

$$\nabla_{X_1} = \partial_z - \lambda\partial_x + A_1,$$
$$\nabla_{X_2} = \partial_w - \lambda\partial_y + A_2,$$

and the commutativity condition is

$$\partial_x A_2 = \partial_y A_1,$$
$$\partial_z A_2 - \partial_w A_1 + [A_1, A_2] = 0.$$

This a well known form of ASDYM equations for constant metric g in a special gauge.

General case

1. Geometry

Extended Lax pair gives a general form of ASDYM equations for arbitrary conformally ASD metric g in signature (2,2) (locally, up to transformations of coordinates and a gauge).

2. Integrability

Extended Lax pair belongs to the hierarchy which unites ASDYM hierarchy and generic 4-dimensional dispersionless hierarchy. Lax-Sato equations and dressing scheme can be constructed for this hierarchy.

Geometry

Given: conformally ASD metric g with signature (2,2) (ASD conformal structure) and ASD gauge field with a connection form $\mathbf{A}$. The corresponding gauge curvature form is $\mathbf{F} = \mathrm{d}\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$, it satisfies the ASDYM equation

$$\mathbf{F} = - * \mathbf{F}$$

First step:
following Dunajski, Ferapontov and Kruglikov, we find local coordinates (z, w, x, y) such that ASD conformal structure is locally represented by a metric

$$\frac{1}{2}g = dwdx - dzdy - F_y dw^2 - (F_x - G_y)dwdz + G_x dz^2,$$

Second step:
notice that for this metric due to ASDYM equation we have

$$F_{34} = 0,$$

where we have used inverse matrix to metric g defined by symmetric bivector

$$\frac{1}{2}Q = \partial_w\partial_x - \partial_z\partial_y + F_y\,\partial_x^2 + (G_y - F_x)\,\partial_x\partial_y - G_x\,\partial_y^2$$

$\det g = \det Q = 1$ (for this metric $F^{12} = F_{34}$). Then it is possible to choose a gauge such that

$$A_3 = A_4 = 0,$$

and we have only two nontrivial gauge field components A_1, A_2.

Third step:
we will prove that ASDYM equations for A_1, A_2 for the metric g coincide with Lie algebraic part of compatibility equations for extended Lax pair.

Tetrad of one-forms

The conformal structure is represented by (DFK)

$$g = 2(\mathbf{e}^{00'}\mathbf{e}^{11'} - \mathbf{e}^{01'}\mathbf{e}^{10'}),$$

where the tetrad of one-forms is

$$\begin{aligned}
\mathbf{e}^{00'} &= dw, \\
\mathbf{e}^{10'} &= dz, \\
\mathbf{e}^{01'} &= dy - G_y dw - G_x dz, \\
\mathbf{e}^{11'} &= dx - F_y dw - F_x dz.
\end{aligned}$$

Tetrad of vector fields

The dual tetrad of vector fields is

$$\begin{aligned}
\mathbf{e}_{00'} &= \partial_w + F_y\partial_x + G_y\partial_y, \quad (+A_2)\\
\mathbf{e}_{10'} &= \partial_z + F_x\partial_x + G_x\partial_y, \quad (+A_1)\\
\mathbf{e}_{01'} &= \partial_y,\\
\mathbf{e}_{11'} &= \partial_x,
\end{aligned}$$

symmetric bivector reads

$$Q = 2(\mathbf{e}_{00'}\mathbf{e}_{11'} - \mathbf{e}_{01'}\mathbf{e}_{10'}).$$

ASDYM equations for this tetrad take the form

$$F_{00'10'} = 0, \quad F_{00'11'} = F_{10'01'}$$

For gauge field curvature $\mathbf{F}$ in the tetrad basis we use a standard formula

$$\mathbf{F}(\mathbf{u},\mathbf{v}) = \nabla_{\mathbf{u}}\nabla_{\mathbf{v}} - \nabla_{\mathbf{v}}\nabla_{\mathbf{u}} - \nabla_{[\mathbf{u},\mathbf{v}]}$$

for arbitrary vector fields $\mathbf{u}$, $\mathbf{v}$. Taking into account the structure of tetrade and the fact that for our gauge $A_3 = A_4 = 0$, we see that the third term doesn't contain a gauge field, and for the curvature components we get

$$\begin{aligned}
&F_{00'10'} = (\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2],\\
&F_{00'11'} = -\partial_x A_2, \quad F_{10'01'} = -\partial_y A_1
\end{aligned}$$

Thus ASDYM equations read

$$\begin{aligned}
&(\partial_w + F_y\partial_x + G_y\partial_y)A_1 - (\partial_z + F_x\partial_x + G_x\partial_y)A_2 - [A_1, A_2] = 0,\\
&\partial_x A_2 = \partial_y A_1,
\end{aligned}$$

which coincides with the Lie algebraic part of commutativity condition for extended vector fields Lax pair.

Integration. The dressing scheme. Vector fields

1. First step. The vector fields part. Nonlinear vector Riemann-Hilbert problem (e.g. on the unit circle, here we don't discuss the question of reductions)

$$
\begin{aligned}
\Psi^0_{\text{in}} &= F_0(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),\\
\Psi^1_{\text{in}} &= F_1(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),\\
\Psi^2_{\text{in}} &= F_2(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),
\end{aligned}
$$

the expansions at infinity are

$$
\Psi^0_{\text{out}} = \lambda + \sum_{n=1}^{\infty} \Psi^0_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},
$$

$$
\Psi^1_{\text{out}} = \sum_{n=0}^{\infty} t^1_n(\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^1_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}
$$

$$
\Psi^2_{\text{out}} = \sum_{n=0}^{\infty} t^2_n(\Psi^0)^n + \sum_{n=1}^{\infty} \Psi^2_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n},
$$

inside the unit circle the functions are analytic.

Ψ^0, Ψ^1, Ψ^2 will give the wave fuctions for the hierarchy of commuting vector fields, defined through coefficients of expansion of these functions.

Integration. The dressing scheme. Matrix part

2. Second step. Consider a matrix Riemann-Hilbert problem

$$\Phi_{\text{in}} = \Phi_{\text{out}} R(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}}),$$

Φ is normalized by 1 at infinity and analytic inside the unit circle,

$$\Phi_{\text{out}} = 1 + \sum_{n=1}^{\infty} \Phi_n(\mathbf{t}^1, \mathbf{t}^2)\lambda^{-n}$$

Expansions of Ψ, Φ give coefficients for extended Lax pair, Φ is a wave function. A general wave function is given by the expression $\Phi F(\Psi^0, \Psi^1, \Psi^2)$, F is arbitrary matrix function.
For constant metric g corresponding to trivial vector fields we have

$$\Psi^0 = \lambda, \quad \Psi^1 = x + \lambda z, \quad \Psi^2 = y + \lambda w,$$

and we get standard Riemann-Hilbert problem for ASDYM.
Remark. A dressing scheme based on $\bar{\partial}$ problem can be also developed.

From the dressing scheme to the hierarchy

The vector fields part of the dressing scheme implies analyticity in the complex plane of the form (no dicontinuity on the unit circle)

$$\omega = \left| \frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0 \wedge \mathrm{d}\Psi^1 \wedge \mathrm{d}\Psi^2,$$

where $x_1 = t_0^1$, $x_2 = t_0^2$ are lowest times of the hierarchy, and from matrix Riemann problem we get analyticity of the matrix-valued form

$$\Omega = \omega \wedge \mathrm{d}\Phi \cdot \Phi^{-1}.$$

Analyticity of these forms imply the relations

$$(\omega_{\text{out}})_- = \left(\left| \frac{D(\Psi^0_{\text{out}}, \Psi^1_{\text{out}}, \Psi^2_{\text{out}})}{D(\lambda, x_1, x_2)} \right|^{-1} \mathrm{d}\Psi^0_{\text{out}} \wedge \mathrm{d}\Psi^1_{\text{out}} \wedge \mathrm{d}\Psi^2_{\text{out}} \right)_- = 0,$$

$$(\Omega_{\text{out}})_- = (\omega_{\text{out}} \wedge \mathrm{d}\Phi_{\text{out}} \cdot \Phi^{-1}_{\text{out}})_- = 0$$

for the series Ψ^0_{out}, Ψ^1_{out}, Ψ^2_{out}, Φ_{out}. These relations are generating relations for the hierarchy in terms of formal series, they are equivalent to the complete set of Lax-Sato equations of the hierarchy.
First relation gives Lax-Sato equations for the hierarchy of commuting polynomial in λ vector fields (here we drop subscript 'out' for the series):

$$\partial^k_n \Psi = \sum_{i=0}^{2} \left(\left(\frac{D(\Psi^0, \Psi^1, \Psi^2)}{D(\lambda, x_1, x_2)} \right)^{-1}_{ik} (\Psi^0)^n \right)_+ \partial_i \Psi,$$

where $1 \leqslant n < \infty$, $k = 1, 2$, $\partial_0 = \partial_\lambda$, $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$,
$\Psi = (\Psi^0, \Psi^1, \Psi^2)$.

The second generating relation gives Lax-Sato equations for Φ on the vector field background in terms of extended polynomial vector fields,

$$
\begin{aligned}
\partial_n^k \Psi &= V_n^k(\lambda)\Psi, \\
\partial_n^k \Phi &= \left(V_n^k(\lambda) - ((V_n^k(\lambda)\Phi) \cdot \Phi^{-1})_+ \right) \Phi
\end{aligned}
$$

First flows give exactly the extended Lax pair for ASDYM equations on ASD background, if we identify $z = t_1^1$, $w = t_1^2$, $x = x_1$, $y = x_2$.

Questions

- Solutions!
- Higher-dimensional case – what is the geometry?
- Lower-dimensional cases and reductions – known integrable systems on the background?

THANK YOU!