Periods of meromorphic quadratic differentials and Goldman bracket

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References

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Main equation

- C_g - Riemann surface of genus g.
- "Schrödinger equation" on C_g:

$$\varphi'' - u\varphi = 0$$

where φ is a $(-1/2)$-differential (locally), and
$-2u$ - meromorphic projective connection on C_g with n simple poles.

- Parametrization of space of all "potentials":

$$\varphi'' + (\frac{1}{2}S_0 + Q)\varphi = 0$$

S_0 - "base" projective connection, Q - meromorphic quadratic differential with n simple poles.

Canonical symplectic structure on $T^*\mathcal{M}_g$

- Moduli space of pairs (C_g, Q) is $\mathcal{Q}_{g,n} = T^*\mathcal{M}_{g,n}$; $\dim\mathcal{Q}_{g,n} = 6g - 6 + 2n$
- $\{q_i\}_{i=1}^{3g-3+n}$ - any complex coordinates on $\mathcal{M}_{g,n}$ (say, $3g-3$ entries of period matrix and $(v_1/v_2)(y_k)$); $\{v_i\}$ - normalized abelian differentials. $\{d\,q_i\}_{i=1}^{3g-3+n}$ - basis in cotangent space; $\{p_i\}_{i=1}^{3g-3+n}$ - coordinates of cotangent vector in this basis.
- Symplectic structure and symplectic potential:

$$\omega_{can} = \sum_{i=1}^{3g-3+n} dp_i \wedge dq_i \qquad \theta_{can} = \sum_{i=1}^{3g-3+n} p_i\, dq_i$$

Homological Darboux coordinates

- Let all zeros of Q be simple: $\{x_i\}_{i=1}^{4g-4+n}$. Canonical cover (spectral, Hitchin, Seiberg-Witten...) $\widehat{C}$:

$$v^2 = Q$$

  in T^*C_g; $4g-4+2n$ branch points at $\{x_i, y_i\}$; genus $4g-3+n$; involution $\mu : \widehat{C} \to \widehat{C}$
- Decomposition of $H_1(\widehat{C}, \mathbb{Z})$ into even and odd parts:

$$H_1(\widehat{C}, \mathbb{Z}) = H_- \oplus H_+$$

  where $\dim H_+ = 2g$, $\dim H_- = 6g-6+2n$. Generators of H_-: $\{a_i^-, b_i^-\}_{i=1}^{3g-3+n}$; intersection $a_i^- \circ b_j^- = \delta_{ij}$.
- Homological coordinates $A_i = \int_{a_i^-} v$, $B_i = \int_{b_i^-} v$.

Canonical cover

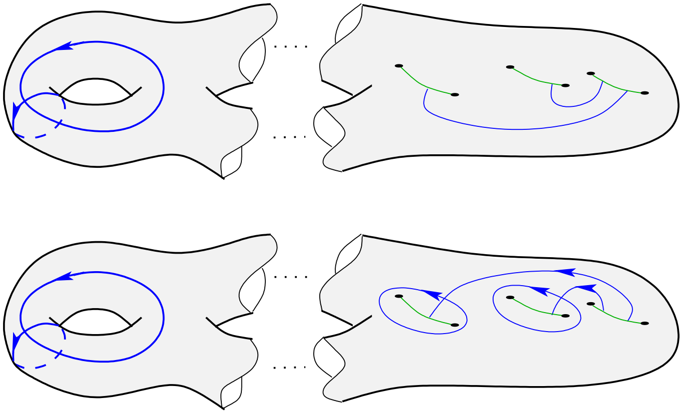

Figure: Canonical basis of cycles on the canonical cover $\widehat{C}$

Homological and canonical symplectic structures

- Homological symplectic structure on $T^*\mathcal{M}_g$:

$$\omega_{hom} = \sum_{i=1}^{3g-3+n} dA_i \wedge dB_i$$

- Theorem 1.

$$\omega_{hom} = \omega_{can}$$

Thus (A_i, B_i) are Darboux coordinates for ω_{can} on the main stratum of $T^*\mathcal{M}_g$ (all zeros of Q are simple).

Symplectic structure on the space of projective connections

- Space $\mathbb{S}_g$: pairs (C_g, S), S holomorphic projective connection on C_g. Affine bundle over $\mathcal{M}_g$.
- Given the "base" projective connection S_0 on C_g which holomorphically depends on moduli of C_g, write any S as $S = S_0 + 2Q$, for some holomorphic quadratic diff. Q.
- The map $F^{S_0} : \mathcal{Q}_g \to \mathbb{S}_g$ is used to induce symplectic structure on $\mathbb{S}_g$ from ω_{can}.
- Equivalence: $S_0 \equiv S_1$ if corresponding symplectic structures on $\mathbb{S}_g$ coincide. Generating function G_{01}:

$$\delta_\mu G = \int_{C_g} \mu(S_1 - S_0)$$

Equivalent projective connections

- Schottky projective connection $S_{Sch}(\cdot) = \{w, \cdot\}$, where w is the Schottky uniformization coordinate; $\{\cdot,\cdot\}$- Schwarzian derivative.
- main example: Bergman projective connection S_B. Canonical bimeromorphic differential $B(x, y)$ on $\mathbb{C}_g$: $\oint_{a_\alpha} B(\cdot, y) = 0$,

$$B(x,y) = \left(\frac{1}{(\xi(x) - \xi(y))^2} + \frac{1}{6} S_B(\xi(x)) + \dots \right) d\xi(x) d\xi(y)$$

  B depends on Torelli marking (choice of canonical basis in homologies on $\mathcal{C}$)
- Generating function from S_{Sch} to S_B: Zograf's F-function $F = \mathcal{Z}'_B(1)$; $\mathcal{Z}_B$- Bowen's zeta-function of Schottky group.
- Generating function corresponding to change of Torelli marking defining S_B is given by $\det(C\Omega + D)$ (cocycle of determinant of Hodge vector bundle).

Main tool: Variational formulas

For any $s_i \in H_-$ define $s_i^* \in H_-$ ($s_i \circ s_j^* = \delta_{ij}$); $\mathcal{P}_i = \int_{s_i} v$. Then

$$\frac{\partial B(x,y)}{\partial \mathcal{P}_i} = \frac{1}{2}\int_{t \in s_i^*} \frac{B(x,t)(B(t,y)}{v(t)}$$

where $z(x)$ and $z(y)$ are kept constant.

$$\frac{\partial v_j(x)}{\partial \mathcal{P}_i} = \frac{1}{2}\int_{t \in s_i^*} \frac{v_j(t)(B(t,x)}{v(t)}$$

$$\frac{\partial \Omega_{jk}}{\partial \mathcal{P}_i} = \frac{1}{2}\int_{t \in s_i^*} \frac{v_j v_k}{v}$$

Poisson bracket for potential $u(z)$

- Let $S_0 = S_B$; $\psi = \phi\sqrt{v}$; $z(x) = \int_{x_0}^{x} v$ - "flat" coordinate on C_g and $\widehat{C}$.
- Main equation: $\psi_{zz} - u(z)\psi = 0$ where

$$u(z) = -1 - \frac{1}{2}\frac{S_B - S_v}{Q}$$

and $S_v(\cdot) = \{\int^x v, \cdot\}$

- Invariant matrix form (on $\widehat{C}$): $\quad d\Psi = \begin{pmatrix} 0 & v \\ uv & 0 \end{pmatrix} \Psi$
- Define $h(x,y) = \frac{B^2(x,y)}{Q(x)Q(y)}$

Poisson bracket for potential $u(z)$ (continued)

$$\frac{4\pi i}{3}\{u(z), u(\zeta)\} = \mathcal{L}_z h^{(\zeta)}(z) - \mathcal{L}_\zeta h^{(z)}(\zeta)$$

where

$$\mathcal{L}_z = \frac{1}{2}\partial_z^3 - 2u(z)\partial_z - u_z(z)$$

is known as "Lenard" operator in KdV theory;

$$h^{(y)}(x) = \int_{x_1}^{x} h(y, \cdot)v(\cdot)$$

The first example of holomorphic Poisson bracket on a Riemann surface (get as a Dirac bracket from Atiyah-Bott symplectic structure??).

Monodromy representation and Goldman bracket

- Fundamental group: $\pi_1(C_g \setminus \{y_i\}_{i=1}^n, x_0)$ with generators $(\gamma_i, \alpha_j, \beta_j)$ and relation $\prod_{j=1}^g \alpha_j\beta_j\alpha_j^{-1}\beta_j^{-1} \prod_{i=1}^n \gamma_i = id$
- Monodromy matrices: $M_{\alpha_i}, M_{\beta_i}, M_{y_i}$ with relation

$$\prod_{i=1}^{n} M_{y_i} \prod_{j=1}^{g} M_{\beta_j}^{-1} M_{\alpha_j}^{-1} M_{\beta_j} M_{\alpha_j} = I$$

- Goldman's bracket on character variety $V_{g,n}$:

$$\{\mathrm{tr} M_\gamma,\ \mathrm{tr} M_{\tilde{\gamma}}\} = \frac{1}{2} \sum_{p \in \gamma \cap \tilde{\gamma}} (\mathrm{tr} M_{\gamma_p \tilde{\gamma}} - \mathrm{tr} M_{\gamma_p \tilde{\gamma}^{-1}})$$

Relation to results of S.Kawai, Math Ann (1996)

- Kawai: "canonical symplectic structure on $T^*\mathcal{M}_g$ implies Goldman bracket if $S_0 = S_{Bers}$".
- Together with our results: S_B and S_{Bers} are in the same equivalence class; implies existence of generating function.
- Conjecture:

$$G = -6\pi i \log \frac{\mathcal{Z}'[\Gamma_{C_0,\eta}](1)}{\det(\Omega - \overline{\Omega}_0)}$$

  where $\mathcal{Z}$ is the Selberg zeta-function $\mathcal{Z}(s) = \prod_\gamma \prod_{m=0}^{\infty}(1 - q_\gamma^{s+m})$ corresponding to quasi-fuchsian group $\Gamma_{C_0,\eta}$; Ω and Ω_0 are period matrices of C_0 and C.

Tau-function of Scrödinger equation

- Motivation: Jimbo-Miwa tau-function for Schlesinger system: $\frac{d\Psi}{dx} = \sum_{i=1}^{N} \frac{A_i}{x-x_i}\Psi$:

$$\frac{\partial \log \tau_{JM}}{\partial x_i} = \frac{1}{2}\mathrm{res}|_{x_i} \frac{\mathrm{tr}(d\Psi\Psi^{-1})^2}{dx}$$

- A straightforward analog of this definition in the case of Schrödinger equation (no isomonodromy!)

$$\frac{\partial \log \tau}{\partial \mathcal{P}_{s_i}} := \frac{1}{4\pi i}\int_{s_i^*}\left(\frac{\mathrm{tr}(d\Psi\Psi^{-1})^2}{v} + 2v\right)$$

where $\mathcal{P}_{s_i} = \int_{s_i} v$; this gives rise to Bergman tau-function

$$\frac{\partial \log \tau}{\partial \mathcal{P}_{s_i}} = -\frac{1}{4\pi i}\int_{s_i^*}\frac{S_B - S_v}{v}$$

$$\tau^\sigma = \det{}^6(C\Omega + D)\,\tau \qquad \tau(\epsilon Q) = \epsilon^{1/6(5g-5+n)}\tau(Q)$$

Open: "Yang-Yang" function

- (φ_i, l_i) - complexified Fenchel-Nielsen Darboux coordinates on character variety $V_{g,n}$.
- $$\omega_{can} = \sum_i d\, l_i \wedge d\varphi_i = \sum_i dp_i \wedge dq_i$$
- $$dG_{YY} = \sum_i l_i d\varphi_i - \sum_i p_i dq_i$$

  (Nekrasov-Rosly-Shatashvili); G_{YY} - "Yang-Yang" function (depends on pants decomposition on the Character variety side; transforms with dilogarithms; depends also on Torelli marking; transforms as a section of Hodge line bundle).

Simplest example: genus 0 with 4 simple poles

Poles: $0, 1, t, \infty$; $B(x, y) = \frac{dx\,dt}{(x-t)^2}$, $S_B = 0$;

$$Q = \frac{\mu}{x(x-1)(x-t)}(dx)^2$$

Poisson structure:

$$\{\mu, t\} = \frac{t(1-t)}{4\pi i}$$

Equation (Heun):

$$\varphi'' + \frac{\mu}{x(x-1)(x-t)}\varphi = 0$$

Homological coordinates:

$$\sqrt{\mu}\int_{a,b} \frac{dx}{\sqrt{x(x-1)(x-t)}}$$