# Integrability obstructions of certain homogeneous Hamiltonian systems in 2D curved spaces

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# Problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} = \mathbf{v}(\mathbf{x}, \mathbf{a}), \qquad \mathbf{x} \in \mathbb{R}^n, \qquad t \in \mathbb{R}.$$

**a** – parameters e.g. masses, quantities characterising forces

## Questions

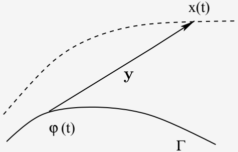
- how to find general solution?
- how to prove that system is integrable/non-integrable?
- how to find values of parameters for that system is solvable/integrable

# Integrability of original and variational equations

In a system

$$\dot{\mathbf{x}} = \mathbf{v}(\mathbf{x}), \qquad \mathbf{x} = (x_1, \ldots x_n)^T, \tag{DS}$$

with known non-stationary particular solution $\boldsymbol{\varphi}(t)$ the substitution $\mathbf{x} = \boldsymbol{\varphi}(t) + \mathbf{y}$ is made




Variational equations (VE)

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbf{y} = A(t)\mathbf{y}, \quad A(t) = \frac{\partial \mathbf{v}}{\partial \mathbf{x}}(\boldsymbol{\varphi}(t)).$$

## Implication

If the system (DS) possesses $k$ functionally independent meromorphic (holomorphic) first integrals $F_1, \ldots, F_k$, then variational equations (VE) have $k$ functionally independent rational (polynomial) first integrals.

# Linear equations and Galois theory

- For a linear system of dimension $n$ (or a scalar equation of degree $n$)

$$y^{(n)} + f_1 y^{(n-1)} + \cdots + f_{n-1} y' + f_n y = 0, \quad f_i \in \mathbb{F}.$$

  the problem of existence of a general solution obtained by a combination of quadratures, exponential of quadratures and algebraic functions was considered by Liouville, Picard, Vessiot, Kolchin, Ritt ,...

- Result of solvability conditions formulated by means of differential Galois group.
- In general solutions do not belong to the differential field of coefficients $\mathbb{F}$
- The smallest differential field containing all solutions is called the Picard-Vessiot extension $\mathbb{L}$

# Linear equations and Galois theory

The differential Galois group $\mathcal{G}$ of the differential equation is the group of all automorphisms of the Picard-Vessiot extension which leaves field of coefficients $\mathbb{F}$ fixed and which commutes with the derivation.

For any solution $y$ and any $\phi \in \mathcal{G}_\partial(\mathbb{L}/\mathbb{F})$ function $\phi y$ is again the solution because

$$\phi\left(y^{(n)} + f_1 y^{(n-1)} + \cdots + f_{n-1}y' + f_n y\right) = \phi(0), \quad f_i \in \mathbb{F},$$

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\phi(y) + f_1\frac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\phi(y) + f_{n-1}\frac{\mathrm{d}}{\mathrm{d}z}\phi(y) + f_n\phi(y) = 0.$$

If $\mathbf{Y}$ is a fundamental matrix of $\dot{\mathbf{y}} = A(z)\mathbf{y}$, then $\phi(\mathbf{Y}) = \mathbf{YG}$, where $\mathbf{G} \in \mathcal{G}_\partial(\mathbb{L}/\mathbb{F})$ is a constant matrix

Question: Is it possible to connect the solvability result for linear system with the integrability of our non-linear system?

# Correspondence between first integrals of the system and invariants of DGG

## Theorem

*If system has k functionally independent first integrals which are meromorphic in a connected neighbourhood of a non-equilibrium solution* $\boldsymbol{\varphi}(t)$*, then the differential Galois group* $\mathcal{G}$ *of the variational equations along* $\boldsymbol{\varphi}(t)$ *has k functionally independent rational invariants i.e. such function* $f \in \mathbb{C}(\mathbf{y})$ *that*

$$\phi(f) = f, \quad \text{for all} \quad \phi \in \mathcal{G}$$

## Fact

The differential Galois group $\mathcal{G}$ of a system of linear equations is a linear algebraic group, so in particular it is also a Lie group.

# Differential Galois integrability obstruction

- Hamiltonian system

$$\dot{z} = \mathbb{J}H'(z), \quad \mathbb{J} = \begin{pmatrix} 0 & \mathbb{I}_n \\ -\mathbb{I}_n & 0 \end{pmatrix}, \quad z = [q, p]^T$$

$H(z)$ – a holomorphic function

- Let $t \to \boldsymbol{\phi}(t) \in \mathbb{C}^{2n}$ be a non-equilibrium solution of the system.
- Variational equations along $\boldsymbol{\phi}(t)$ have the form

$$\dot{Y} = \mathbb{J}H''(\varphi(t))Y.$$

- We can attach to variational equations the differential Galois group $\mathcal{G}$ that is a subgroup of $\mathrm{Sp}(2n, \mathbb{C})$..

**Theorem (Morales-Ruiz and Ramis)**

*Assume that a Hamiltonian system is meromorphically integrable in the Liouville sense in a neigbourhood of a phase curve $\boldsymbol{\Gamma}$. Then the identity component of the differential Galois group of NVEs associated with $\boldsymbol{\Gamma}$ is Abelian*

# Application of differential Galois obstructions. Example

$$V=\frac{1}{2}(p_1^2+p_2^2+p_3^2)+\frac{1}{2}(\omega^2x_1^2+x_2^2+x_3^2)-\frac{\mu}{\sqrt{x_1^2+x_2^2+x_3^2}}$$

restriction of the systems to invariant manifold given by $x_3=p_3=0$

$$V=\frac{1}{2}(p_1^2+p_2^2)+\frac{1}{2}(\omega^2x_1^2+x_2^2)-\frac{\mu}{\sqrt{x_1^2+x_2^2}}$$

Hamilton equations

$$\dot{x}_1=p_1,\quad \dot{p}_1=-\omega^2x_1-\frac{\mu x_1}{(x_1^2+x_2^2)^{3/2}},$$
$$\dot{x}_2=p_2,\quad \dot{p}_2=-x_2-\frac{\mu x_2}{(x_1^2+x_2^2)^{3/2}}$$

invariant manifolds

$$\mathcal{N}_i=\left\{(x_1,x_2,p_1,p_2)\in\mathbb{C}^4\,|\,x_i=p_i=0\right\}$$

for $i=1,2$

# Particular solutions and variational equations

particular solution I

$$\dot{x}_1 = p_1, \qquad \dot{p}_2 = -\omega^2 x_1 - \frac{\mu}{x_1^2}$$

Variational equations

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{P}_1\\ \dot{P}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{2\mu}{x_1^3} - \omega^2 & 0 & 0 & 0\\ 0 & -1 - \frac{\mu}{x_1^3} & 0 & 0\end{bmatrix} \begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{P}_1\\ \dot{P}_2\end{bmatrix}$$

rationalization of NVE: $z = x_1(t)$

$$X_2'' + p_2 X_2' + q_2 X_2 = 0,$$

$$p_2 = \frac{\mu + z^3\omega^2}{z(-2ez - 2\mu + z^3\omega^2)}$$

$$q_2 = \frac{\mu + z^3}{z^2(2ez + 2\mu - z^3\omega^2)}$$

particular solution II

$$\dot{x}_2 = p_2, \qquad \dot{p}_2 = -x_2 - \frac{\mu}{x_2^2}$$

Variational equations

$$\begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{P}_1\\ \dot{P}_2\end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\mu x_2^3 - \omega^2 & 0 & 0 & 0\\ 0 & -1 + \frac{2\mu}{x_2^3} & 0 & 0\end{bmatrix} \begin{bmatrix}\dot{X}_1\\ \dot{X}_2\\ \dot{P}_1\\ \dot{P}_2\end{bmatrix}$$

rationalization of NVE: $z = x_2(t)$

$$X_1'' + p_1 X_2' + q_1 X_1 = 0,$$

$$p_1 = \frac{\mu + z^3}{z(-2ez - 2\mu + z^3)}$$

$$q_1 = \frac{\mu + z^3\omega^2}{z^2(2ez + 2\mu - z^3)}$$

# Variational equations

By a change of the variable

$$X_i = w \exp\left[-\frac{1}{2}\int p_i(s)\mathrm{d}s\right].$$

we obtain the standard reduced forms

$$w'' = r_i(z)w, \qquad r_i(z) = \frac{1}{2}p_i'(z) + \frac{1}{4}p_i(z)^2 - q_i(z).$$

Coefficients

$$r_1 = -\frac{3\mu^2 + z^6\omega^2(-4+\omega^2) + 4ez^4(2+\omega^2) + 2z^3\mu(4+5\omega^2)}{4z^2(2ez+2\mu-z^3\omega^2)^2},$$

$$r_2 = \frac{-3\mu^2 - 4ez^4(1+2\omega^2) + z^6(-1+4\omega^2) - 2z^3\mu(5+4\omega^2)}{4z^2(-2ez+z^3-2\mu)^2}$$

Differences of exponents at $z_1 = 0$, $z_2 = s_1$, $z_3 = s_2$, $z_4 = s_3$ and $z_5 = \infty$:

for particular solution I: $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{\omega}\right\}$,

for particular solution II: $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 2\omega\right\}$

# Kovacic algorithm

Let $\mathcal{G}$ be the differential Galois group of reduced equation. Then one of four cases can occur.

Case I $\mathcal{G}$ is conjugate to a subgroup of the triangular group

$$\mathcal{T} = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \;\middle|\; a \in \mathbb{C}^*, b \in \mathbb{C} \right\};$$

in this case equation has an exponential solution of the form $y = P \exp \int \omega$, where $P \in \mathbb{C}[z]$ and $\omega \in \mathbb{C}(z)$,

Case II $\mathcal{G}$ is conjugate to a subgroup of

$$\mathcal{D}^{\dagger} = \left\{ \begin{bmatrix} c & 0 \\ 0 & c^{-1} \end{bmatrix} \;\middle|\; c \in \mathbb{C}^* \right\} \cup \left\{ \begin{bmatrix} 0 & c \\ -c^{-1} & 0 \end{bmatrix} \;\middle|\; c \in \mathbb{C}^* \right\};$$

in this case equation has a solution of the form $y = \exp \int \omega$, where $\omega$ is algebraic over $\mathbb{C}(z)$ of degree 2,

Case III $\mathcal{G}$ is primitive and finite; in this case all solutions of equation are algebraic, thus $y = \exp \int \omega$, where $\omega$ belongs to an algebraic extension of $\mathbb{C}(z)$ of degree $n = 4, 6$ or 12.

Case IV $\mathcal{G} = \mathrm{SL}(2, \mathbb{C})$ and equation has no Liouvillian solution.

# Necessary conditions for cases I, II and III for our example

Solution I

C 1. $\omega = \frac{2}{m}$, $m \in \mathbb{N}$ and $\omega \leq 2$

C 2. $\omega = \frac{4}{m}$, $m \geq 4$ and $\omega \leq 1$

C 3. $\omega = \frac{2q}{m}$, $q = 1, \ldots, 6$, $m \geq 6$ and $\omega \leq 2$

Solution II

C 1. $\frac{1}{\omega} = \frac{2}{m}$, $m \in \mathbb{N}$

C 2. $\frac{1}{\omega} = \frac{4}{m}$, $m \geq 4$

C 3. $\frac{1}{\omega} = \frac{2q}{m}$, $q = 1, \ldots, 6$, $m \geq 6$

Finite number of choices for $\omega$

$$\omega \in \left\{2, \frac{3}{2}, \frac{5}{4}, 1, \frac{1}{2}, \ldots\right\}$$

# Integrability of homogeneous Hamiltonian equations

Natural Hamiltonian in flat space

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(\mathbf{q}), \qquad (\mathbf{q}, \mathbf{p}) \in \mathbb{C}^{2n},$$

with $V$ — homogeneous of degree $k \in \mathbb{Z}$

$$V(\lambda q_1, \ldots, \lambda q_n) = \lambda^k V(q_1, \ldots, q_n)$$

- how to check integrability of such class of potentials in a wide class of functions (without restrictions on degree of a first integral with respect to momenta)?
- what is a counterpart of this class in curved spaces?

# Particular solutions and Variational equations

## Definition

Darboux point $\mathbf{d} \in \mathbb{C}^n$ is a non-zero solution of

$$V'(\mathbf{d}) = \mathbf{d}$$

Particular solution

$$\mathbf{q}(t) = \varphi(t)\mathbf{d}, \ \mathbf{p}(t) = \dot{\varphi}(t)\mathbf{d} \quad \text{provided} \quad \ddot{\varphi} = -\varphi^{k-1}.$$

$$\ddot{\mathbf{x}} = -\varphi(t)^{k-2} V''(\mathbf{d})\mathbf{x},$$

where $V''(\mathbf{d})$ is the Hessian of $V$ calculated at $\mathbf{d}$. If $V''(\mathbf{d})$ is diagonalisable

$$\ddot{\eta}_i = \lambda_i \varphi(t)^{k-2} \eta_i, \qquad i = 1, \ldots, n,$$

where $\lambda_i$ for $i = 1, \ldots, n$ are eigenvalues of $V''$.

By homogeneity of $V$, $\lambda_n = k - 1$.

# Transformation into hypergeometric equation

The Yoshida transformation

$$t \longrightarrow z := \frac{1}{\varepsilon}\varphi(t)^k.$$

Variational equations after transformations:

$$z(1-z)\eta_i'' + \left(\frac{k-1}{k} - \frac{3k-2}{2k}z\right)\eta_i' + \frac{\lambda_i}{2k}\eta_i = 0,$$

where $i = 1, \ldots, n$, have the form of a direct product of hypergeometric differential equation for which the differences of exponents at $z = 0$, $z = 1$ and $z = \infty$ are

$$\rho = \frac{1}{k}, \quad \sigma = \frac{1}{2}, \quad \tau = \frac{1}{2k}\sqrt{(k-1)^2 + 8k\lambda_i}.$$

# Solvability of Riemann $P$ equation. Kimura theorem

### Theorem

*The identity component of the differential Galois group of the Riemann $P$ equation is solvable iff*

A. *at least one of the four numbers $\rho+\sigma+\tau$, $-\rho+\sigma+\tau$, $\rho-\sigma+\tau$, $\rho+\sigma-\tau$ is an odd integer, or*
B. *the numbers $\rho$ or $-\rho$ and $\sigma$ or $\sigma$ and $\tau$ or $-\tau$ belong (in an arbitrary order) to some of appropriate fifteen families forming the so-called Schwarz's table fifteen families*

| | | | | |
|---|---|---|---|---|
| 1 | $1/2+l$ | $1/2+s$ | arbitrary complex number | |
| 2 | $1/2+l$ | $1/3+s$ | $1/3+q$ | |
| 3 | $2/3+l$ | $1/3+s$ | $1/3+q$ | $l+s+q$ even |
| 4 | $1/2+l$ | $1/3+s$ | $1/4+q$ | |
| 5 | $2/3+l$ | $1/4+s$ | $1/4+q$ | $l+s+q$ even |
| 6 | $1/2+l$ | $1/3+s$ | $1/5+q$ | |
| 7 | $2/5+l$ | $1/3+s$ | $1/3+q$ | $l+s+q$ even |
| 8 | $2/3+l$ | $1/5+s$ | $1/5+q$ | $l+s+q$ even |
| 9 | $1/2+l$ | $2/5+s$ | $1/5+q$ | |
| 10 | $3/5+l$ | $1/3+s$ | $1/5+q$ | $l+s+q$ even |
| 11 | $2/5+l$ | $2/5+s$ | $2/5+q$ | $l+s+q$ even |
| 12 | $2/3+l$ | $1/3+s$ | $1/5+q$ | $l+s+q$ even |
| 13 | $4/5+l$ | $1/5+s$ | $1/5+q$ | $l+s+q$ even |
| 14 | $1/2+l$ | $2/5+s$ | $1/3+q$ | |
| 15 | $3/5+l$ | $2/5+s$ | $1/3+q$ | $l+s+q$ even |

where $l, s, q \in \mathbb{Z}$.

# Morales-Ramis Theorem

- find all non-zero solutions of

$$V'(\mathbf{d}) = \mathbf{d}$$

- calculate eigenvalues $\lambda_1, \dots, \lambda_n$ of $V''(\mathbf{d})$

**Theorem**

*If the Hamiltonian system with homogeneous potential is meromorphically integrable, then each $(k, \lambda_i)$ belong to the following list:*

# Morales-Ramis table

| case | $k$ | $\lambda$ |
|---|---|---|
| 1. | $\pm 2$ | $\lambda$ |
| 2. | $k$ | $p+\frac{k}{2}p(p-1)$ |
| 3. | $k$ | $\frac{1}{2}\left(\frac{k-1}{k}+p(p+1)k\right)$ |
| 4. | 3 | $-\frac{1}{24}+\frac{1}{6}(1+3p)^2, \quad -\frac{1}{24}+\frac{3}{32}(1+4p)^2$ <br> $-\frac{1}{24}+\frac{3}{50}(1+5p)^2, \quad -\frac{1}{24}+\frac{3}{50}(2+5p)^2$ |
| 5. | 4 | $-\frac{1}{8}+\frac{2}{9}(1+3p)^2$ |

# Morales-Ramis table

| case | $k$ | $\lambda$ | |
|---|---|---|---|
| 6. | 5 | $-\frac{9}{40}+\frac{5}{18}(1+3p)^2$, | $-\frac{9}{40}+\frac{2}{5}(1+5p)^2$ |
| 7. | $-3$ | $\frac{25}{24}-\frac{1}{6}(1+3p)^2$, | $\frac{25}{24}-\frac{3}{32}(1+4p)^2$ |
| | | $\frac{25}{24}-\frac{3}{50}(1+5p)^2$, | $\frac{25}{24}-\frac{6}{25}(1+5p)^2$ |
| 8. | $-4$ | $\frac{9}{8}-\frac{2}{9}(1+3p)^2$ | |
| 9. | $-5$ | $\frac{49}{40}-\frac{5}{18}(1+3p)^2$, | $\frac{49}{40}-\frac{2}{5}(1+5p)^2$ |

where $p$ is an integer and $\lambda$ an arbitrary complex number.

Morales Ruiz, J. J., *Differential Galois theory and non-integrability of Hamiltonian systems*, volume 179 of *Progress in Mathematics*, Birkhäuser Verlag, Basel, 1999.

# Relations between eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$

$\lambda_i(\mathbf{d}) := \Lambda_i(\mathbf{d}) + 1$, are the non-trivial eigenvalues of $V''(\mathbf{d})$.
$\tau_i$ is the elementary symmetric polynomial of degree $i$ in $(n-1)$ variables

## Theorem

*For a generic homogeneous $V \in \mathbb{C}[\mathbf{q}]$ of degree $k > 2$ we have*

$$\sum_{[\mathbf{d}]\in\mathcal{D}^\star(V)} \frac{\tau_1(\boldsymbol{\Lambda}(\mathbf{d}))^r}{\tau_{n-1}(\boldsymbol{\Lambda}(\mathbf{d}))} = (-1)^n(n+k-2)^r, \quad 0 \leq r \leq n-1,$$

*or, alternatively*

$$\sum_{[\mathbf{d}]\in\mathcal{D}^\star(V)} \frac{\tau_r(\boldsymbol{\Lambda}(\mathbf{d}))}{\tau_{n-1}(\boldsymbol{\Lambda}(\mathbf{d}))} = (-1)^{n-r-1}\sum_{i=0}^{r}\binom{n-r-1}{r-i}(k-1)^i.$$

## Classification program

- relations give only finite choices of $(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n = k-1)$, from these $\lambda$s we reconstruct potentials
- no new apart already known integrable potentials for $n=2$ and $2 < k \leq 6$
- for $n = k = 3$ it was found 10 integrable potentials e.g.

$$V_{10} = \frac{4\sqrt{2}q_1^3}{3} + \frac{5q_1q_2^2}{2\sqrt{2}} + q_2^2q_3 + \frac{1}{3}q_3^3$$

with additional first integrals of degree 4 and 6 in momenta.


Maciejewski, A. J. and M. Przybylska, (2004): All meromorphically integrable 2D Hamiltonian systems with homogeneous potentials of degree 3, *Phys. Lett. A* 327(5-6):461–473.

Maciejewski, A. J. and M. Przybylska, (2005): Darboux points and integrability of Hamiltonian systems with homogeneous polynomial potential, *J. Math. Phys.* 46(6):062901, 33 pages.

Przybylska M., (2009): Darboux points and integrability of homogenous Hamiltonian systems with three and more degrees of freedom, *Regul. Chaotic Dyn.*14(2):263–311 and *Regul. Chaotic Dyn.*14(3):349–388.

## What is analogue of homogeneous systems in curved spaces?

no obvious answer

### Nakagawa and Yoshida

$$H = T(\mathbf{p}) + V(\mathbf{q}).$$

where $T$ and $V$ are homogeneous functions of integer degrees.
To find a straight line particular solution one must solve overdetermined system of nonlinear equations

$$T'(\mathbf{c}) = \mathbf{c}, \qquad V'(\mathbf{c}) = \mathbf{c},$$

### Our first proposition

$$H = T + V, \qquad T = \frac{1}{2}r^{m-k}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right), \qquad V = r^m U(\varphi),$$

where $m$ and $k$ are integers, and $k \neq 0$.

# Main integrability theorem. Auxiliary sets

$$\mathcal{I}_0(k,m) := \left\{ \frac{1}{k}(mp+1)(2mp+k) \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_1(k,m) := \left\{ \frac{1}{2k}(mp-2)(mp-k) \;\middle|\; p = 2r+1, r \in \mathbb{Z} \right\},$$

$$\mathcal{I}_2(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{2}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_3(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{3}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_4(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{4}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_5(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{1}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_6(k,m) := \left\{ \frac{1}{8k}\left[4m^2\left(p+\frac{2}{5}\right)^2 - (k-2)^2\right] \;\middle|\; p \in \mathbb{Z} \right\},$$

$$\mathcal{I}_a(k,m) := \mathcal{I}_0(k,m) \cup \mathcal{I}_1(k,m) \cup \mathcal{I}_2(k,m).$$

# Main integrability theorem. Main theorem

## Theorem

*Assume that $U(\varphi)$ is a complex meromorphic function and there exists $\varphi_0 \in \mathbb{C}$ such that $U'(\varphi_0) = 0$ and $U(\varphi_0) \neq 0$. If the Hamiltonian system defined by Hamiltonian*

$$H = T + V, \qquad T = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right), \qquad V = r^m U(\varphi),$$

*is integrable in the Liouville sense, then number*

$$\lambda := 1 + \frac{U''(\varphi_0)}{kU(\varphi_0)},$$

*belongs to set $\mathcal{I}(k, m)$ which is defined by the following table*

# Main integrability theorem. Integrability table

| No. | $k$ | $m$ | $\mathcal{I}(k,m)$ |
|---|---|---|---|
| 1 | $k=-2(mp+1)$ | $m$ | $\mathbb{C}$ |
| 2 | $k\in\mathbb{Z}\setminus\{0\}$ | $m$ | $\mathcal{I}_{\mathrm{a}}(k,m)$ |
| 3 | $k=2(mp-1)\pm\frac{1}{3}m$ | $3q$ | $\bigcup_{i=0}^{6}\mathcal{I}_i(k,m)$ |
| 4 | $k=2(mp-1)\pm\frac{1}{2}m$ | $2q$ | $\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_4(k,m)$ |
| 5 | $k=2(mp-1)\pm\frac{3}{5}m$ | $5q$ | $\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_6(k,m)$ |
| 6 | $k=2(mp-1)\pm\frac{1}{5}m$ | $5q$ | $\mathcal{I}_{\mathrm{a}}(k,m)\cup\mathcal{I}_3(k,m)\cup\mathcal{I}_5(k,m)$ |

Table : Integrability table. Here $k,m,p,q\in\mathbb{Z}$ and $k\neq 0$.

# Example

$$H = \frac{1}{2} r^{m-k} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^m \cos\varphi. \quad (1)$$

It corresponds to $U(\varphi) = -\cos\varphi$. As $U'(\varphi) = \sin\varphi$, we take $\varphi_0 = 0$.
Since $U''(0)/U(0) = -1$, we have $\lambda = (k-1)/k$.
Comparison of this value with forms of $\lambda$ in sets $\mathcal{I}_j(k, m)$, $j = 0, \ldots, 6$ gives:

$$\begin{aligned} &1. \quad m = 1, && k = -5, && l = 6, \\ &2. \quad m = -1, && k = 1, && l = -2, \\ &3. \quad m = 1, && k = 1, && l = 0, \\ &4. \quad m = -1, && k = -5, && l = 4, \end{aligned} \quad (2)$$

$$\begin{aligned} &5. \quad m = 2, && k = 1, && l = 1, \\ &6. \quad m = -2, && k = 1, && l = -3, \\ &7. \quad m = 2, && k = -5, && l = 7, \\ &8. \quad m = -2, && k = -5, && l = 3. \end{aligned} \quad (3)$$

# Example. Superintegrable cases

**Case 1.**

$$H = \frac{1}{2} r^6 \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r \cos\varphi,$$

$$F_1 := r^2 p_\varphi^2 \cos(2\varphi) - r^3 p_r p_\varphi \sin(2\varphi) + r^{-1} \sin\varphi \sin(2\varphi),$$

$$F_2 := r^2 p_\varphi^2 \sin(2\varphi) + r^3 p_r p_\varphi \cos(2\varphi) - r^{-1} \sin\varphi \cos(2\varphi).$$

**Case 2.**

$$H = \frac{1}{2} r^{-2} \left( p_r^2 + \frac{p_\varphi^2}{r^2} \right) - r^{-1} \cos\varphi,$$

$$F_1 := r^{-2} p_\varphi^2 \cos(2\varphi) + r^{-1} p_r p_\varphi \sin(2\varphi) + r \sin\varphi \sin(2\varphi),$$

$$F_2 := -r^{-2} p_\varphi^2 \sin(2\varphi) + r^{-1} p_r p_\varphi \cos(2\varphi) + r \sin\varphi \cos(2\varphi).$$

# Example. Superintegrable cases

**Case 3.**

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r\cos\varphi,$$

$$F_1 := r^{-1}p_\varphi^2\cos\varphi + p_r p_\varphi\sin\varphi + \frac{1}{2}r^2\sin^2\varphi,$$

$$F_2 := \left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi + r^{-1}p_r p_\varphi\cos(2\varphi) - r\sin\varphi.$$

**Case 4.**

$$H = \frac{1}{2}r^4\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-1}\cos\varphi,$$

$$F_1 := rp_\varphi^2\cos\varphi - r^2p_r p_\varphi\sin\varphi + \frac{1}{2}r^{-2}\sin^2\varphi,$$

$$F_2 := r^4\left(p_r^2 - r^{-2}p_\varphi^2\right)\cos\varphi\sin\varphi - r^3p_r p_\varphi\cos(2\varphi) - r^{-1}\sin\varphi.$$

# Example. Integrable cases

- In cases with parameters given in (3) we have integrable as well as non-integrable systems.
- Namely cases 5 and 8 are integrable.

**Case 5.**

$$H = \frac{1}{2}r\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^2\cos\varphi,$$

$$F := r^{-1}(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^2(1+\cos^2\varphi) + 2p_rp_\varphi\sin\varphi.$$

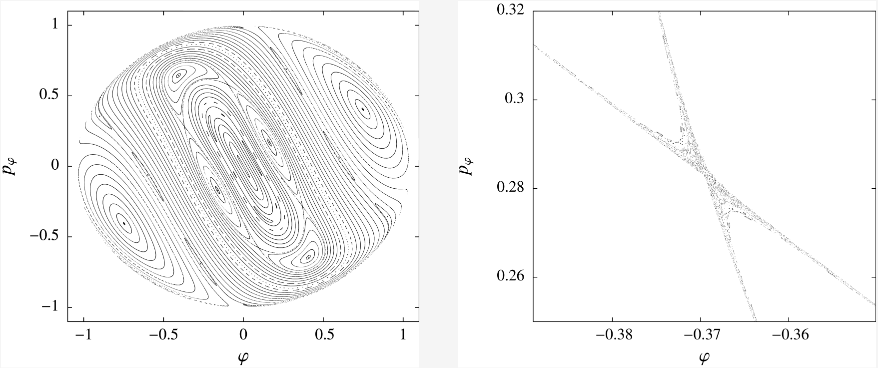
**Case 8.**

$$H = \frac{1}{2}r^3\left(p_r^2 + \frac{p_\varphi^2}{r^2}\right) - r^{-2}\cos\varphi,$$

$$F := r(p_\varphi^2 - r^2p_r^2)\cos\varphi + r^{-2}(1+\cos^2\varphi) - 2r^2p_rp_\varphi\sin\varphi.$$

- Poincaré sections for Hamiltonian systems with parameters given in cases 6 and 7 in (3) show chaotic area.

# Example. Non-integrable case 6

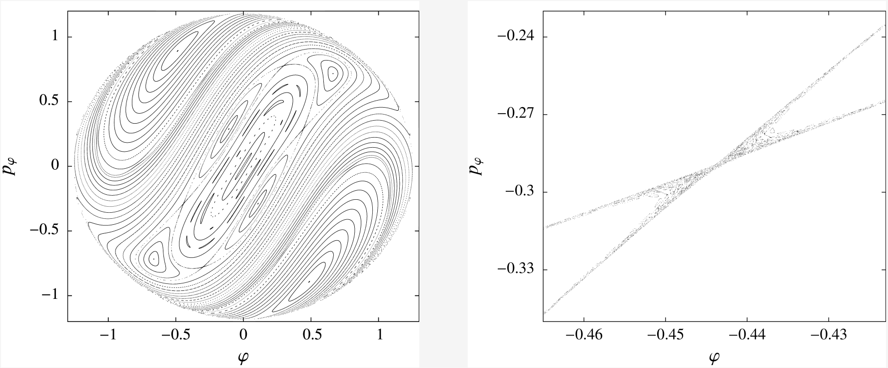



(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = -2, k = 1$ corresponding to case 6

# Example. Non-integrable case 7

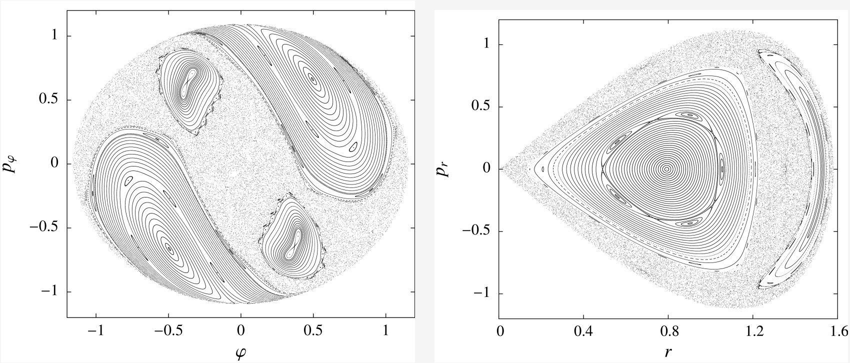



(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) magnification of region around unstable periodic solution

Figure : Poincaré cross sections on energy level $E = -0.3$ for Hamiltonian system given by (1) with $m = 2, k = -5$ corresponding to case 7

# Example 2. Non-integrable cases for family $k = -2(mp + 1)$

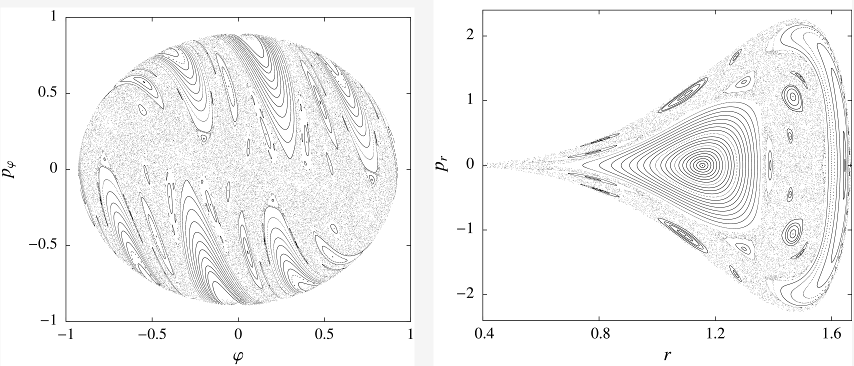



(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) section plane $\varphi = 0$ with coordinates $(r, p_r)$

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = -2, k = 2$

## Example. Non-integrable cases for family $k = -2(mp+1)$

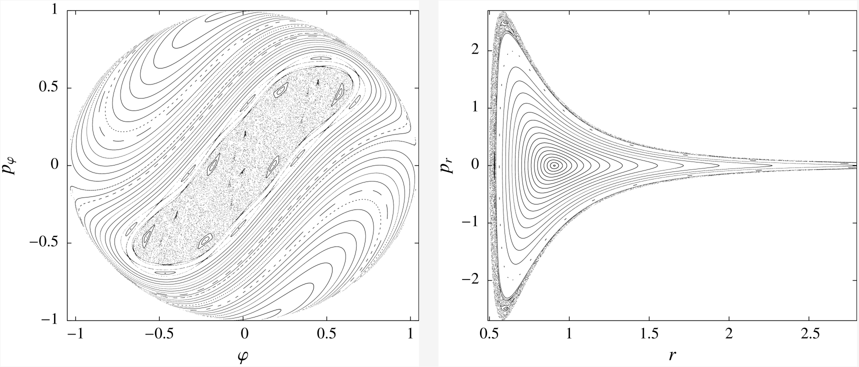



(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) section plane $\varphi = 0$ with coordinates $(r, p_r)$

Figure : Poincaré cross sections on energy level $E = -0.6$ for Hamiltonian system given by (1) with $m = -1, k = 8$

# Example. Non-integrable cases for family $k = -2(mp + 1)$




(a) section plane $r = 1$ with coordinates $(\varphi, p_\varphi)$

(b) section plane $\varphi = 0$ with coordinates $(r, p_r)$

Figure : Poincaré cross sections on energy level $E = -0.5$ for Hamiltonian system given by (1) with $m = 1, k = -6$

# Another analogue in curved spaces

$n$ dimensional constant curvature spaces $S^n_{[\kappa]}$: the sphere $\mathbb{S}^n$ for $\kappa > 0$, Euclidean space $\mathbb{E}^n$ for $\kappa = 0$ and and hyperbolic space $\mathbb{H}^n$ for $\kappa < 0$

$$\mathrm{C}_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ 1 & \text{for} \quad \kappa = 0, \\ \cosh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0, \end{cases}$$

$$\mathrm{S}_\kappa(x) := \begin{cases} \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}x) & \text{for} \quad \kappa > 0, \\ x & \text{for} \quad \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}x) & \text{for} \quad \kappa < 0. \end{cases}$$

These functions satisfy the following identities

$$\mathrm{C}_\kappa^2(x) + \kappa\,\mathrm{S}_\kappa^2(x) = 1, \quad \mathrm{S}_\kappa'(x) = \mathrm{C}_\kappa(x), \quad \mathrm{C}_\kappa'(x) = -\kappa\,\mathrm{S}_\kappa(x).$$

# Another analogue in curved spaces.

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\varphi^2}{\mathrm{S}_\kappa(r)^2}\right) + \mathrm{S}_\kappa^k(r)U(\varphi), \tag{4}$$

with $k \in \mathbb{Z}$. Hamilton equations

$$\dot{r} = p_r, \qquad \dot{p}_r = \frac{p_\varphi^2}{\mathrm{S}_\kappa^3(r)}\,\mathrm{C}_\kappa(r) - k\,\mathrm{S}_\kappa^{k-1}(r)\,\mathrm{C}_\kappa(r),$$
$$\dot{\varphi} = \frac{p_\varphi}{\mathrm{S}_\kappa^2(r)}, \qquad \dot{p}_\varphi = -k\,\mathrm{S}_\kappa^{k-1}(r)U'(\varphi),$$

have an invariant plane given by $\varphi(t) = \varphi_0 = \mathrm{const}$ with $U'(\varphi_0) = 0$ and $p_\varphi = 0$.
We consider a particular solution on the energy level $H = e$

$$H = \frac{1}{2}p_r^2 + \mathrm{S}_\kappa^k(r)U(\varphi_0).$$

# Case $n = 2$. Integrability theorem

## Theorem

*If the Hamiltonian system governed by Hamilton function (4) is meromorphically integrable, then at each Darboux point the pair $(k, \lambda)$ belongs to one of the following list*

| case | $k$ | $\lambda$ |
|---|---|---|
| 1 | $k$ | $-\dfrac{(k-2p)(p-1)}{k}$ |
| 2 | $k$ | $-\dfrac{(k+4p)[k-4(1+p)]}{8k}$ |
| 3 | $-2+4p$ | *arbitrary* |
| 4 | $k=2q-1$ | $-\dfrac{(-2+3k+12p)[3k-2(5+6p)]}{72k}$ |

*Here $p$ is an arbitrary integer.*

# Variational equations

If $[R, \Phi, P_R, P_\Phi]^T$ denote the variations of variables $[r, \varphi, p_r, p_\varphi]^T$, then variational equations on the invariant plane take the form

$$\begin{bmatrix} \dot{R} \\ \dot{\Phi} \\ \dot{P}_R \\ \dot{P}_\Phi \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathrm{S}_\kappa^{-2}(r) \\ k\,\mathrm{S}_\kappa^{k-2}(r)[k\kappa\,\mathrm{S}_\kappa^2(r) - (k-1)]U(\varphi_0) & 0 & 0 & 0 \\ 0 & -\mathrm{S}_\kappa^k(r)U''(\varphi_0) & 0 & 0 \end{bmatrix} \begin{bmatrix} R \\ \Phi \\ P_R \\ P_\Phi \end{bmatrix}$$

Normal part of variational equations is given by the following closed subsystem

$$\dot{\Phi} = \mathrm{S}_\kappa^{-2}(r)P_\Phi, \qquad \dot{P}_\Phi = -\mathrm{S}_\kappa^k(r)U''(\varphi_0)\Phi,$$

or rewritten as a one second order equation

$$\ddot{\Phi} + a(r, p_r)\dot{\Phi} + b(r, p_r)\Phi = 0,$$

$$a(r, p_r) = 2\frac{\mathrm{C}_\kappa(r)}{\mathrm{S}_\kappa(r)}p_r, \quad b(r, p_r) = \mathrm{S}_\kappa^{k-2}(r)U''(\varphi_0).$$

# Rationalizaton of normal variational equations

- Using the change of independent variable $t \mapsto z = \mathrm{S}_\kappa(r)/\sqrt{\kappa}$ we can transform it into a linear equation with rational coefficients

$$\Phi'' + p(z)\Phi' + q(z)\Phi = 0, \;\; p(z) = \frac{\ddot{z} + \dot{z}a}{\dot{z}^2} = \frac{z}{z^2-1} + 2\frac{(k+2)z^k - u^k}{z(z^k - u^k)},$$

$$q(z) = \frac{b}{\dot{z}^2} = \frac{k(\lambda-1)z^{k-2}}{2(z^2-1)(z^k-u^k)},$$

where $u$ is defined by relation $e\kappa^{k/2} = Bu^k$.

- We choose $u = 0$ that is equivalent to take zero energy level. The the variational equation reduces to

$$\Phi'' + \frac{(k+6)z^2 - 4 - k}{2z(z^2-1)}\Phi' + \frac{k(\lambda-1)}{2z^2(z^2-1)}\Phi = 0.$$

- Transformation $z \mapsto y = z^2$ convert it into the Riemann $P$ equation

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}y^2} + \left[\frac{k+6}{4y} + \frac{1}{2(y-1)}\right]\frac{\mathrm{d}\Phi}{\mathrm{d}y} + \left[\frac{k(1-\lambda)}{8y^2} + \frac{k(\lambda-1)}{8y(y-1)}\right]\Phi = 0,$$

$$\rho = \frac{1}{4}\sqrt{(k-2)^2 + 8k\lambda}, \quad \tau = \frac{1}{2}, \quad \sigma = \frac{k+4}{4}.$$

# Examples

- Case $k = 1$

**Lemma**

*If Hamiltonian system with $\kappa \neq 0$ and $k = 1$ is integrable then either $\lambda = 1$, or $\lambda = 0$.*

- If $k = -2$ Hamiltonian system is integrable. The additional first integral has the following form

$$G = \frac{p_\varphi^2}{2} + U(\varphi). \tag{5}$$

  In fact, in this case system is separable in variables $(r, \varphi)$. For such systems one can ask about its superintegrability and in our work it was shown that the necessary condition is that $\lambda = 1 - s^2$, where $s$ is a non-zero rational number.
- for $k = 2$ also does not give give obstructions for integrability
- for $|k| > 2$ as well as for $k = -1$ there is no an effective way to perform analysis till the end.

# Examples

- Hamiltonian with potential

$$V(r,\varphi) = S_\kappa^k(r)\cos^k\varphi,$$

for which $\lambda = 0$, so the necessary conditions for integrability are fulfilled and additional first integral is

$$I_\kappa = p_r \sin\varphi + p_\varphi \cos\varphi\sqrt{\kappa}\cot\sqrt{\kappa}r, \qquad \kappa \neq 0.$$

- Limit

$$I_0 = \lim_{\kappa\to 0} I_\kappa = p_r\sin\varphi + r^{-1}p_\varphi\cos\varphi,$$

gives the first integral for the case $\kappa = 0$.

- If $\kappa = 0$ and $k = 1$, then there exists additional independent first integral quadratic in momenta

$$I_2 = \left(p_r^2 - \frac{p_\varphi^2}{r^2}\right)\cos\varphi\sin\varphi + r^{-1}p_r p_\varphi\cos(2\varphi) - r\sin\varphi.$$

Thus, in this case the system is maximally super-integrable.

# Examples

For potential

$$V(r,\varphi) = S_\kappa^k(r)\cosh\varphi,$$

we have $\lambda = (k+1)/k$. Comparison with integrability table gives $k=-1$, and $k=-3$.

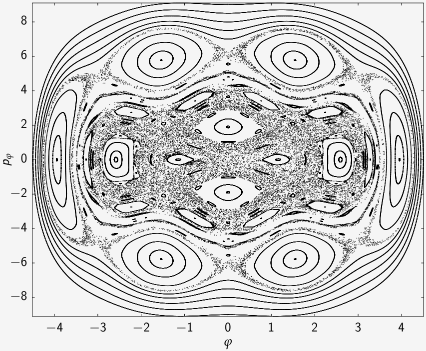



Figure : The Poincaré cross section on energy level $e=50$, for $k=-1$ and $\kappa=1$. The cross-section plane is $r=\pi/2$ with $p_r>0$

# Examples

For $k = -3$ chaotic region was not easily visible. To see it we made the change of variable $\varphi = \mathrm{i}\psi$, $p_\varphi = -\mathrm{i}p_\psi$ and then the Hamiltonian takes the form

$$H = \frac{1}{2}\left(p_r^2 - \frac{p_\psi^2}{S_\kappa(r)^2}\right) + S_\kappa^k(r)\cos\psi$$

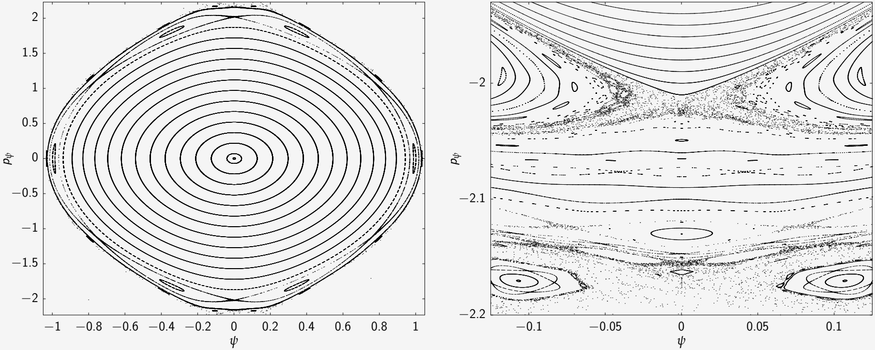



Figure : The Poincaré cross section on energy level $e = 50$, for $k = -3$ and $\kappa = 1$. The cross-section plane is $r = \pi/2$ with $p_r > 0$

# References


- Szumiński, W., Maciejewski, A. J. and M. Przybylska, (2015): Note on integrability of certain homogeneous Hamiltonian systems, *Phys. Lett. A* 379(45-46), 2970–2976.

- Rañada M. F., Superintegrable systems with a position dependent mass : Kepler-related and Oscillator-related systems, arXiv:math-ph/1605.02336

- Fordy A. P., A note on some superintegrable Hamiltonian systems arXiv:nlin.SI/1601.03079

- Kimura, T. (1969/1970): On Riemann's equations which are solvable by quadratures, *Funkcial. Ekvac.* 12, 269–281.

- Ballesteros, A., F. J. Herranz, M. Santander, and T. Sanz-Gil, (2003): Maximal superin- tegrability on N -dimensional curved spaces, *J. Phys. A: Math. Gen.* 36(7), L93.

- Maciejewski, A. J., Szumiński, W. and M. Przybylska, (2016): Note on integrability of certain homogeneous Hamiltonian systems in 2D constant curvature spaces, submitted to Phys. Lett. A