Orthogonal Polynomials and Integrable Systems

Peter Clarkson

School of Mathematics, Statistics and Actuarial Science
University of Kent, Canterbury, CT2 7NF, UK
P.A.Clarkson@kent.ac.uk

Geometric and Algebraic Aspects of Integrability
LMS-EPSRC Durham Symposium
2 August 2016

References

- **P A Clarkson**, "On Airy solutions of the second Painlevé equation", *Stud. Appl. Math.*, **137** (2016) 93–109
- **P A Clarkson & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, **39** (2014) 223–254
- **P A Clarkson & K Jordaan**, "Properties of generalized Freud polynomials", arXiv:1606.06026
- **P A Clarkson, K Jordaan & A Kelil**, "A generalized Freud weight", *Stud. Appl. Math.*, **136** (2016) 288–320
- **P A Clarkson, A F Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *J. Difference Equ. Appl.*, **22** (2016) 656–675

Outline

1. Introduction
   - Painlevé equations
   - Orthogonal polynomials
2. Semi-classical orthogonal polynomials

$$w(x;t) = |x|^{\nu} \exp(-x^2 + tx), \qquad x, t \in \mathbb{R}, \quad \nu > -1$$
$$w(x;t) = |x|^{2\nu+1} \exp(-x^4 + tx^2), \qquad x, t \in \mathbb{R}, \quad \nu > 0$$

3. Semi-classical orthogonal polynomials on complex contours

$$w(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right), \qquad t > 0$$

   on the curve $\mathcal{C}$ from $\mathrm{e}^{2\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{-2\pi\mathrm{i}/3}\infty$.

4. The nonlinear difference system

$$x_n + x_{n+1} = y_n^2 - t \qquad\qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$
$$x_n(y_n + y_{n-1}) = n$$

5. Airy Solutions of the second Painlevé equation and related equations
6. Conclusions

Painlevé Equations

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 6q^2 + z \qquad \mathrm{P_I}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 2q^3 + zq + a \qquad \mathrm{P_{II}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{aq^2+b}{z} + cq^3 + \frac{d}{q} \qquad \mathrm{P_{III}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-a)q + \frac{b}{q} \qquad \mathrm{P_{IV}}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \frac{1}{z}\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{(q-1)^2}{z^2}\left(aq + \frac{b}{q}\right) + \frac{cq}{z} + \frac{dq(q+1)}{q-1} \qquad \mathrm{P_V}$$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-z}\right)\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{q-z}\right)\frac{\mathrm{d}q}{\mathrm{d}z} + \frac{q(q-1)(q-z)}{z^2(z-1)^2}\left\{a + \frac{bz}{q^2} + \frac{c(z-1)}{(q-1)^2} + \frac{dz(z-1)}{(q-z)^2}\right\} \qquad \mathrm{P_{VI}}$$

with a, b, c and d arbitrary constants.

Painlevé σ-Equations

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma = 0 \qquad \mathrm{S}_{\mathrm{I}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}\beta^2 \qquad \mathrm{S}_{\mathrm{II}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2} - \frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma\right) + 4z\vartheta_\infty\frac{\mathrm{d}\sigma}{\mathrm{d}z} = z^2\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - 2\sigma + 2\vartheta_0\right) \qquad \mathrm{S}_{\mathrm{III}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S}_{\mathrm{IV}}$$

$$\left(z\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - \left[2\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^2 - z\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j\right) = 0 \qquad \mathrm{S}_{\mathrm{V}}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left[z(z-1)\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right]^2 + \left[\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left\{2\sigma - (2z-1)\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right\} + \kappa_1\kappa_2\kappa_3\kappa_4\right]^2 = \prod_{j=1}^{4}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + \kappa_j^2\right) \qquad \mathrm{S}_{\mathrm{VI}}$$

where β, $\vartheta_0, \vartheta_\infty$ and $\kappa_1, \ldots, \kappa_4$ are arbitrary constants.

Special function solutions of Painlevé equations

	Number of (essential) parameters	Special function	Number of parameters	Associated orthogonal polynomial
$\mathrm{P_I}$	0	—		
$\mathrm{P_{II}}$	1	**Airy** $\mathrm{Ai}(z), \mathrm{Bi}(z)$	0	—
$\mathrm{P_{III}}$	2	**Bessel** $J_\nu(z), I_\nu(z), K_\nu(z)$	1	—
$\mathrm{P_{IV}}$	2	**Parabolic** $D_\nu(z)$	1	**Hermite** $H_n(z)$
$\mathrm{P_V}$	3	**Kummer** $M(a,b,z), U(a,b,z)$ **Whittaker** $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$	2	**Associated Laguerre** $L_n^{(k)}(z)$
$\mathrm{P_{VI}}$	4	**hypergeometric** ${}_2F_1(a,b;c;z)$	3	**Jacobi** $P_n^{(\alpha,\beta)}(z)$

Properties of the Second Painlevé Equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + a \qquad \mathrm{P_{II}}$$

- **Hamiltonian structure**
- **Airy solutions**

University of Kent

Hamiltonian Representation

$\mathrm{P_{II}}$ can be written as the **Hamiltonian system**

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathcal{H}_{\rm II}}{\partial p} = p - q^2 - \tfrac12 z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathcal{H}_{\rm II}}{\partial q} = 2qp + a + \tfrac12 \qquad \mathrm{H_{II}}$$

where $\mathcal{H}_{\rm II}(q,p,z;a)$ is the Hamiltonian defined by

$$\mathcal{H}_{\rm II}(q,p,z;a) = \tfrac12 p^2 - (q^2 + \tfrac12 z)p - (a+\tfrac12)q$$

Eliminating p then q satisfies $\mathrm{P_{II}}$ whilst eliminating q yields

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac12\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac12(a+\tfrac12)^2 \qquad \mathrm{P_{34}}$$

Theorem

(**Okamoto [1986]**)

The function $\sigma(z;a) = \mathcal{H}_{\rm II} \equiv \tfrac12 p^2 - (q^2+\tfrac12 z)p - (a+\tfrac12)q$ *satisfies*

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac14(a+\tfrac12)^2 \qquad \mathrm{S_{II}}$$

and conversely

$$q(z;a) = \frac{2\sigma''(z) + a + \frac12}{4\sigma'(z)}, \qquad p(z;a) = -2\frac{\mathrm{d}\sigma}{\mathrm{d}z}$$

is a solution of $\mathrm{H_{II}}$.

Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n + \tfrac{1}{2} \qquad \mathrm{P_{II}}$$

$$p_n\frac{\mathrm{d}^2 p_n}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2 + 2p_n^3 - zp_n^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S_{II}}$$

Theorem

Let

$$\varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(\zeta) + \sin(\vartheta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/3}z$$

with ϑ an arbitrary constant, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ Airy functions, and $\tau_n(z)$ be the Wronskian, a Hankel determinant, given by

$$\tau_n(z;\vartheta) = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right) = \det\left[\frac{\mathrm{d}^{j+k}\varphi}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1}$$

then

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}, \quad p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta), \quad \sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

respectively satisfy $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$, with $n \in \mathbb{Z}$.

Properties of the Fourth Painlevé Equation and the Fourth Painlevé σ-Equation

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2-a)q + \frac{b}{q} \qquad \mathrm{P_{IV}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 - 4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right)^2 + 4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z} + 2\vartheta_\infty\right) = 0 \qquad \mathrm{S_{IV}}$$

- **Parabolic Cylinder Function Solutions**

University of Kent

Parabolic Cylinder Function Solutions of $\mathrm{P_{IV}}$

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon) = \mathcal{W}\left(\varphi_\nu(z;\varepsilon), \varphi_\nu'(z;\varepsilon), \ldots, \varphi_\nu^{(n-1)}(z;\varepsilon)\right) = \det\left[\frac{\mathrm{d}^{j+k}\varphi_\nu}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1}$$

for $n \geq 1$*, where* $\tau_{\nu,0}(z;\varepsilon) = 1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1$$

Then solutions of $\mathrm{P_{IV}}$

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = \frac{1}{2q}\left(\frac{\mathrm{d}q}{\mathrm{d}z}\right)^2 + \frac{3}{2}q^3 + 4zq^2 + 2(z^2 - a)q + \frac{b}{q}$$

are given by

$$q_{\nu,n}^{[1]}(z;a_1,b_1) = -2z + \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \quad (a_1,b_1) = \left(\varepsilon(2n-\nu), -2(\nu+1)^2\right)$$

$$q_{\nu,n}^{[2]}(z;a_2,b_2) = \varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu,n+1}(z;\varepsilon)}{\tau_{\nu+1,n}(z;\varepsilon)}, \qquad (a_2,b_2) = \left(-\varepsilon(n+\nu), -2(\nu-n+1)^2\right)$$

$$q_{\nu,n}^{[3]}(z;a_3,b_3) = -\varepsilon\frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{\nu+1,n}(z;\varepsilon)}{\tau_{\nu,n}(z;\varepsilon)}, \qquad (a_3,b_3) = \left(\varepsilon(2\nu-n+1), -2n^2\right)$$

Parabolic Cylinder Function Solutions of $\mathrm{S_{IV}}$

Theorem

Suppose $\tau_{\nu,n}(z;\varepsilon)$ *is given by*

$$\tau_{\nu,n}(z;\varepsilon)=\mathcal{W}\left(\varphi_\nu(z;\varepsilon),\varphi_\nu'(z;\varepsilon),\ldots,\varphi_\nu^{(n-1)}(z;\varepsilon)\right)=\det\left[\frac{\mathrm{d}^{j+k}\varphi_\nu}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1}$$

for $n\geq 1$*, where* $\tau_{\nu,0}(z;\varepsilon)=1$ *and* $\varphi_\nu(z;\varepsilon)$ *satisfies*

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2}-2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z}+2\varepsilon\nu\varphi_\nu=0,\qquad \varepsilon^2=1$$

Then solutions of $\mathrm{S_{IV}}$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2-4\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z}-\sigma\right)^2+4\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+2\vartheta_0\right)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}+2\vartheta_\infty\right)=0$$

are given by

$$\sigma_{\nu,n}(z)=\frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_{\nu,n}(z;\varepsilon),\qquad (\vartheta_0,\vartheta_\infty)=\big(\varepsilon(\nu-n+1),-\varepsilon n\big)$$

$$\frac{\mathrm{d}^2\varphi_\nu}{\mathrm{d}z^2} - 2\varepsilon z\frac{\mathrm{d}\varphi_\nu}{\mathrm{d}z} + 2\varepsilon\nu\varphi_\nu = 0, \qquad \varepsilon^2 = 1 \tag{$*$}$$

- **If** $\nu \notin \mathbb{Z}$

$$\varphi_\nu(z;\varepsilon) = \begin{cases} \left\{\cos(\vartheta)D_\nu(\sqrt{2}\,z) + \sin(\vartheta)D_\nu(-\sqrt{2}\,z)\right\}\exp\left(\tfrac{1}{2}z^2\right), & \varepsilon = 1 \\ \left\{\cos(\vartheta)D_{-\nu-1}(\sqrt{2}\,z) + \sin(\vartheta)D_{-\nu-1}(-\sqrt{2}\,z)\right\}\exp\left(-\tfrac{1}{2}z^2\right), & \varepsilon = -1 \end{cases}$$

- **If** $\nu = n \in \mathbb{Z}$, **with** $n \geq 0$

$$\varphi_n(z;\varepsilon) = \begin{cases} \cos(\vartheta)H_n(z) + \sin(\vartheta)\exp(z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \varepsilon = 1 \\ \cos(\vartheta)H_n(\mathrm{i}z) + \sin(\vartheta)\exp(-z^2)\dfrac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \varepsilon = -1 \end{cases}$$

- **If** $\nu = -n \in \mathbb{Z}$, **with** $n \geq 1$

$$\varphi_{-n}(z;\varepsilon) = \begin{cases} \cos(\vartheta)H_{n-1}(\mathrm{i}z)\exp(z^2) + \sin(\vartheta)\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfc}(z)\exp(z^2)\right\}, & \varepsilon = 1 \\ \cos(\vartheta)H_{n-1}(z)\exp(-z^2) + \sin(\vartheta)\dfrac{\mathrm{d}^{n-1}}{\mathrm{d}z^{n-1}}\left\{\operatorname{erfi}(z)\exp(-z^2)\right\}, & \varepsilon = -1 \end{cases}$$

with ϑ an arbitrary constant, $D_\nu(\zeta)$ the **parabolic cylinder function**, $H_n(z)$ the **Hermite polynomial**, $\operatorname{erfc}(z)$ the **complementary error function** and $\operatorname{erfi}(z)$ the **imaginary error function**.

Orthogonal Polynomials

- Some History
- Monic orthogonal polynomials
- Semi-classical orthogonal polynomials

Some History

- The relationship between semi-classical orthogonal polynomials and integrable equations dates back to **Shohat [1939]** and **Freud [1976]**.
- **Fokas, Its & Kitaev [1991, 1992]** identified these integrable equations as **discrete Painlevé equations**.
- **Magnus [1995]** considered the **Freud weight**

$$w(x;t) = \exp\left(-x^4 + tx^2\right), \qquad x,t \in \mathbb{R},$$

and showed that the coefficients in the **three-term recurrence relation** can be expressed in terms of solutions of

$$q_n(q_{n-1} + q_n + q_{n+1}) + 2tq_n = n$$

which is **discrete** $\mathrm{P_I}$ ($\mathrm{dP_I}$), as shown by **Bonan & Nevai [1984]**, and

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}t^2} = \frac{1}{2q_n}\left(\frac{\mathrm{d}q_n}{\mathrm{d}t}\right)^2 + \frac{3}{2}q_n^3 + 4tq_n^2 + 2(t^2 + \tfrac{1}{2}n)q_n - \frac{n^2}{2q_n}$$

which is $\mathrm{P_{IV}}$ with $a = -\frac{1}{2}n$ and $b = -\frac{1}{2}n^2$. The connection between the Freud weight and solutions of $\mathrm{dP_I}$ and $\mathrm{P_{IV}}$ is due to **Kitaev [1988]**.

Monic Orthogonal Polynomials

Let $P_n(x)$, $n = 0, 1, 2, \ldots$, be the **monic orthogonal polynomials** of degree n in x, with respect to the positive weight $w(x)$, such that

$$\int_a^b P_m(x)P_n(x)\,w(x)\,\mathrm{d}x = h_n\delta_{m,n}, \quad h_n > 0, \qquad m,n = 0,1,2,\ldots$$

One of the important properties that orthogonal polynomials have is that they satisfy the **three-term recurrence relation**

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x)$$

where the recurrence coefficients are given by

$$\alpha_n = \frac{\widetilde{\Delta}_{n+1}}{\Delta_{n+1}} - \frac{\widetilde{\Delta}_n}{\Delta_n}, \qquad \beta_n = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2}$$

with

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-1} \\ \mu_1 & \mu_2 & \ldots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-2} \end{vmatrix}, \qquad \widetilde{\Delta}_n = \begin{vmatrix} \mu_0 & \mu_1 & \ldots & \mu_{n-2} & \mu_n \\ \mu_1 & \mu_2 & \ldots & \mu_{n-1} & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \ldots & \mu_{2n-3} & \mu_{2n-1} \end{vmatrix}$$

and $\mu_k = \displaystyle\int_a^b x^k\,w(x)\,\mathrm{d}x$ are the **moments** of the weight $w(x)$.

Further Properties

- The Hankel determinant

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_{n-1} \\ \mu_1 & \mu_2 & \dots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-2} \end{vmatrix}, \qquad \mu_k = \int_a^b x^k \, w(x) \, \mathrm{d}x$$

also has the integral representation

$$\Delta_n = \frac{1}{n!} \int_a^b \overset{n}{\cdots} \int_a^b \prod_{\ell=1}^{n} w(x_\ell) \prod_{1 \le j < k \le n} (x_j - x_k)^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_n, \qquad n \ge 1$$

which is the **partition function** in random matrix theory.

- The monic polynomials $P_n(x)$ can be uniquely expressed as

$$P_n(x) = \frac{1}{\Delta_n} \begin{vmatrix} \mu_0 & \mu_1 & \dots & \mu_n \\ \mu_1 & \mu_2 & \dots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \dots & \mu_{2n-1} \\ 1 & x & \dots & x^n \end{vmatrix}$$

Semi-classical Orthogonal Polynomials

Consider the **Pearson equation** satisfied by the weight $w(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sigma(x)w(x)] = \tau(x)w(x)$$

- **Classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with $\deg(\sigma) \leq 2$ and $\deg(\tau) = 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
Hermite	$\exp(-x^2)$	1	$-2x$
Laguerre	$x^\nu \exp(-x)$	x	$1+\nu-x$
Jacobi	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$	$\beta-\alpha-(2+\alpha+\beta)x$

- **Semi-classical orthogonal polynomials**: $\sigma(x)$ and $\tau(x)$ are polynomials with either $\deg(\sigma) > 2$ or $\deg(\tau) > 1$

	$w(x)$	$\sigma(x)$	$\tau(x)$
Airy	$\exp(-\frac{1}{3}x^3+tx)$	1	$t-x^2$
semi-classical Hermite	$x^\nu \exp(-x^2+tx)$	x	$1+\nu+tx-2x^2$
Generalized Freud	$x^{2\nu+1}\exp(-x^4+tx^2)$	x	$2\nu+2+2tx^2-4x^4$

If the weight has the form

$$w(x;t) = w_0(x)\exp(tx)$$

where the integrals $\displaystyle\int_{-\infty}^{\infty} x^k w_0(x)\exp(tx)\,\mathrm{d}x$ exist for all $k \geq 0$.

- The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the **Toda system**

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_n - \beta_{n+1}, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1})$$

- The kth moment is given by

$$\mu_k(t) = \int_{-\infty}^{\infty} x^k w_0(x)\exp(tx)\,\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} w_0(x)\exp(tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$$

- Since $\mu_k(t) = \dfrac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$, then $\Delta_n(t)$ and $\widetilde{\Delta}_n(t)$ can be expressed as Wronskians

$$\Delta_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right) = \det\left[\frac{\mathrm{d}^{j+k}\mu_0}{\mathrm{d}t^{j+k}}\right]_{j,k=0}^{n-1}$$

$$\widetilde{\Delta}_n(t) = \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-2}\mu_0}{\mathrm{d}t^{n-2}}, \frac{\mathrm{d}^{n}\mu_0}{\mathrm{d}t^{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\Delta_n(t)$$

Semi-classical Hermite Weight

$$w(x;t) = |x|^{\nu} \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \qquad \nu > -1$$

- **PAC & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, **39** (2014) 223–254

Semi-classical Hermite weight

Consider the **semi-classical Hermite weight**

$$w(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

- **If $\nu \notin \mathbb{N}$, then the moment $\mu_0(t;\nu)$ is given by**

$$\mu_0(t;\nu) = \int_{-\infty}^{\infty} w(x;t)\,\mathrm{d}x = \frac{\Gamma(\nu+1)\exp(\tfrac{1}{8}t^2)}{2^{(\nu+1)/2}}\Big\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) + D_{-\nu-1}\big(\tfrac{1}{2}\sqrt{2}\,t\big)\Big\}$$

  since the **parabolic cylinder function** $D_\nu(\zeta)$ has the integral representation

$$D_\nu(\zeta) = \frac{\exp(-\tfrac{1}{4}\zeta^2)}{\Gamma(-\nu)}\int_0^\infty s^{-\nu-1}\exp(-\tfrac{1}{2}s^2 - \zeta s)\,\mathrm{d}s$$

- **If $\nu = 2N$, with $N \in \mathbb{N}$ then**

$$\mu_0(t;2N) = \int_{-\infty}^{\infty} x^{2N} w(x;t)\,\mathrm{d}x = \sqrt{\pi}\,\left(\tfrac{1}{2}\mathrm{i}\right)^{2N} H_{2N}\big(\tfrac{1}{2}\mathrm{i}t\big)\exp\left(\tfrac{1}{4}t^2\right)$$

  with $H_n(z)$ the **Hermite polynomial**.

- **If $\nu = 2N+1$, with $N \in \mathbb{N}$ then**

$$\mu_0(t;2N+1) = \int_{-\infty}^{\infty} x^{2N}|x| w(x;t)\,\mathrm{d}x = \sqrt{\pi}\,\frac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\left\{\mathrm{erf}\big(\tfrac{1}{2}t\big)\,\exp\left(\tfrac{1}{4}t^2\right)\right\}$$

  with $\mathrm{erf}(z)$ the **error function**.

- **The moment $\mu_k(t;\nu)$ is given by**

$$\mu_k(t;\nu) = \int_{-\infty}^{\infty} x^k |x|^\nu \exp(-x^2 + tx)\,\mathrm{d}x$$
$$= \frac{\mathrm{d}^k}{\mathrm{d}t^k}\left(\int_{-\infty}^{\infty} |x|^\nu \exp(-x^2 + tx)\,\mathrm{d}x\right) = \frac{\mathrm{d}^k\mu_0}{\mathrm{d}t^k}$$

- **The Hankel determinant $\Delta_n(t;\nu)$ is given by**

$$\Delta_n(t;\nu) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\mu_0(t;\nu)\right]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

where

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\left\{D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right) + D_{-\nu-1}\left(\tfrac{1}{2}\sqrt{2}\,t\right)\right\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\,\left(\tfrac{1}{2}\mathrm{i}\right)^{2N} H_{2N}\left(\tfrac{1}{2}\mathrm{i}t\right)\exp\left(\tfrac{1}{4}t^2\right), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\left\{\mathrm{erf}\left(\tfrac{1}{2}t\right)\,\exp\left(\tfrac{1}{4}t^2\right)\right\}, & \nu = 2N+1 \end{cases}$$

Theorem (PAC & Jordaan [2014])

The recurrence coefficients $\alpha_n(t;\nu)$ and $\beta_n(t;\nu)$ in the three-term recurrence relation

$$xP_n(x;t) = P_{n+1}(x;t) + \alpha_n(t;\nu)P_n(x;t) + \beta_n(t;\nu)P_{n-1}(x;t),$$

for monic polynomials orthogonal with respect to the semi-classical Hermite weight

$$w(x;t) = |x|^\nu \exp(-x^2 + tx), \qquad x \in \mathbb{R}, \quad \nu > -1$$

are given by

$$\alpha_n(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\Delta_{n+1}(t;\nu)}{\Delta_n(t;\nu)}, \qquad \beta_n(t;\nu) = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \Delta_n(t;\nu)$$

where $\Delta_n(t;\nu)$ is the Hankel determinant

$$\Delta_n(t;\nu) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\mu_0(t;\nu)\right]_{j,k=0}^{n-1} \equiv \mathcal{W}\left(\mu_0, \frac{\mathrm{d}\mu_0}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\mu_0}{\mathrm{d}t^{n-1}}\right)$$

with

$$\mu_0(t;\nu) = \begin{cases} \dfrac{\Gamma(\nu+1)\exp(\frac{1}{8}t^2)}{2^{(\nu+1)/2}}\left\{D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right) + D_{-\nu-1}\left(\tfrac{1}{2}\sqrt{2}\,t\right)\right\}, & \nu \notin \mathbb{N} \\ \sqrt{\pi}\left(\tfrac{1}{2}\mathrm{i}\right)^{2N} H_{2N}\left(\tfrac{1}{2}\mathrm{i}t\right)\exp\left(\tfrac{1}{4}t^2\right), & \nu = 2N \\ \sqrt{\pi}\,\dfrac{\mathrm{d}^{2N+1}}{\mathrm{d}t^{2N+1}}\left\{\mathrm{erf}\left(\tfrac{1}{2}t\right)\exp\left(\tfrac{1}{4}t^2\right)\right\}, & \nu = 2N+1 \end{cases}$$

Recurrence coefficients for $w(x;t) = x^2 \exp(-x^2 + tx)$

$$\alpha_0(t) = \tfrac{1}{2}t + \frac{2t}{t^2+2}$$

$$\alpha_1(t) = \tfrac{1}{2}t + \frac{4t^3}{t^4+12} - \frac{2t}{t^2+2}$$

$$\alpha_2(t) = \tfrac{1}{2}t + \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72} - \frac{4t^3}{t^4+12}$$

$$\alpha_3(t) = \tfrac{1}{2}t + \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720} - \frac{6t(t^4+12-4t^2)}{t^6-6t^4+36t^2+72}$$

$$\alpha_4(t) = \tfrac{1}{2}t + \frac{10t(t^8+216t^4+720-24t^6-480t^2)}{t^{10}-30t^8+360t^6-1200t^4+3600t^2+7200} - \frac{8t^3(t^4+60-12t^2)}{t^8-16t^6+120t^4+720}$$

$$\beta_1(t) = \tfrac{1}{2} - \frac{2(t^2-2)}{(t^2+2)^2}$$

$$\beta_2(t) = 1 - \frac{4t^2(t^2-6)(t^2+6)}{(t^4+12)^2}$$

$$\beta_3(t) = \tfrac{3}{2} - \frac{6(t^4-12t^2+12)(t^6+6t^4+36t^2-72)}{(t^6-6t^4+36t^2+72)^2}$$

$$\beta_4(t) = 2 - \frac{8t^2(t^4-20t^2+60)(t^8+72t^4-2160)}{(t^8-16t^6+120t^4+720)^2}$$

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = [x - \alpha_n(t)]P_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

then

$$P_1(x;t) = x - \frac{t(t^2+6)}{2(t^2+2)}$$

$$P_2(x;t) = x^2 - \frac{t(t^4+4t^2+12)}{t^4+12}x + \frac{t^6+6t^4+36t^2-72}{4(t^4+12)}$$

$$P_3(x;t) = x^3 - \frac{3t(t^6-2t^4+20t^2+120)}{2(t^6-6t^4+36t^2+72)}x^2 + \frac{3(t^8+40t^4-240)}{4(t^6-6t^4+36t^2+72)}x - \frac{t(t^8+72t^4-2160)}{8(t^6-6t^4+36t^2+72)}$$

$$\begin{aligned} P_4(x;t) = x^4 &- \frac{2t(t^8-12t^6+72t^4+240t^2+720)}{t^8-16t^6+120t^4+720}x^3 \\ &+ \frac{3(t^{10}-10t^8+80t^6+1200t^2-2400)}{2(t^8-16t^6+120t^4+720)}x^2 \\ &- \frac{t(t^{10}-10t^8+120t^6-240t^4-1200t^2-7200)}{2(t^8-16t^6+120t^4+720)}x \\ &+ \frac{t^{12}-12t^{10}+180t^8-480t^6-3600t^4-43200t^2+43200}{16(t^8-16t^6+120t^4+720)} \end{aligned}$$

Generalized Freud Weight

$$w(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}, \quad \nu > 0$$

- **PAC, K Jordaan & A Kelil**, "A generalized Freud weight", *Stud. Appl. Math.*, **136** (2016) 288–320
- **P A Clarkson & K Jordaan**, "Properties of generalized Freud polynomials", arXiv:1606.06026

Generalized Freud weight

For the **generalized Freud weight**

$$w(x;t) = |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right), \qquad x \in \mathbb{R}$$

the moments are

$$\begin{aligned}\mu_0(t;\nu) &= \int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x \\ &= \int_0^{\infty} y^{\nu+1} \exp\left(-y^2 + ty\right) \mathrm{d}y \\ &= 2^{-(\nu+1)/2}\Gamma(\nu+1)\, \exp(\tfrac{1}{8}t^2) D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)\end{aligned}$$

$$\begin{aligned}\mu_{2n}(t;\nu) &= \int_{-\infty}^{\infty} x^{2n}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x \\ &= (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\int_{-\infty}^{\infty} |x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x\right) = (-1)^n \frac{\mathrm{d}^n \mu_0}{\mathrm{d}t^n}\end{aligned}$$

$$\mu_{2n+1}(t;\nu) = \int_{-\infty}^{\infty} x^{2n+1}|x|^{2\nu+1} \exp\left(-x^4 + tx^2\right) \mathrm{d}x = 0$$

for $n = 1, 2, \ldots$. When $\nu = n \in \mathbb{Z}^+$, then

$$D_{-n-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big) = \tfrac{1}{2}\sqrt{2\pi}\, \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left\{\left[1 + \operatorname{erf}\left(\tfrac{1}{2}t\right)\right] \exp\left(\tfrac{1}{8}t^2\right)\right\},$$

where $\operatorname{erf}(z)$ is the **error function**.

Theorem (PAC, Jordaan & Kelil [2016])

The recurrence coefficient $\beta_n(t)$ *in the three-term recurrence relation*

$$xP_n(x;t) = P_{n+1}(x;t) + \beta_n(t)P_{n-1}(x;t),$$

is given by

$$\beta_{2n}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t;\nu+1)}{\tau_n(t;\nu)}, \qquad \beta_{2n+1}(t;\nu) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_{n+1}(t;\nu)}{\tau_n(t;\nu+1)}$$

where $\tau_n(t;\nu)$ *is the Wronskian given by*

$$\tau_n(t;\nu) = \mathcal{W}\left(\phi_\nu, \frac{\mathrm{d}\phi_\nu}{\mathrm{d}t}, \ldots, \frac{\mathrm{d}^{n-1}\phi_\nu}{\mathrm{d}t^{n-1}}\right)$$

with

$$\phi_\nu(t) = \mu_0(t;\nu) = \frac{\Gamma(\nu+1)}{2^{(\nu+1)/2}}\ \exp\left(\tfrac{1}{8}t^2\right) D_{-\nu-1}\left(-\tfrac{1}{2}\sqrt{2}\,t\right)$$

Remark: The function $S_n(t;\nu) = \dfrac{\mathrm{d}}{\mathrm{d}t}\ln\tau_n(t;\nu)$ satisfies

$$4\left(\frac{\mathrm{d}^2S_n}{\mathrm{d}t^2}\right)^2 - \left(t\frac{\mathrm{d}S_n}{\mathrm{d}t} - S_n\right)^2 + 4\frac{\mathrm{d}S_n}{\mathrm{d}t}\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n\right)\left(2\frac{\mathrm{d}S_n}{\mathrm{d}t} - n - \nu\right) = 0$$

which is equivalent to $\mathrm{S_{IV}}$, the $\mathrm{P_{IV}}$ σ-equation, so

$$\beta_{2n}(t;\nu) = S_n(t;\nu+1) - S_n(t;\nu), \qquad \beta_{2n+1}(t;\nu) = S_{n+1}(t;\nu) - S_n(t;\nu+1)$$

Theorem

The recurrence coefficients $\beta_n(t)$ satisfy the equation

$$\frac{\mathrm{d}^2\beta_n}{\mathrm{d}t^2} = \frac{1}{2\beta_n}\left(\frac{\mathrm{d}\beta_n}{\mathrm{d}t}\right)^2 + \tfrac{3}{2}\beta_n^3 - t\beta_n^2 + (\tfrac{1}{8}t^2 - \tfrac{1}{2}a_n)\beta_n + \frac{b_n}{16\beta_n} \qquad (1)$$

which is equivavlent to $\mathrm{P_{IV}}$, where the parameters a_n and b_n are given by

$$a_{2n} = -2\nu - n - 1, \qquad a_{2n+1} = \nu - n$$
$$b_{2n} = -2n^2, \qquad b_{2n+1} = -2(\nu + n + 1)^2$$

Further $\beta_n(t)$ satisfies the nonlinear difference equation

$$\beta_{n+1} + \beta_n + \beta_{n-1} = \tfrac{1}{2}t + \frac{2n + (2\nu+1)[1-(-1)^n]}{8\beta_n} \qquad (2)$$

which is the general **discrete** $\mathrm{P_I}$.

Remark: The link between the differential equation (1) and the difference equation (2) is given by the **Bäcklund transformations**

$$\beta_{n+1} = \frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}, \qquad \beta_{n-1} = -\frac{1}{2\beta_n}\frac{\mathrm{d}\beta_n}{\mathrm{d}t} - \tfrac{1}{2}\beta_n + \tfrac{1}{4}t + \frac{c_n}{4\beta_n}$$

with $c_n = \tfrac{1}{2}n + \tfrac{1}{4}(2\nu+1)[1-(-1)^n]$.

The first few recurrence coefficients are:

$$\beta_1(t) = \Phi_\nu$$

$$\beta_2(t) = -\frac{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1}{2\Phi_\nu}$$

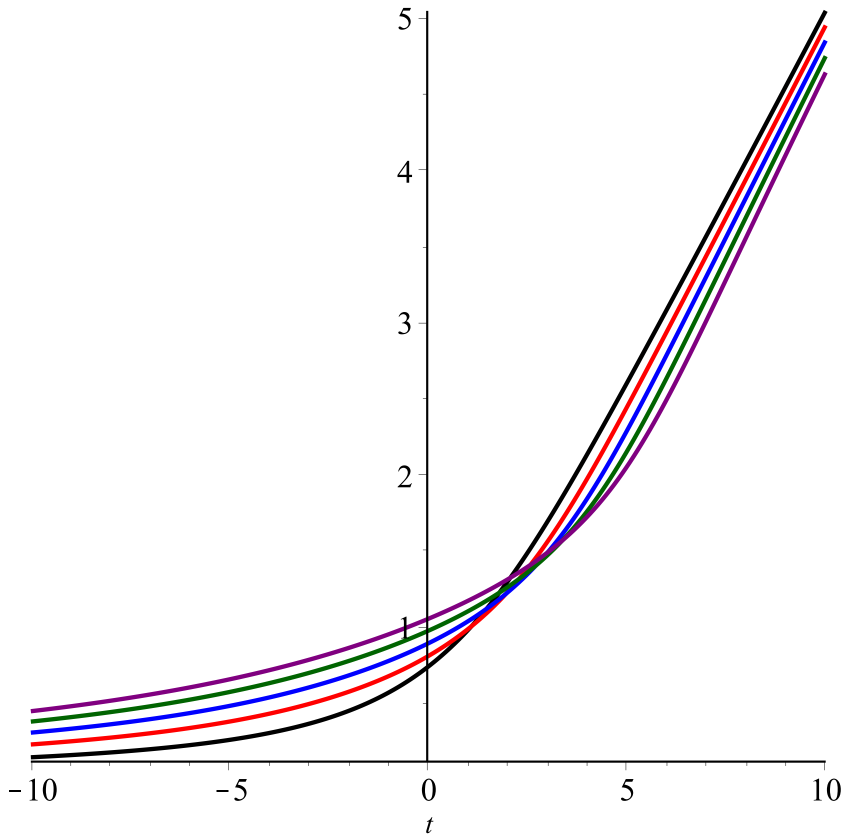
$$\beta_3(t) = -\frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} - \frac{\nu+1}{2\Phi_\nu}$$

$$\beta_4(t) = \frac{t}{2(\nu+2)} + \frac{\Phi_\nu}{2\Phi_\nu^2 - t\Phi_\nu - \nu - 1} + \frac{(\nu+1)(t^2+2\nu+4)\Phi_\nu + (\nu+1)^2 t}{2(\nu+2)[2(\nu+2)\Phi_\nu^2 - (\nu+1)t\Phi_\nu - (\nu+1)^2]}$$

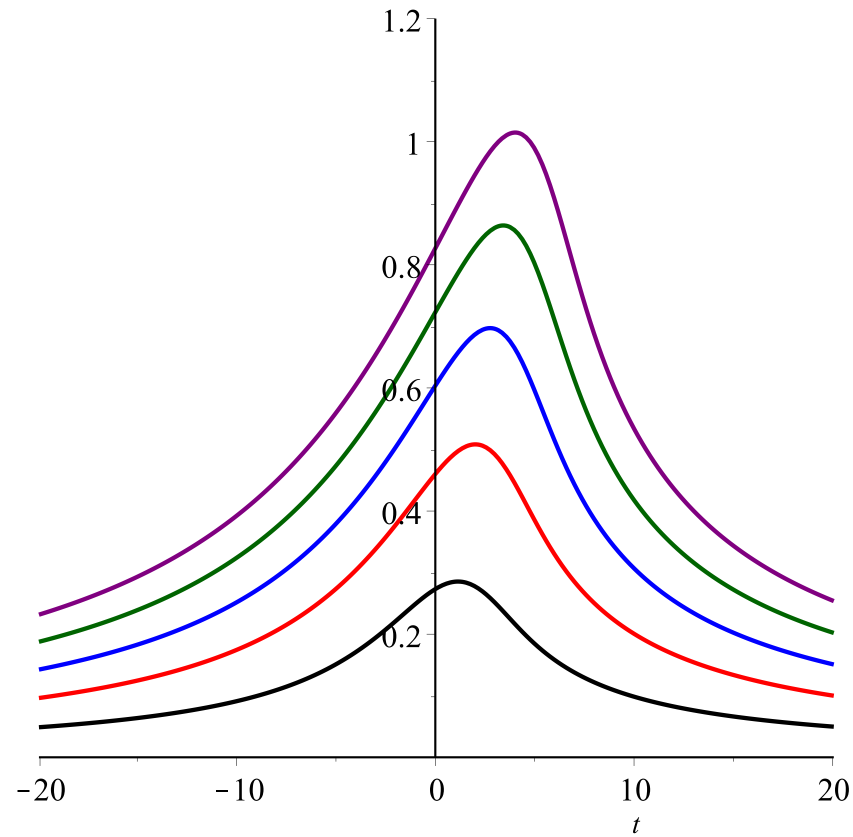
$$\beta_5(t) = -\frac{2\nu t}{\nu+1} - \frac{2(\nu+1)}{t} - \frac{2\nu(2t^2+\nu+1)\Phi_\nu - 4\nu^2 t}{(\nu+1)[(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2]} - \frac{2[\nu t^2 + (\nu+1)(2\nu+1)]\Phi_\nu^2 + 2\nu t(t^2+4\nu+5)\Phi_\nu - 4\nu^2 t^2 - 8\nu^2(\nu+1)}{t[t\Phi_\nu^3(t) + (2t^2 - 2\nu + 1)\Phi_\nu^2 - 6\Phi_\nu \nu t + 4\nu^2]}$$

where

$$\Phi_\nu(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\left\{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)\exp\left(\tfrac{1}{8}t^2\right)\right\} = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{2}\,\frac{D_{-\nu}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}{D_{-\nu-1}\big(-\tfrac{1}{2}\sqrt{2}\,t\big)}.$$



$\beta_{2n-1}(t;\frac{1}{2}), \quad n = 1, 2, \ldots, 5$



$\beta_{2n}(t;\frac{1}{2}), \quad n = 1, 2, \ldots, 5$

Plots of the recurrence coefficients $\beta_{2n-1}(t;\frac{1}{2})$ **and** $\beta_{2n}(t;\frac{1}{2})$, $n = 1, 2, \ldots, 5$, **for** $n = 1$ **(black),** $n = 2$ **(red),** $n = 3$ **(blue),** $n = 4$ **(green) and** $n = 5$ **(purple).**

Hence, using the three-term recurrence relation

$$P_{n+1}(x;t) = xP_n(x;t) - \beta_n(t)P_{n-1}(x;t), \qquad n = 0, 1, 2, \ldots$$

with

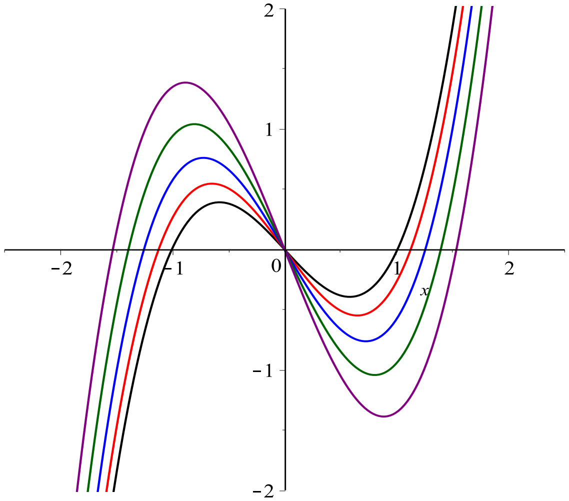
$$P_0(x;t) = 1, \qquad P_{-1}(x;t) = 0, \qquad \beta_0(t) = 0$$

then the first few polynomials are given by

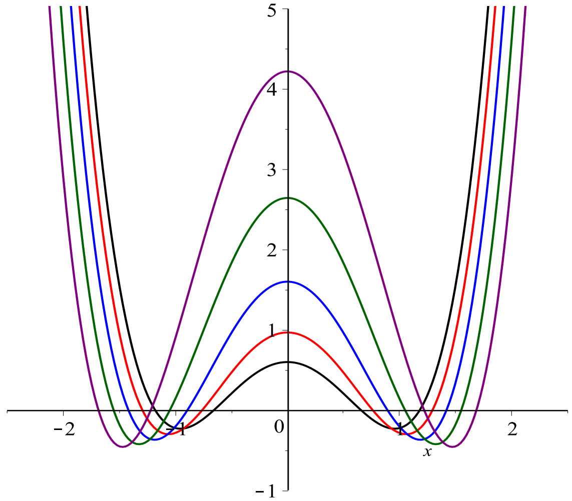
$$\begin{aligned}
P_1(x;t) &= x \\
P_2(x;t) &= x^2 - \Phi_\nu \\
P_3(x;t) &= x^3 + 2\frac{t\Phi_\nu - \nu}{\Phi_\nu}\,x \\
P_4(x;t) &= x^4 + 2\frac{t\Phi_\nu^2 + (2t^2+1)\Phi_\nu - 2\nu t}{\Phi_\nu^2 + 2t\Phi_\nu - 2\nu}\,x^2 - 2\frac{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}{\Phi_\nu^2 + 2t\Phi_\nu - 2\nu} \\
P_5(x;t) &= x^5 + 2\frac{(\nu+2)t\Phi_\nu^2 + \nu(2t^2-1)\Phi_\nu - 2\nu^2 t}{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}\,x^3 \\
&\qquad + \frac{2\left[2t^2 - (\nu+1)^2\right]\Phi_\nu^2 - 4\nu(\nu+3)t\Phi_\nu + 4(\nu+2)\nu^2}{(\nu+1)\Phi_\nu^2 + 2\nu t\Phi_\nu - 2\nu^2}\,x
\end{aligned}$$

where

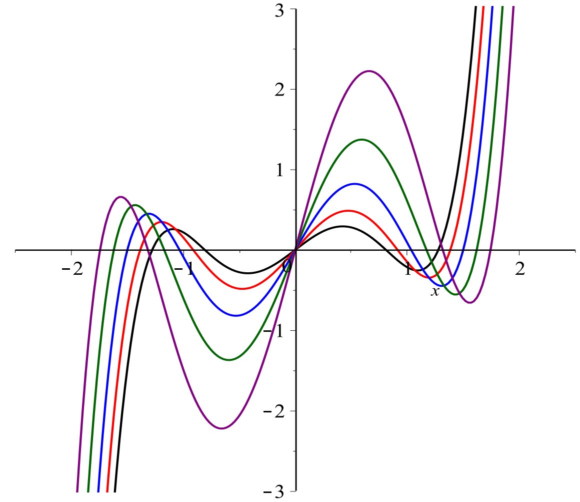
$$\Phi_\nu(t) = 2t - \sqrt{2}\,\frac{D_{1-\nu}\big(\sqrt{2}\,t\big)}{D_{-\nu}\big(\sqrt{2}\,t\big)}$$



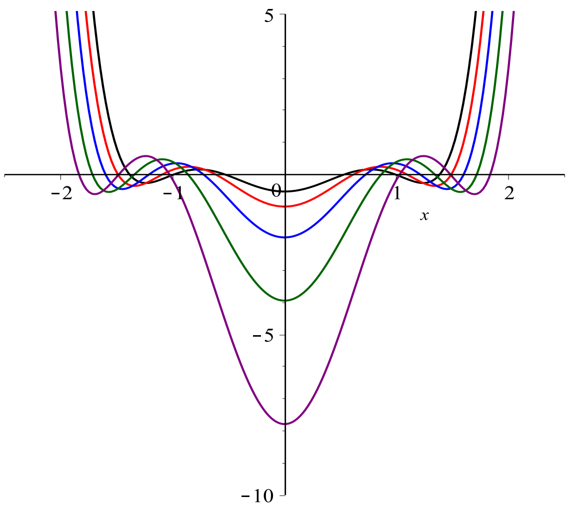
$P_3(x;t;\frac{1}{2})$



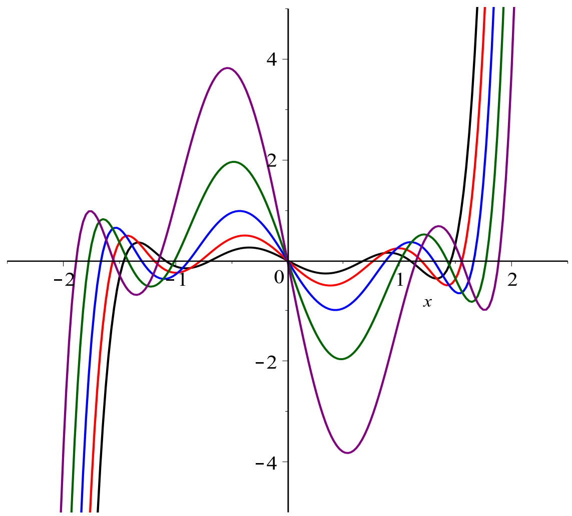
$P_4(x;t;\frac{1}{2})$



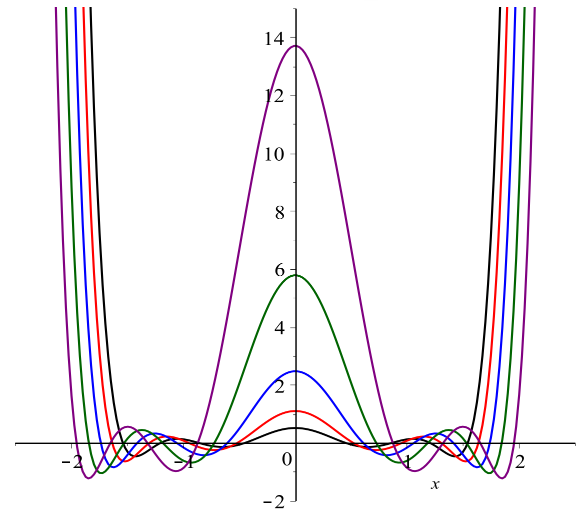
$P_5(x;t;\frac{1}{2})$



$P_6(x;t;\frac{1}{2})$



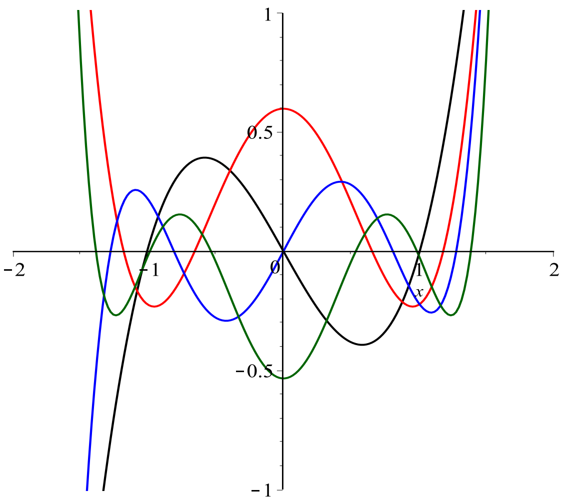
$P_7(x;t;\frac{1}{2})$



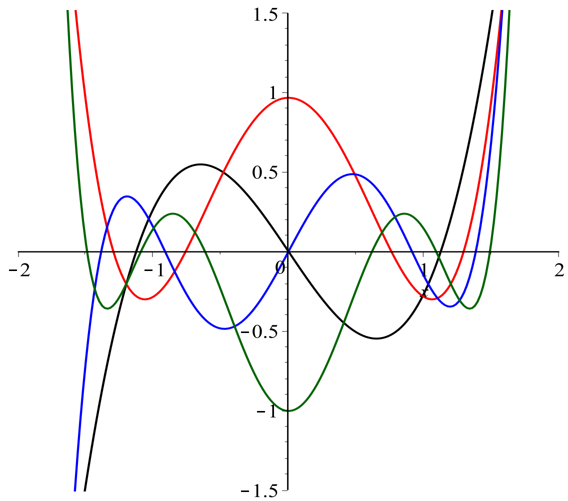
$P_8(x;t;\frac{1}{2})$

Plots of the polynomials $P_n(x;t;\frac{1}{2})$, $n = 3, 4, \ldots, 8$ for $t = 0$ (black), $t = 1$ (red), $t = 2$ (blue), $t = 3$ (green) and $t = 4$ (purple).

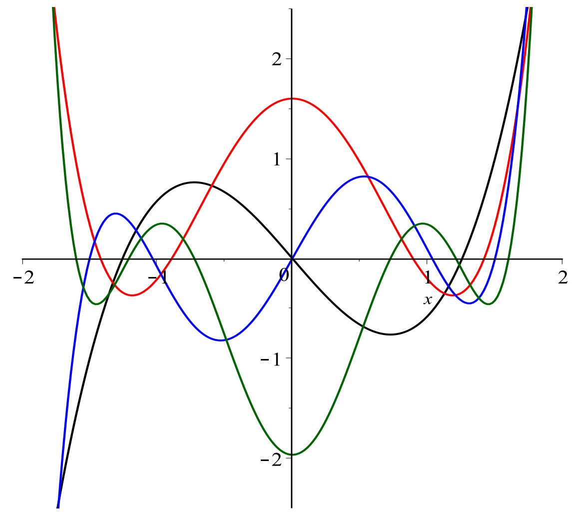
University of Kent



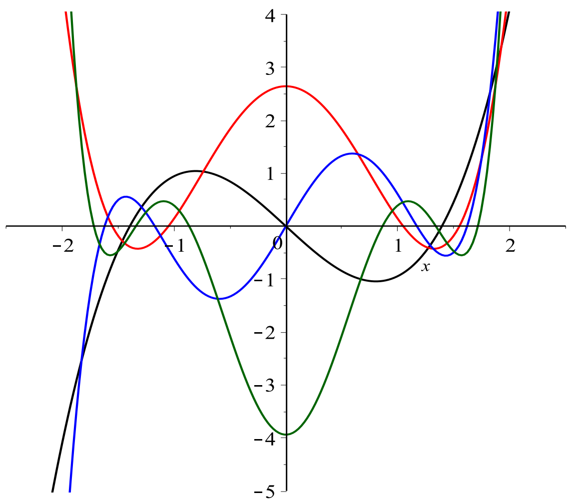
$t = 0$



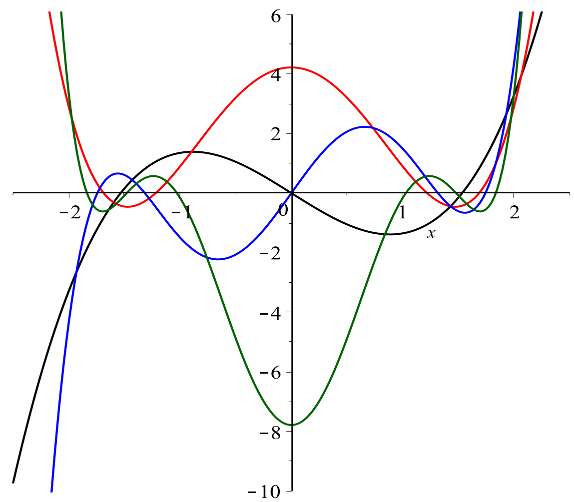
$t = 1$



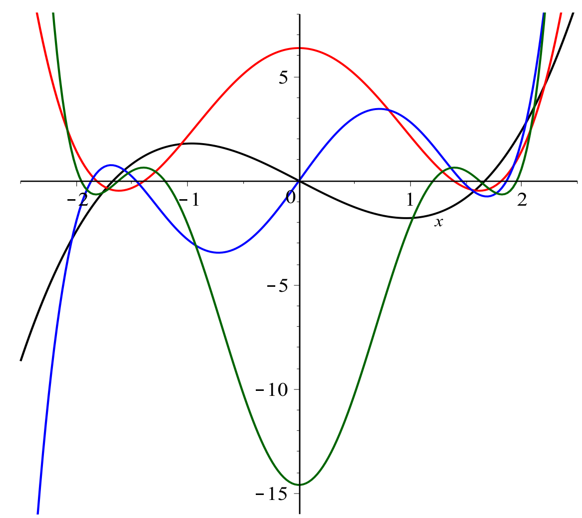
$t = 2$



$t = 3$



$t = 4$



$t = 5$

Plots of the polynomials $P_3(x;t;\frac{1}{2})$ (black), $P_4(x;t;\frac{1}{2})$ (red), $P_5(x;t;\frac{1}{2})$ (blue), $P_6(x;t;\frac{1}{2})$ (green) for $t = 0, 1, \ldots, 5$.

Theorem (PAC, Jordaan & Kelil [2016])

Suppose that the monic polynomials $Q_n(x;t)$ *are generated by the three-term recurrence relation*

$$xQ_n(x;t) = Q_{n+1}(x;t) + \tfrac{1}{4}[1-(-1)^n]t\,Q_{n-1}(x;t),$$

with $Q_0(x;t) = 1$ *and* $Q_1(x;t) = x$ *and the monic polynomials* $P_n(x;t)$ *arise from the generalized Freud weight*

$$w(x;t) = |x|^{2\nu+1}\exp\left(-x^4+tx^2\right) \qquad (1)$$

Then

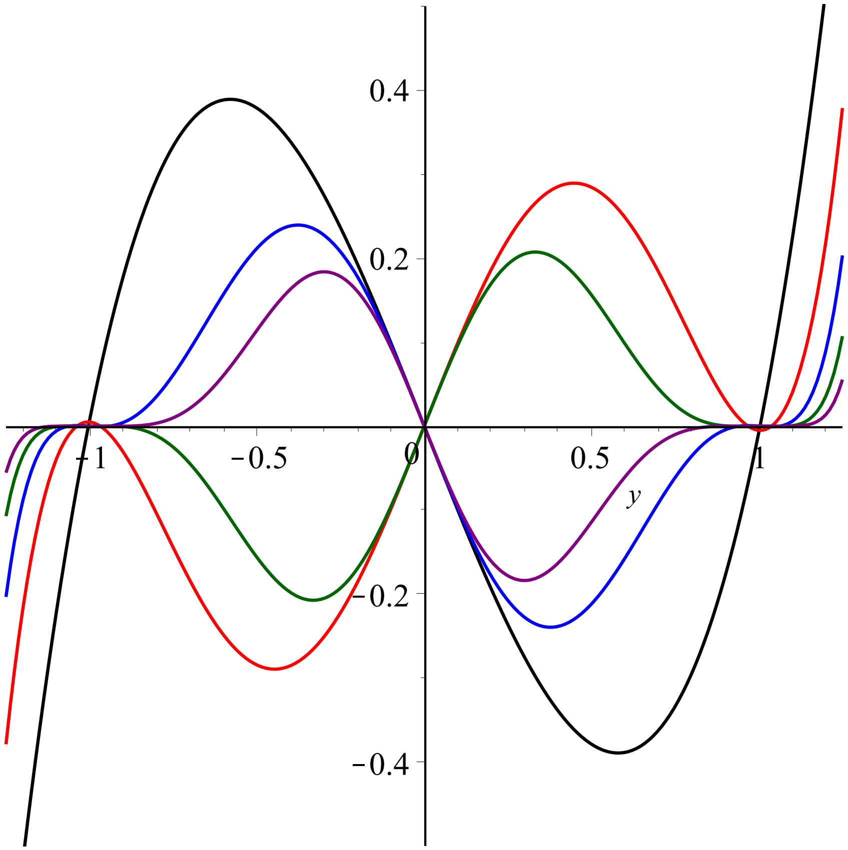
$$Q_{2n}(x;t) = (x^2-\tfrac{1}{2}t)^n$$
$$Q_{2n+1}(x;t) = x(x^2-\tfrac{1}{2}t)^n$$

and in the limit as $t\to\infty$

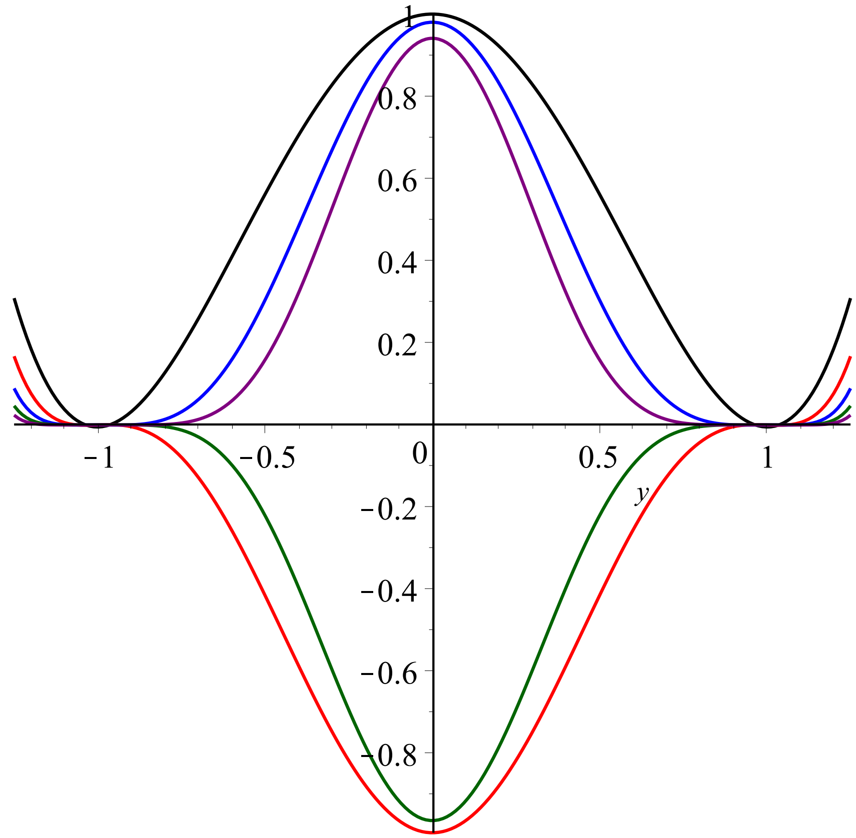
$$P_{2n}(x;t) \to (x^2-\tfrac{1}{2}t)^n = Q_{2n}(x;t)$$
$$P_{2n+1}(x;t) \to x(x^2-\tfrac{1}{2}t)^n = Q_{2n+1}(x;t)$$

This is due to the fact that for the generalized Freud weight (1), as $t\to\infty$

$$\beta_{2n}(t)\to 0, \qquad \beta_{2n+1}(t)\to \tfrac{1}{2}t$$



$\widetilde{P}_{2n-1}(y;t;\frac{1}{2})$

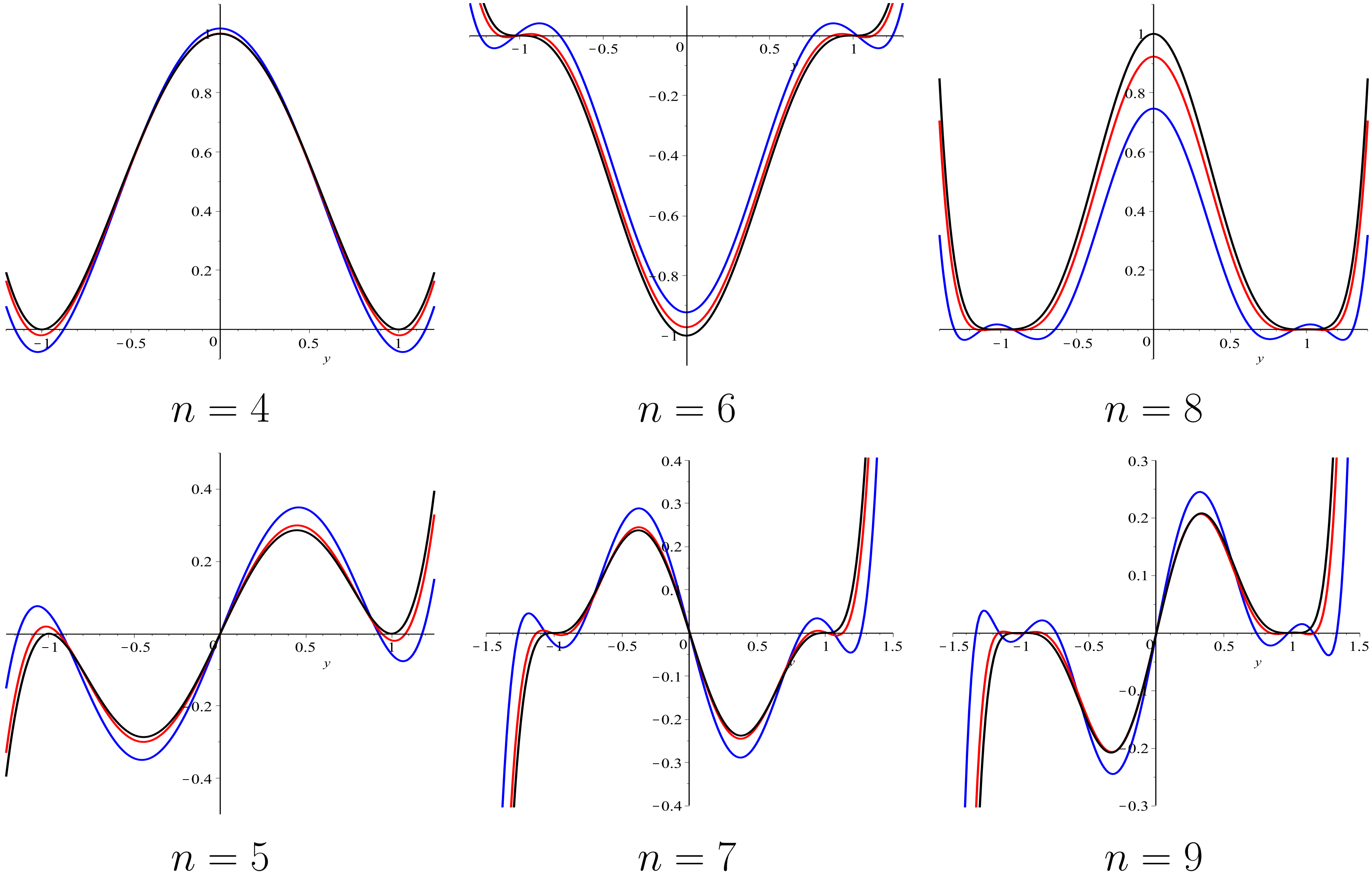


$\widetilde{P}_{2n}(y;t;\frac{1}{2})$

Plots of polynomials $\widetilde{P}_{2n-1}(y;t;\frac{1}{2})$ and $\widetilde{P}_{2n}(y;t;\frac{1}{2})$, for $n=1$ (black), $n=2$ (red), $n=3$ (blue), $n=4$ (green) and $n=5$ (purple), when $t=20$, where

$$\widetilde{P}_n(y;t,\nu) = (\tfrac{1}{2}t)^{n/2} P_n(x;t,\nu), \qquad x = (\tfrac{1}{2}t)^{1/2}\, y$$

University of Kent



Plots of the polynomials $\widetilde{P}_n(y; 5, \frac{1}{2})$ (blue), $\widetilde{P}_n(y; 10, \frac{1}{2})$ (red) and $\widetilde{Q}_n(y)$ (black) for $n = 4, 5, \ldots, 9$, where $\widetilde{Q}_{2n}(y) = (y^2 - 1)^n$ and $\widetilde{Q}_{2n+1}(y) = y(y^2 - 1)^n$.

Semi-classical orthogonal polynomials on complex contours

$$w(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right)$$

- **PAC, A Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *J. Difference Equ. Appl.*, **22** (2016) 656–675

Consider the semi-classical Airy weight

$$w(x;t) = \exp\left(-\tfrac{1}{3}x^3 + tx\right)$$

on the curve $\mathcal{C}$ from $\mathrm{e}^{2\pi\mathrm{i}/3}\infty$ to $\mathrm{e}^{-2\pi\mathrm{i}/3}\infty$. The moments are

$$\mu_0(t) = \int_{\mathcal{C}} \exp\left(-\tfrac{1}{3}x^3 + tx\right)\mathrm{d}x = \mathrm{Ai}(t)$$

$$\mu_k(t) = \int_{\mathcal{C}} x^k \exp\left(-\tfrac{1}{3}x^3 + tx\right)\mathrm{d}x = \frac{\mathrm{d}^k}{\mathrm{d}t^k}\mathrm{Ai}(t) = \mathrm{Ai}^{(k)}(t)$$

where $\mathrm{Ai}(t)$ is the **Airy function**, the Hankel determinant is

$$\Delta_n(t) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\mathrm{Ai}(t)\right]_{j,k=0}^{n-1} = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

with $\Delta_0(t) = 1$, and the recursion coefficients are

$$\alpha_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\Delta_{n+1}(t)}{\Delta_n(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n)}(t)\big)}{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)}$$

$$\beta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\Delta_n(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

with

$$\alpha_0(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\mathrm{Ai}(t) = \frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad \beta_0(t) = 0$$

The recurrence coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the discrete system

$$\begin{aligned}(\alpha_n + \alpha_{n-1})\beta_n - n &= 0\\ \alpha_n^2 + \beta_n + \beta_{n+1} - t &= 0\end{aligned} \tag{1}$$

and the differential system (Toda)

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = \beta_{n+1} - \beta_n, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n(\alpha_n - \alpha_{n-1}) \tag{2}$$

Eliminating α_{n-1} and β_{n+1} between (1) and (2) yields

$$\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} = -\alpha_n - 2\beta_n + t, \qquad \frac{\mathrm{d}\beta_n}{\mathrm{d}t} = 2\alpha_n\beta_n - n \tag{3}$$

Letting $x_n = -\beta_n$ and $y_n = -\alpha_n$ in (1), (2) and (3) yields

$$\begin{aligned}x_n + x_{n+1} &= y_n^2 - t\\ x_n(y_n + y_{n-1}) &= n\end{aligned} \tag{4}$$

$$\frac{\mathrm{d}x_n}{\mathrm{d}t} = x_n(y_{n-1} - y_n), \qquad \frac{\mathrm{d}y_n}{\mathrm{d}t} = x_{n+1} - x_n \tag{5}$$

and

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_ny_n + n \tag{6}$$

Consider the system

$$\frac{\mathrm{d}y_n}{\mathrm{d}t} = y_n^2 - 2x_n - t, \qquad \frac{\mathrm{d}x_n}{\mathrm{d}t} = -2x_n y_n + n$$

- Eliminating x_n yields

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1$$

which is equivalent to

$$\frac{\mathrm{d}^2 q}{\mathrm{d}z^2} = 2q^3 + zq + n + \tfrac{1}{2}$$

i.e. $\mathrm{P_{II}}$ with $a = n + \frac{1}{2}$.

- Eliminating y_n yields

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}$$

which is equivalent to

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = \frac{1}{2v}\left(\frac{\mathrm{d}v}{\mathrm{d}z}\right)^2 - 2v^2 - zv - \frac{n^2}{2v}$$

an equation known as $\mathrm{P_{34}}$.

The Airy solutions of the equations

$$\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = 2y_n^3 - 2ty_n - 2n - 1, \qquad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

$$\frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} = \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n}, \qquad x_0(t) = 0$$

are

$$y_n(t) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t)}{\tau_{n+1}(t)} = \frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)}{\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n)}(t)\big)}$$

$$x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big)$$

where

$$\tau_n(t) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\mathrm{Ai}(t)\right]_{j,k=0}^{n-1} = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big), \qquad n \geq 1$$

and $\tau_0(t) = 1$.

As $t \to \infty$

$$y_n(t) = t^{1/2} + \frac{2n+1}{4\,t} - \frac{12n^2+12n+5}{32\,t^{5/2}} + \frac{(2n+1)(16n^2+16n+15)}{64\,t^4} + \mathcal{O}(t^{-11/2})$$

$$x_n(t) = \frac{n}{2\,t^{1/2}} - \frac{n^2}{4\,t^2} + \frac{5n(4n^2+1)}{64\,t^{7/2}} - \frac{n^2(8n^2+7)}{16\,t^5} + \mathcal{O}(t^{-13/2})$$

An Alternative Discrete Painlevé I Equation

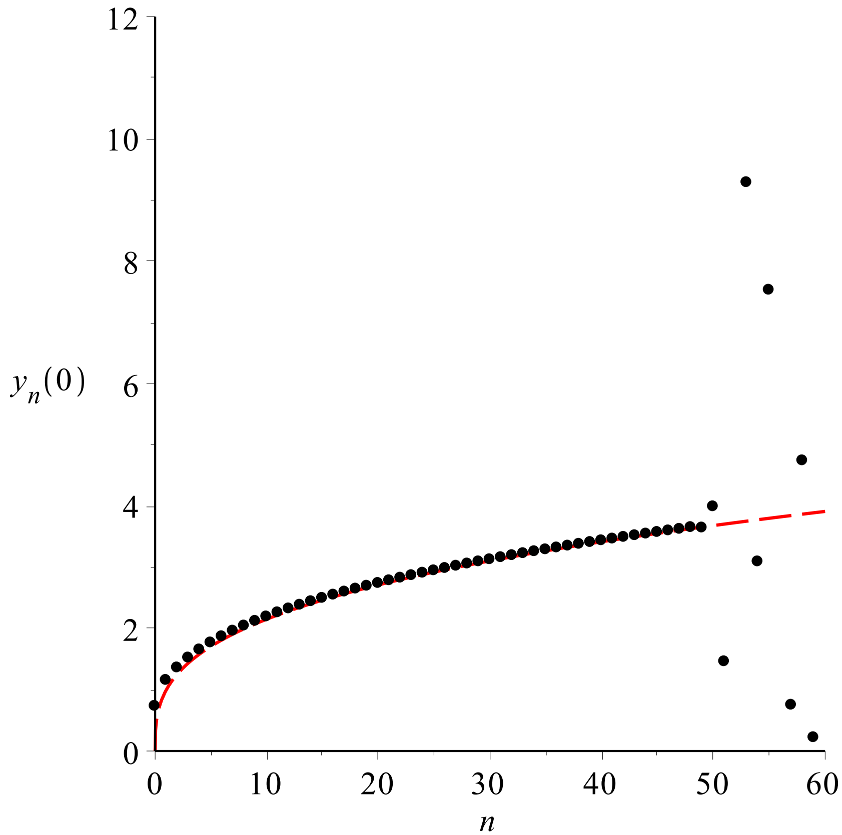
$$x_n + x_{n+1} = y_n^2 - t \qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$
$$x_n(y_n + y_{n-1}) = n$$

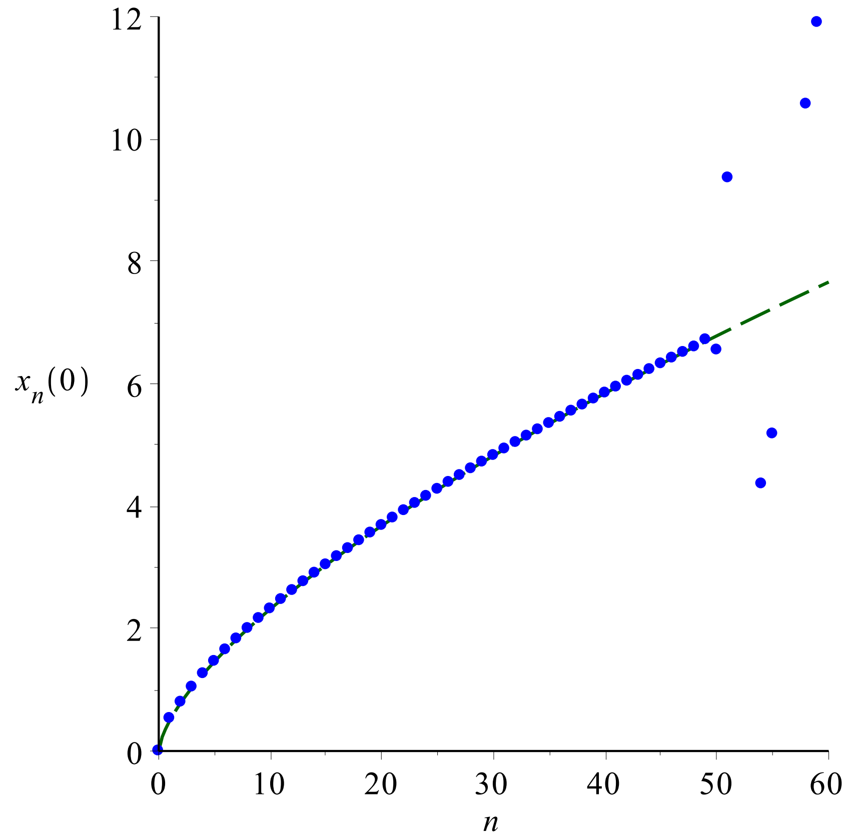
- **PAC, A Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *J. Difference Equ. Appl.*, **22** (2016) 656–675

$$x_n + x_{n+1} = y_n^2 - t$$
$$x_n(y_n + y_{n-1}) = n$$

$$x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

The system is very sensitive to the initial conditions [50 digits]

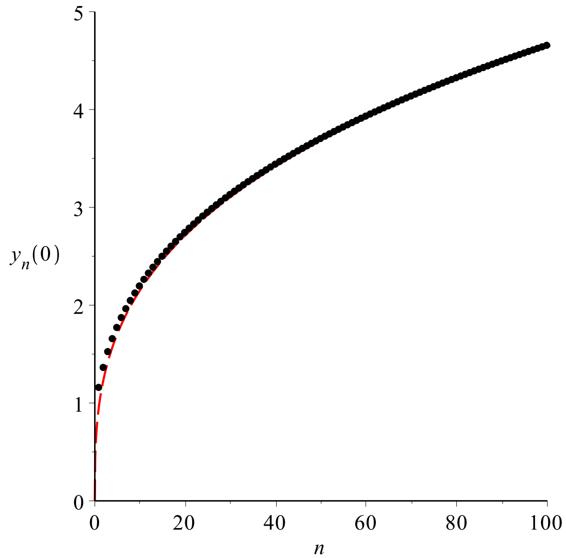




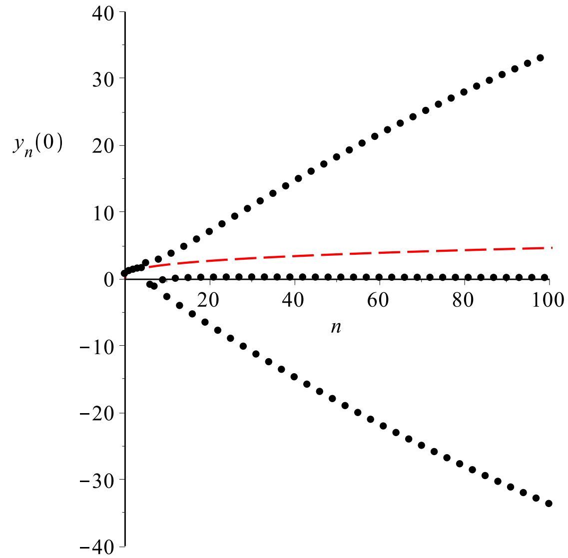
$$y_0(0) = -\frac{\mathrm{Ai}'(0)}{\mathrm{Ai}(0)} = 3^{1/3}\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}$$

$$x_0(0) = 0$$

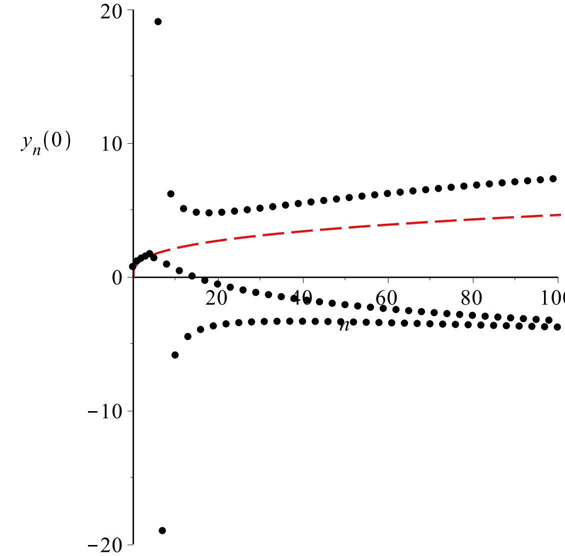
University of Kent



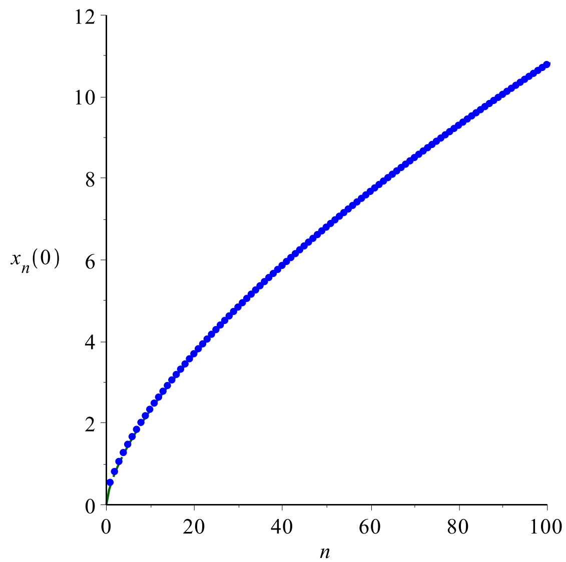
$y_0(0) = 0.7290111\ldots$



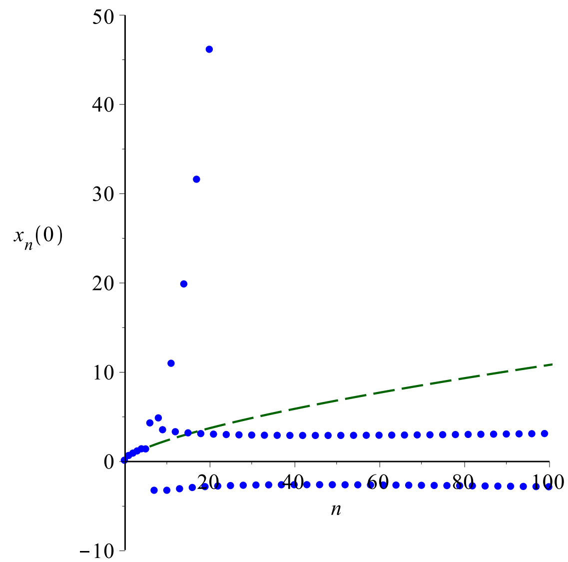
$y_0(0) = 0.729$



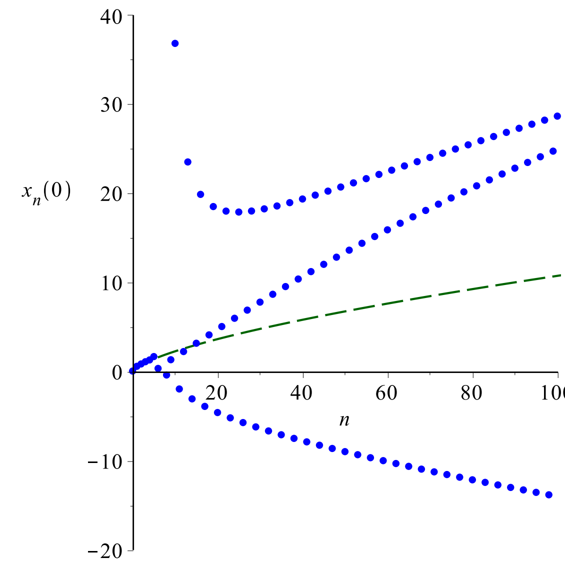
$y_0(0) = 0.72902$



$x_0(0) = 0$



$x_0(0) = 0$



$x_0(0) = 0$

$$\begin{aligned} x_n + x_{n+1} &= y_n^2 - t \\ x_n(y_n + y_{n-1}) &= n \end{aligned} \qquad x_0(t) = 0, \quad y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}$$

Solving for x_n yields

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t$$

which is known as alt-d$\mathrm{P_I}$ (Fokas, Grammaticos & Ramani [1993]).

We have seen that y_n and x_n satisfy

$$\begin{aligned} \frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} &= 2y_n^3 - 2ty_n - 2n - 1 \\ \frac{\mathrm{d}^2 x_n}{\mathrm{d}t^2} &= \frac{1}{2x_n}\left(\frac{\mathrm{d}x_n}{\mathrm{d}t}\right)^2 + 4x_n^2 + 2tx_n - \frac{n^2}{2x_n} \end{aligned}$$

which have “Airy-type” solutions

$$y_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \ln \frac{\tau_n(t)}{\tau_{n+1}(t)}, \qquad x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln \tau_n(t)$$

where

$$\tau_n(t) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}} \mathrm{Ai}(t)\right]_{j,k=0}^{n-1} = \mathcal{W}\big(\mathrm{Ai}(t), \mathrm{Ai}'(t), \ldots, \mathrm{Ai}^{(n-1)}(t)\big), \quad n \geq 1$$

and $\tau_0(t) = 1$.

Theorem (**PAC, Loureiro & Van Assche [2016]**)

For positive values of t, there exists a unique solution of

$$x_n + x_{n+1} = y_n^2 - t$$
$$x_n(y_n + y_{n-1}) = n$$

with $x_0(t) = 0$ for which $x_{n+1}(t) > 0$ and $y_n(t) > 0$ for all $n \geq 0$. This solution corresponds to the initial value

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}.$$

Theorem (**PAC, Loureiro & Van Assche [2016]**)

For positive values of t, there exists a unique solution of

$$\frac{n+1}{y_n + y_{n+1}} + \frac{n}{y_n + y_{n-1}} = y_n^2 - t$$

for which $y_n(t) \geq 0$ for all $n \geq 0$. This solution corresponds to the initial values

$$y_0(t) = -\frac{\mathrm{Ai}'(t)}{\mathrm{Ai}(t)}, \qquad y_1(t) = -y_0(t) + \frac{1}{y_0^2(t) - t}$$

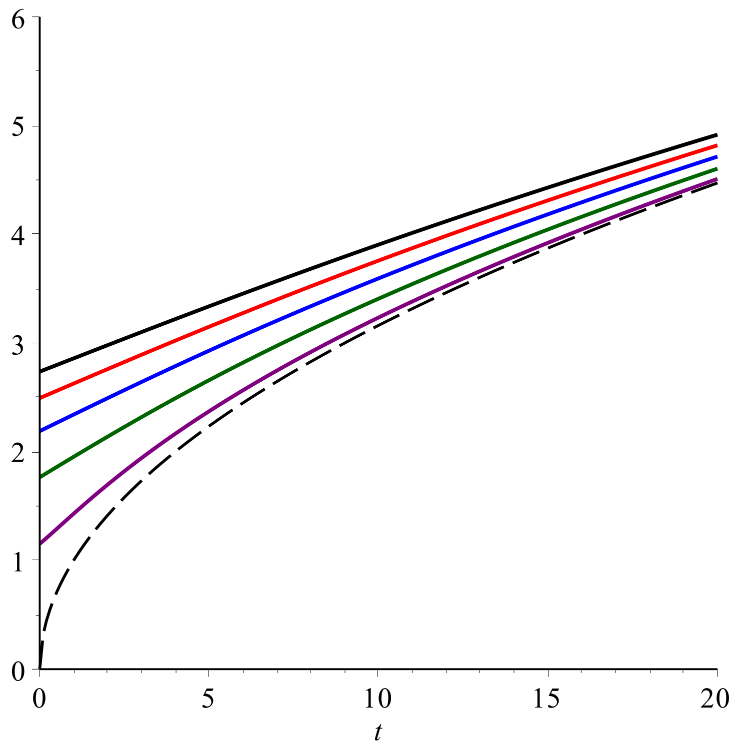
Lemma *If* $0 < t_1 < t_2$ *then*

$$y_n(t_1) < y_n(t_2), \qquad x_n(t_1) > x_n(t_2)$$

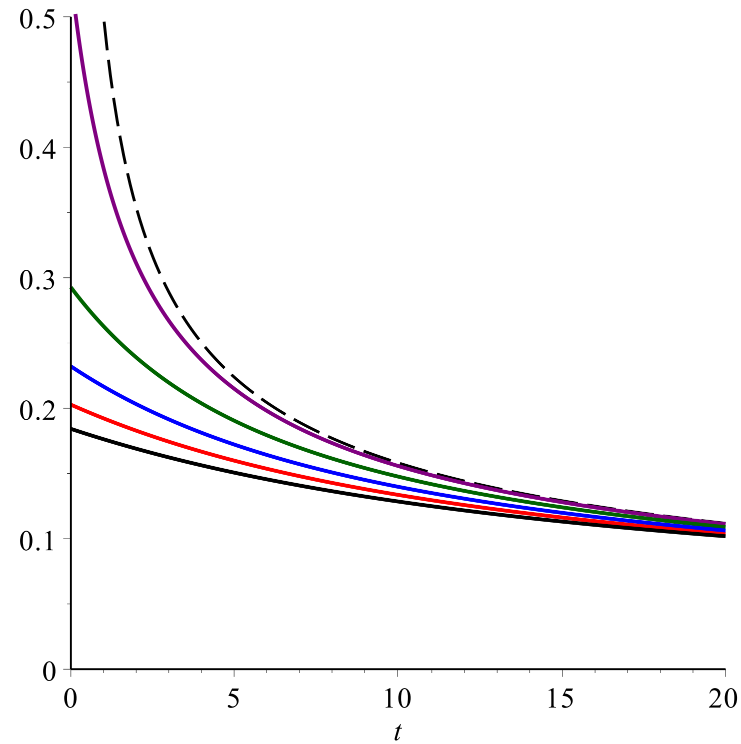
i.e. $y_n(t)$ *is monotonically increasing and* $x_n(t)$ *is monotonically decreasing.*

Lemma *For fixed* t *with* $t > 0$ *then*

$$\sqrt{t} < y_n(t) < y_{n+1}(t), \qquad \frac{1}{2\sqrt{t}} > \frac{x_n(t)}{n} > \frac{x_{n+1}(t)}{n+1}$$



$y_n(t), \quad n = 1, 5, 10, 15, 20$

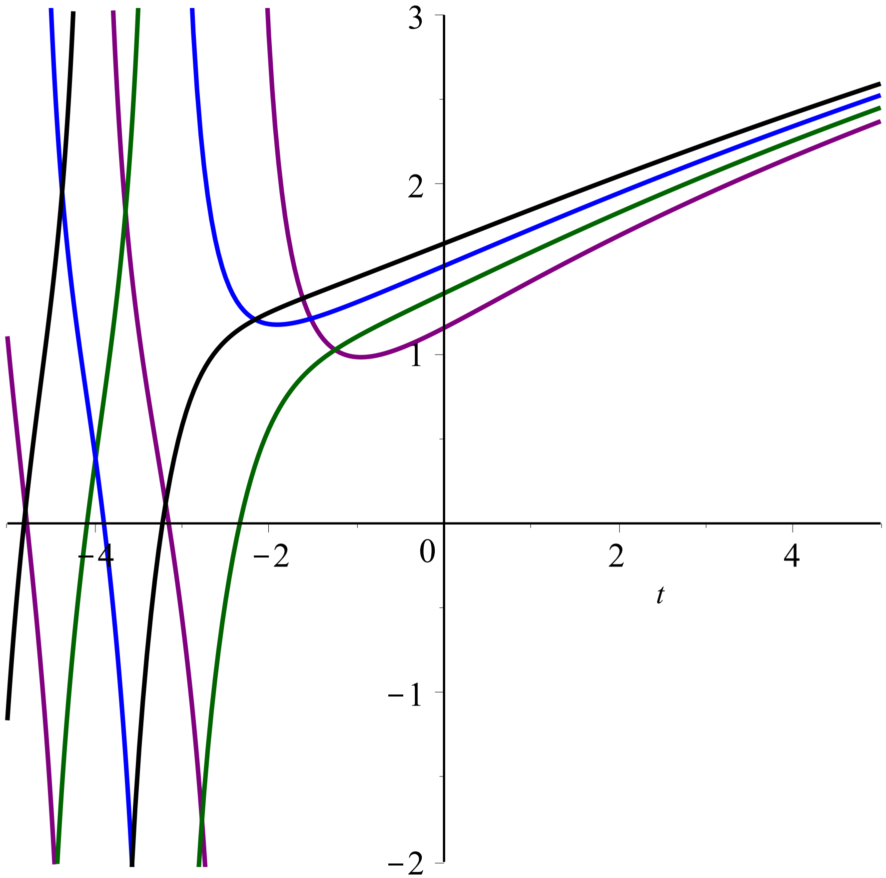


$\frac{1}{n}x_n(t), \quad n = 1, 5, 10, 15, 20$

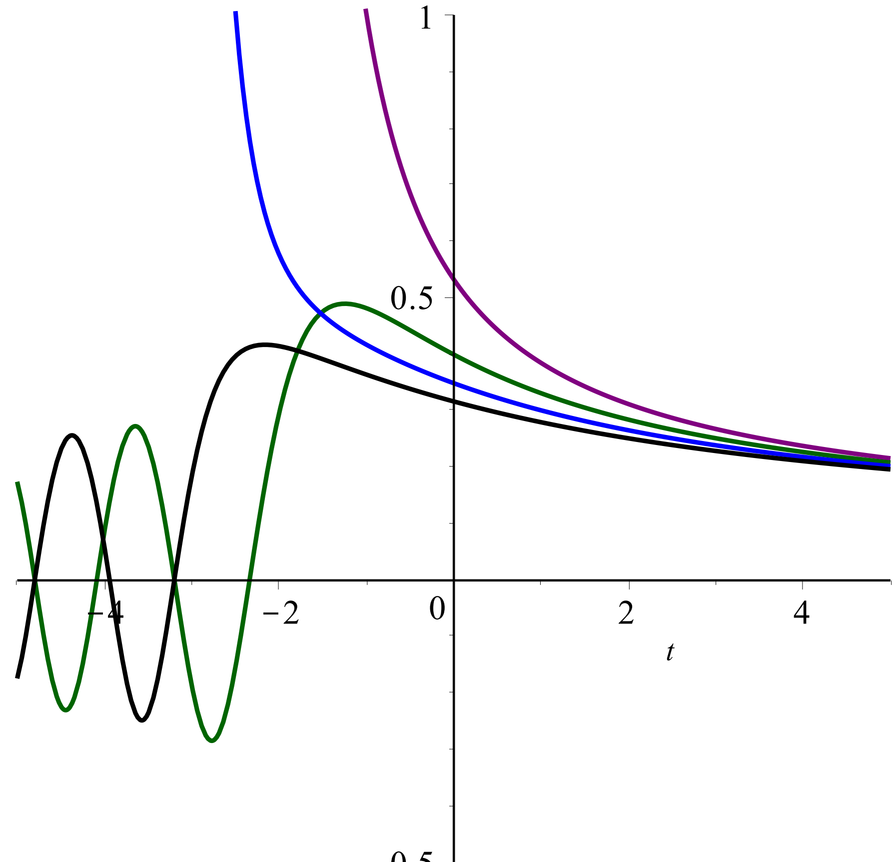
University of Kent

Question: What happens if we don't require that $t > 0$?

$$y_n(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\frac{\tau_n(t)}{\tau_{n+1}(t)}, \qquad x_n(t) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\tau_n(t), \qquad \tau_n(t) = \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}t^{j+k}}\,\mathrm{Ai}(t)\right]_{j,k=0}^{n-1}$$



$y_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

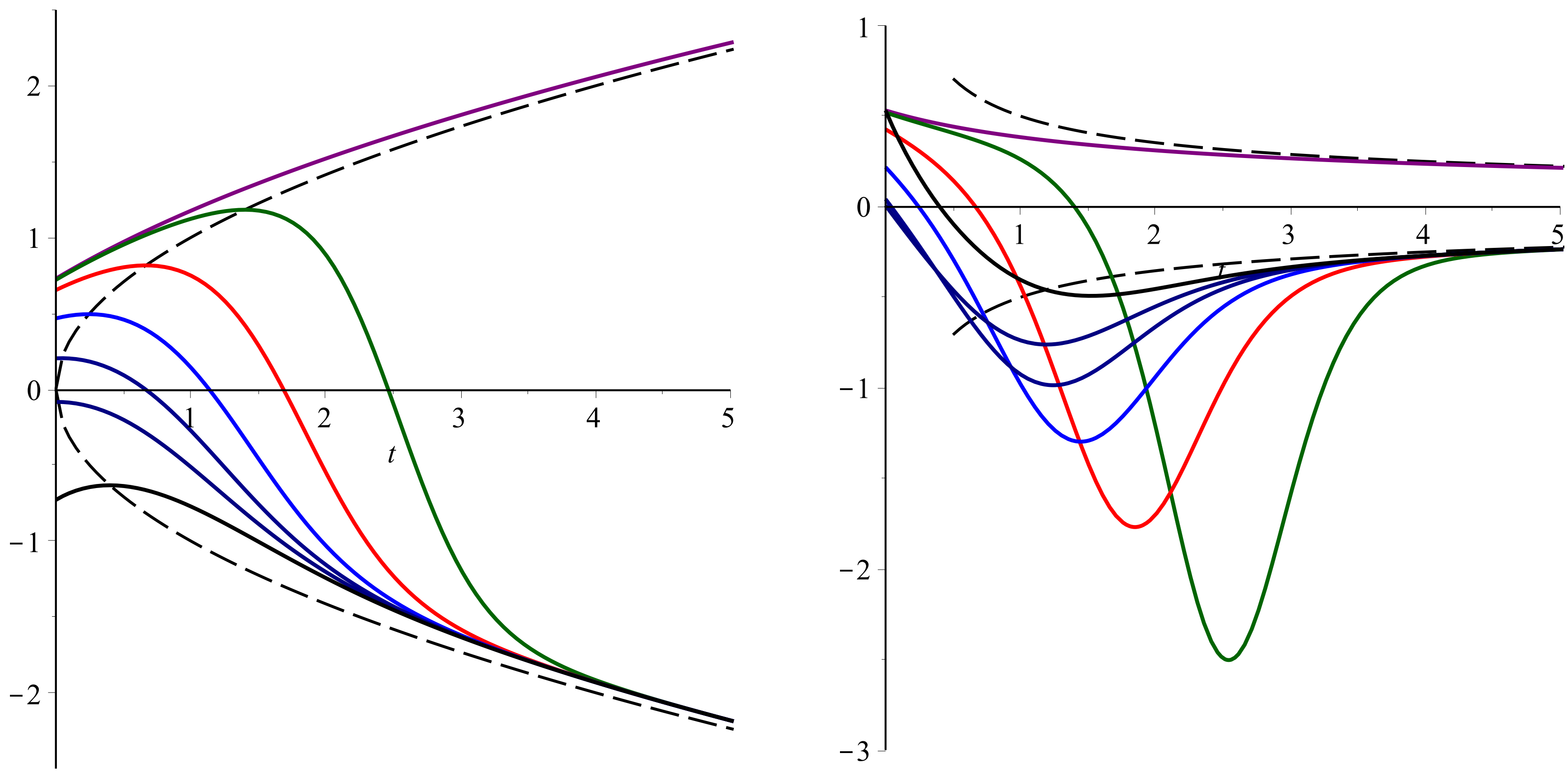


$\frac{1}{n}x_n(t), \quad n = \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}$

Question: What happens if we have a linear combination of $\mathrm{Ai}(t)$ and $\mathrm{Bi}(t)$?

$$y_0(t;\vartheta) = -\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi(t;\vartheta), \qquad x_1(t;\vartheta) = -\frac{\mathrm{d}^2}{\mathrm{d}t^2}\ln\varphi(t;\vartheta)$$

$$\varphi(t;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(t) + \sin(\vartheta)\,\mathrm{Bi}(t)$$



$y_0(t;\vartheta)$ $\qquad$ $x_1(t;\vartheta)$

$$\vartheta = \mathbf{0},\ \tfrac{1}{1000}\boldsymbol{\pi},\ \tfrac{1}{100}\boldsymbol{\pi},\ \tfrac{1}{25}\boldsymbol{\pi},\ \tfrac{1}{10}\boldsymbol{\pi},\ \tfrac{1}{5}\boldsymbol{\pi},\ \tfrac{1}{2}\boldsymbol{\pi}$$

University of Kent

Airy Solutions

$$\frac{\mathrm{d}^2q}{\mathrm{d}z^2} = 2q^3 + zq + \alpha \qquad \mathrm{P_{II}}$$

$$p\frac{\mathrm{d}^2p}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p}{\mathrm{d}z}\right)^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2 \qquad \mathrm{S_{II}}$$

- **PAC**, "On Airy Solutions of the Second Painlevé Equation", *Stud. Appl. Math.*, **137** (2016) 93–109

Airy Solutions of $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n + \tfrac{1}{2} \qquad \mathrm{P_{II}}$$

$$p_n \frac{\mathrm{d}^2 p_n}{\mathrm{d}z^2} = \frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2 + 2p_n^3 - zp_n^2 - \tfrac{1}{2}n^2 \qquad \mathrm{P_{34}}$$

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2 \qquad \mathrm{S_{II}}$$

Theorem

Let

$$\varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(\zeta) + \sin(\vartheta)\,\mathrm{Bi}(\zeta), \qquad \zeta = -2^{-1/3}z$$

with ϑ an arbitrary constant, $\mathrm{Ai}(\zeta)$ and $\mathrm{Bi}(\zeta)$ Airy functions, and $\tau_n(z)$ be the Wronskian

$$\tau_n(z;\vartheta) = \det\left[\frac{\mathrm{d}^{j+k}\varphi}{\mathrm{d}z^{j+k}}\right]_{j,k=0}^{n-1} = \mathcal{W}\left(\varphi, \frac{\mathrm{d}\varphi}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi}{\mathrm{d}z^{n-1}}\right)$$

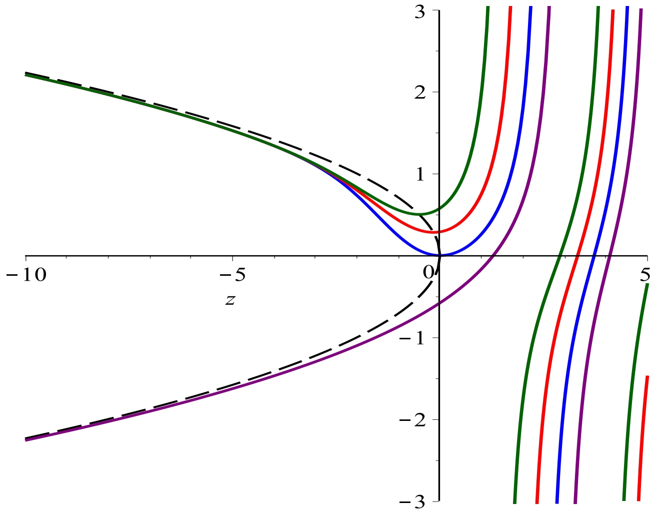
then

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}, \quad p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta), \quad \sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

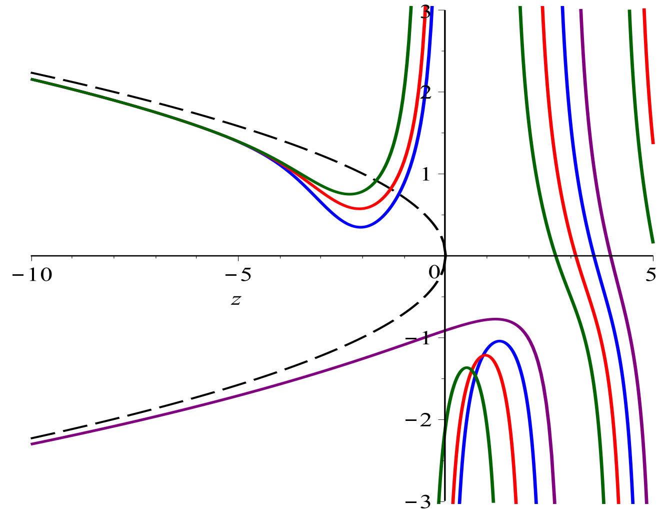
respectively satisfy $\mathrm{P_{II}}$, $\mathrm{P_{34}}$ and $\mathrm{S_{II}}$, with $n \in \mathbb{Z}$.

Airy Solutions of $\mathrm{P_{II}}$

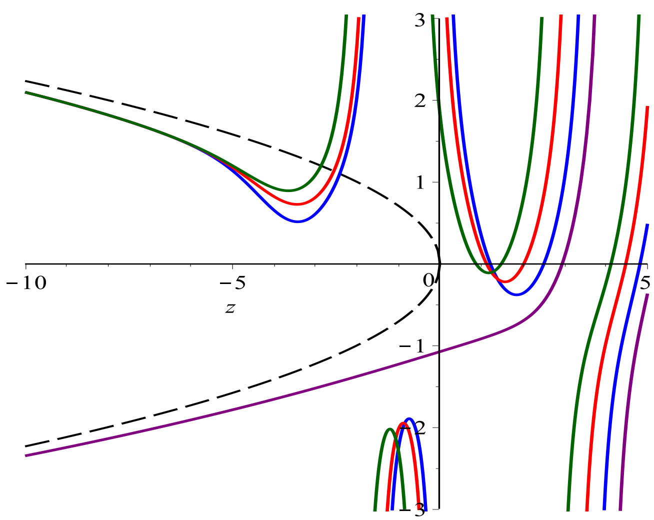
$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_n(z;\vartheta)}{\tau_{n+1}(z;\vartheta)}$$



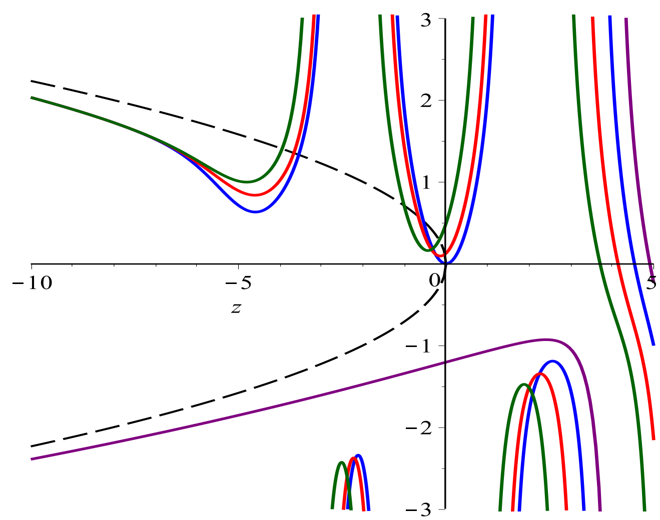
$\boldsymbol{n=0}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=1}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n=2}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



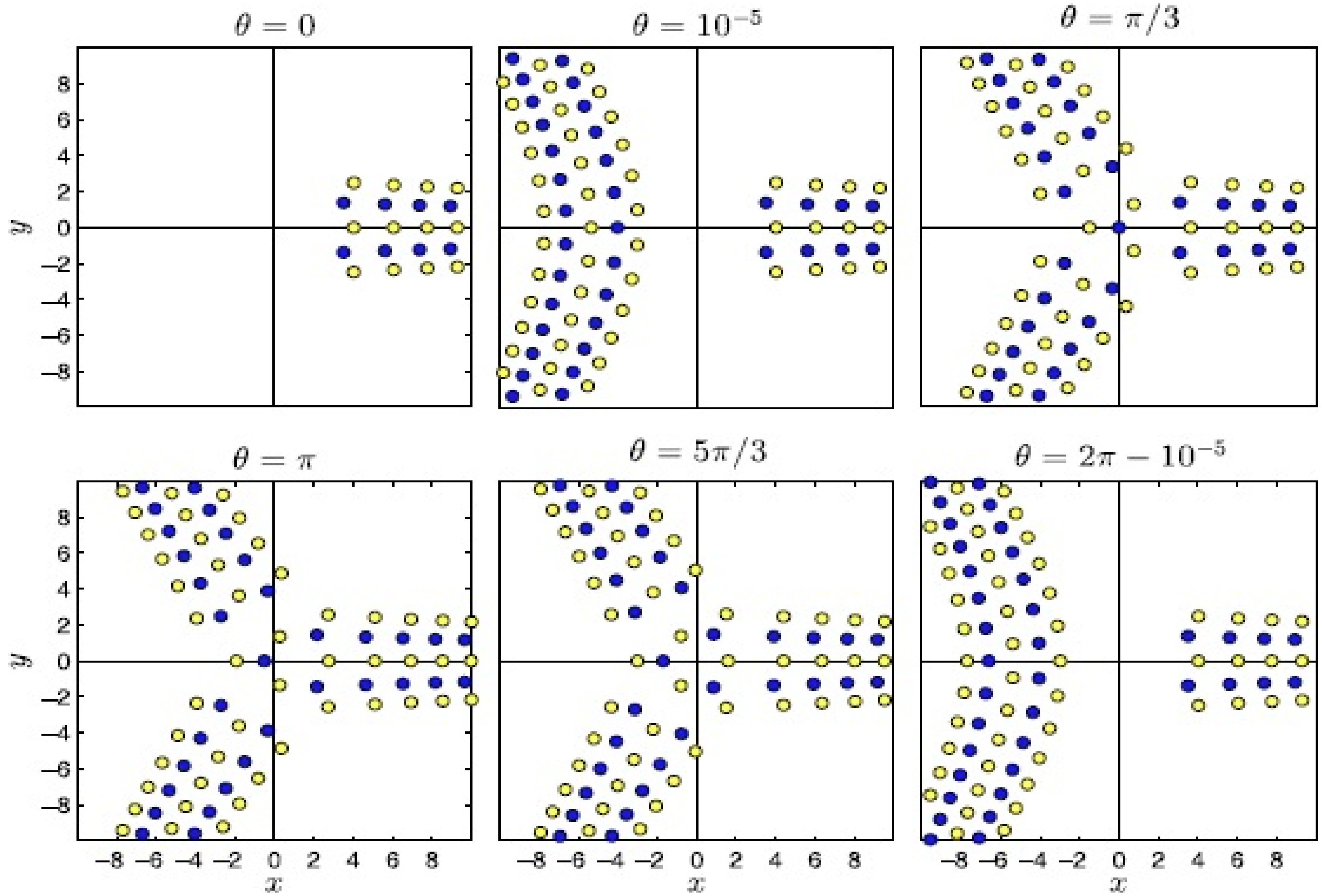
$\boldsymbol{n=3}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

University of Kent

Airy Solutions of $\mathrm{P_{II}}$ with $\alpha = \frac{5}{2}$

(Fornberg & Weideman [2014])

$$q_2(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\mathcal{W}(\varphi,\varphi')}{\mathcal{W}(\varphi,\varphi',\varphi'')}, \qquad \varphi(z;\vartheta) = \cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}z) + \sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}z)$$

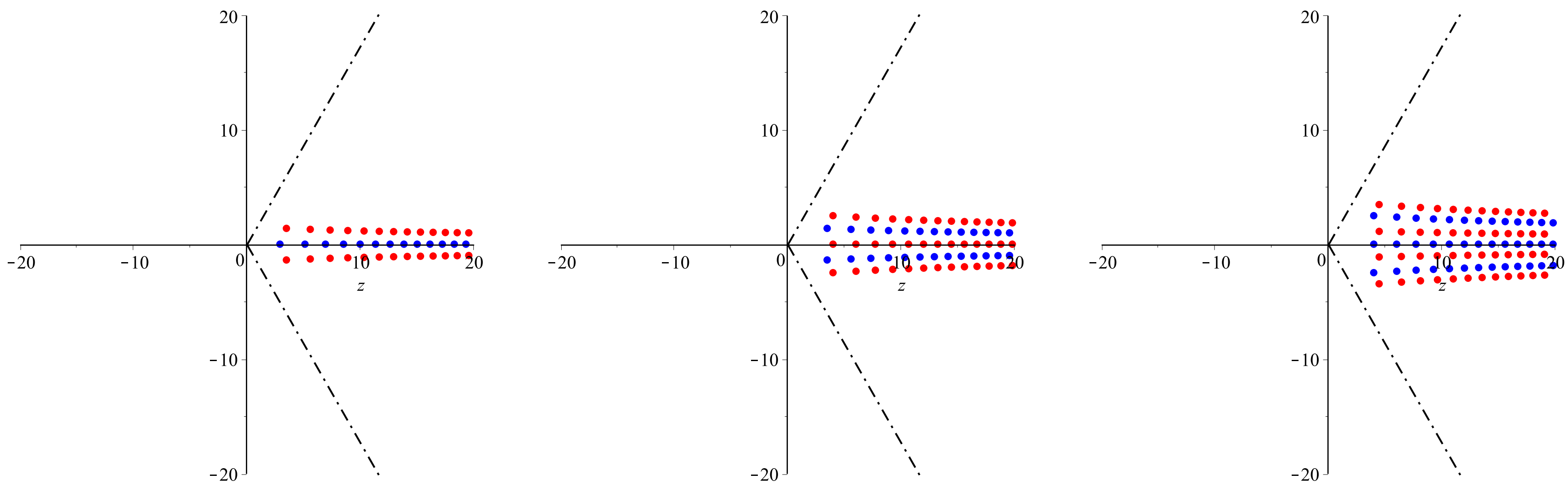


blue/**yellow** denote poles with residue $+1/-1$

Tronquée Solutions of $\mathrm{P_{II}}$ (Airy with $\vartheta = 0$)

$$q_n(z;0) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \frac{\tau_n(z;0)}{\tau_{n+1}(z;0)}, \qquad \tau_n(z;0) = \mathcal{W}\left(\varphi_0, \frac{\mathrm{d}\varphi_0}{\mathrm{d}z}, \ldots, \frac{\mathrm{d}^{n-1}\varphi_0}{\mathrm{d}z^{n-1}}\right)$$

with $\varphi_0 = \varphi(z;0) = \mathrm{Ai}(-2^{-1/3}z)$



$n = 2$ $\qquad$ $n = 3$ $\qquad$ $n = 4$

Plots of the poles of $q_n(z;0)$ for $n = 2, 3, 4$; the **blue** and **red** circles represent poles with residues $+1$ and -1, respectively.

Airy Solutions of $\mathrm{P_{II}}$

$$\frac{\mathrm{d}^2 q_n}{\mathrm{d}z^2} = 2q_n^3 + zq_n + n + \tfrac{1}{2}$$

Theorem

(**PAC [2016]**)

Let $q_n(z;\vartheta)$ *be defined by*

$$q_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\frac{\tau_{n-1}(z;\vartheta)}{\tau_n(z;\vartheta)}, \qquad n \geq 1$$

with

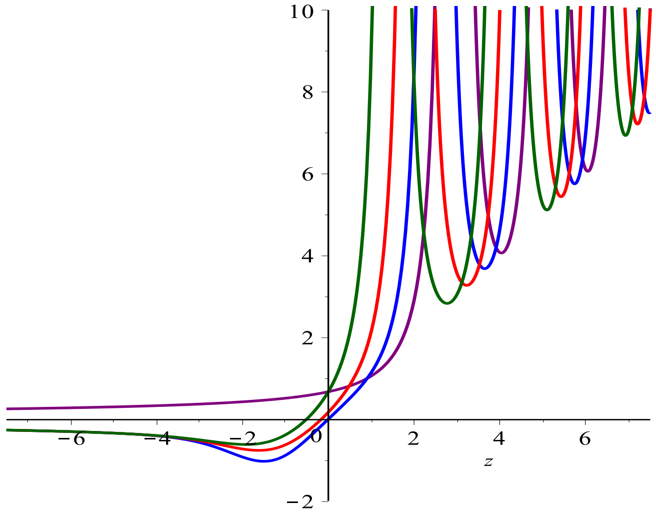
$$\tau_n(z;\vartheta) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}\left\{\cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}\,z) + \sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}\,z)\right\}\right]_{j,k=0}^{n-1}$$

then as $z \to -\infty$

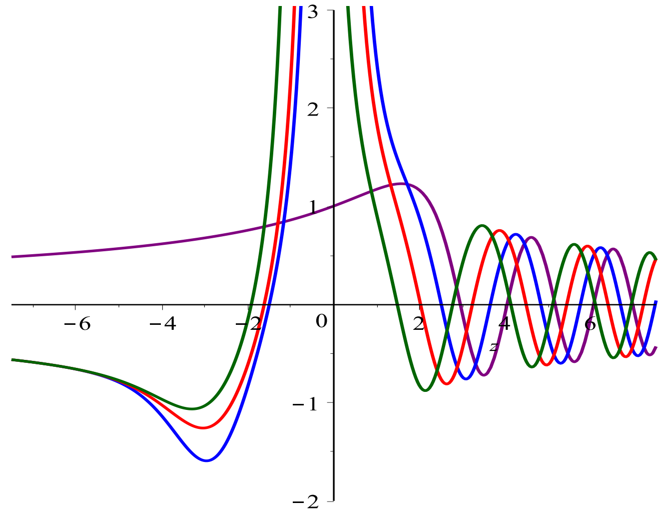
$$q_n(z;\vartheta) = \begin{cases} -\dfrac{(-z)^{1/2}}{\sqrt{2}} + \dfrac{2n-1}{4z} + \dfrac{12n^2-12n+5}{16\sqrt{2}\,(-z)^{5/2}} + \mathcal{O}\left(z^{-4}\right), & \textit{if } \vartheta = 0 \\ \dfrac{(-z)^{1/2}}{\sqrt{2}} + \dfrac{2n-1}{4z} - \dfrac{12n^2-12n+5}{16\sqrt{2}\,(-z)^{5/2}} + \mathcal{O}\left(z^{-4}\right), & \textit{if } \theta \neq 0 \end{cases}$$

Airy Solutions of P_{34}

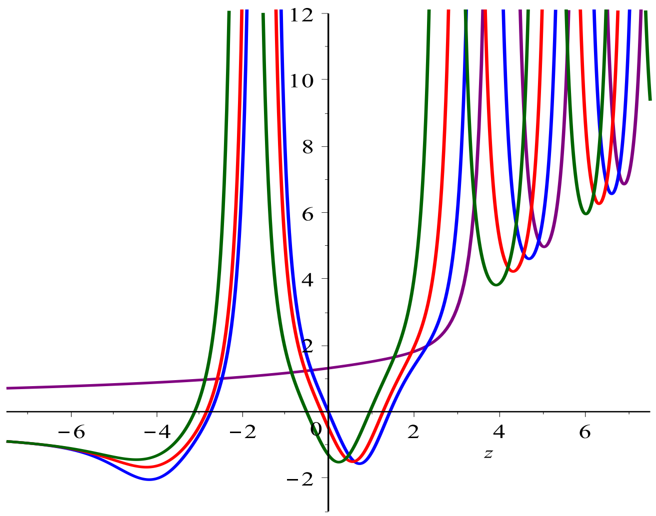
$$p_n(z;\vartheta) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;\vartheta)$$



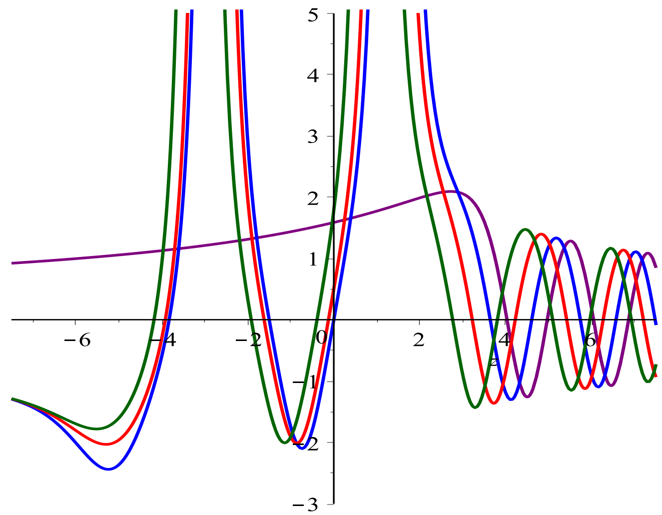
$\boldsymbol{n=1},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



$\boldsymbol{n=2},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



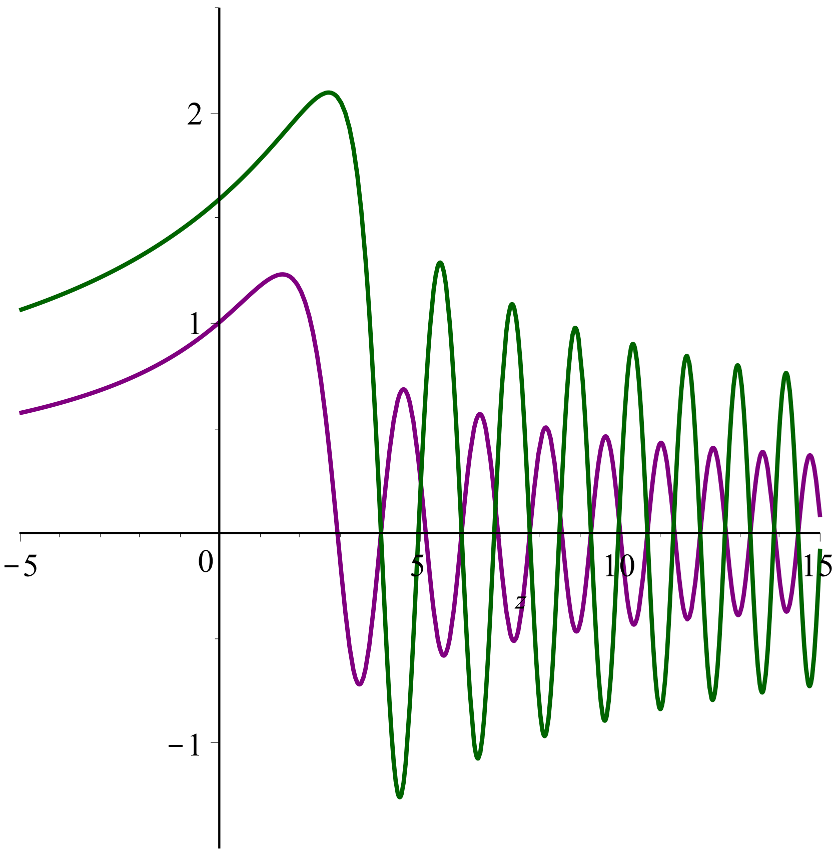
$\boldsymbol{n=3},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$



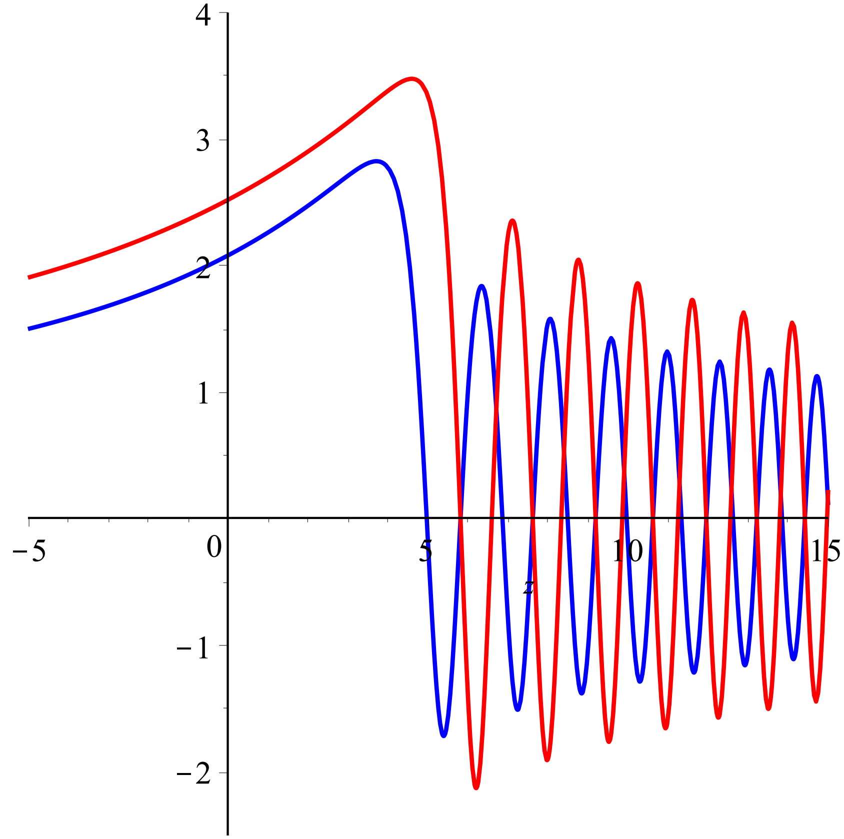
$\boldsymbol{n=4},\quad \vartheta = 0, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi, \pi$

Airy Solutions of P_{34}

$$p_n(z;0) = -2\frac{\mathrm{d}^2}{\mathrm{d}z^2}\ln\tau_n(z;0)$$



$n=2$, $n=4$



$n=6$, $n=8$

University of Kent

Airy Solutions of P_{34}

$$p_n\frac{\mathrm{d}^2p_n}{\mathrm{d}z^2}=\frac{1}{2}\left(\frac{\mathrm{d}p_n}{\mathrm{d}z}\right)^2+2p_n^3-zp_n^2-\tfrac{1}{2}n^2$$

Theorem

(PAC [2016])

Let $p_n(z;\vartheta)$ be defined by

$$p_n(z;\vartheta)=-2\frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

with

$$\tau_n(z;\vartheta)=\det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}\left\{\cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}\,z)+\sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}\,z)\right\}\right]_{j,k=0}^{n-1}$$

then as $z\to-\infty$

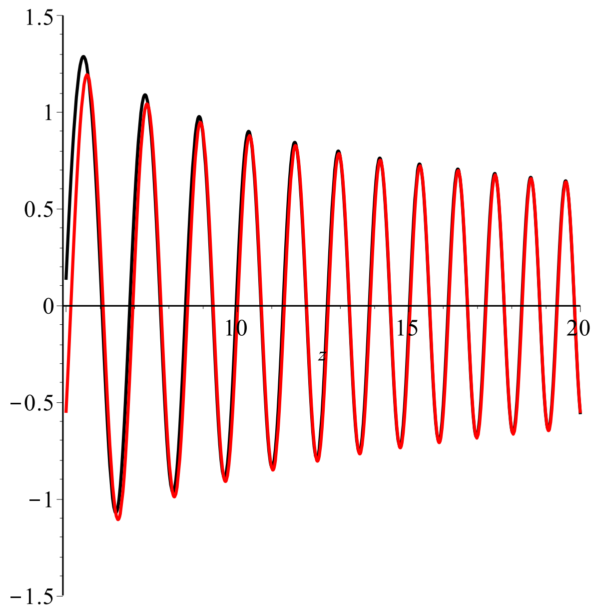
$$p_n(z;\vartheta)=\begin{cases}\dfrac{n}{\sqrt{2}\,(-z)^{1/2}}-\dfrac{n^2}{2z^2}+\dfrac{5n(4n^2+1)}{16\sqrt{2}\,(-z)^{7/2}}+\mathcal{O}\left(z^{-5}\right), & \textit{if}\quad \vartheta=0\\ -\dfrac{n}{\sqrt{2}\,(-z)^{1/2}}-\dfrac{n^2}{2z^2}-\dfrac{5n(4n^2+1)}{16\sqrt{2}\,(-z)^{7/2}}+\mathcal{O}\left(z^{-5}\right), & \textit{if}\quad \theta\neq 0\end{cases}$$

and as $z\to\infty$

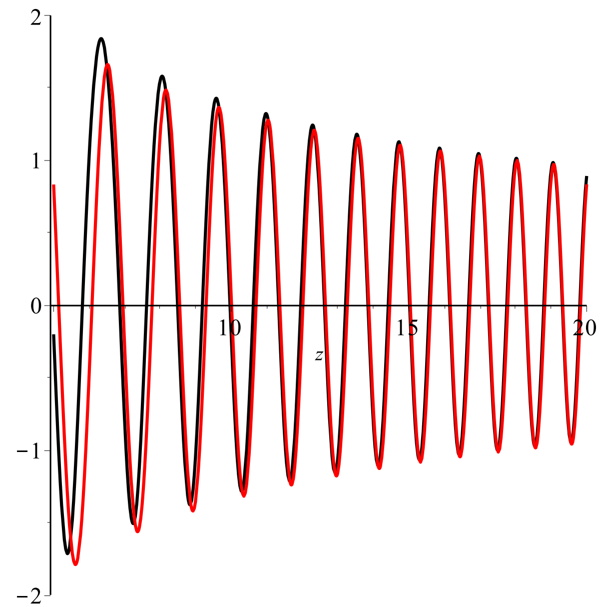
$$p_{2n}(z;0)=n\sqrt{2}\,z^{-1/2}\cos\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2}-n\pi\right)+o\left(z^{-1/2}\right)$$

University of Kent

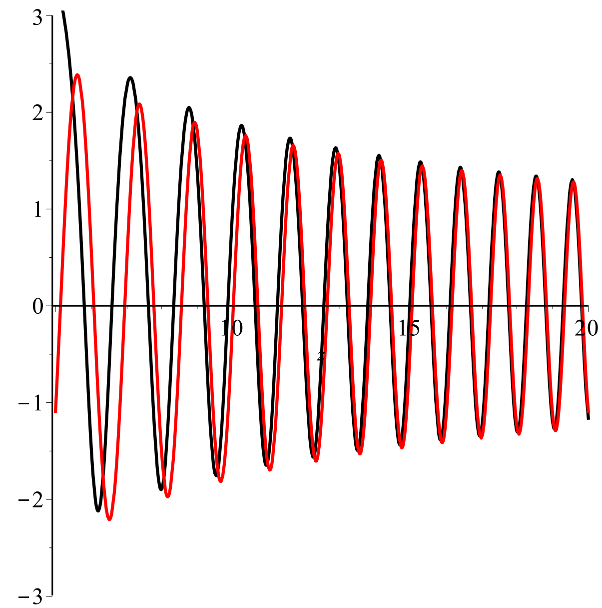
$$p_n(z;0), \qquad \frac{n}{\sqrt{2z}}\cos\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2} - \tfrac{1}{2}n\pi\right)$$



$n = 4$



$n = 6$



$n = 8$

University of Kent

Its, Kuijlaars & Östensson [2008, 2009] discuss solutions of

$$u_\beta \frac{\mathrm{d}^2 u_\beta}{\mathrm{d}t^2} = \tfrac{1}{2}\left(\frac{\mathrm{d}u_\beta}{\mathrm{d}t}\right)^2 + 4u_\beta^3 + 2tu_\beta^2 - 2\beta^2 \tag{1}$$

with β a constant, which is equivalent to P_{34} through the transformation

$$p(z) = 2^{1/3} u_\beta(t), \qquad t = -2^{-1/3} z, \qquad \beta = \tfrac{1}{2} n$$

in a study of the double scaling limit of unitary random matrix ensembles.

Theorem (**Its, Kuijlaars & Östensson [2009]**)

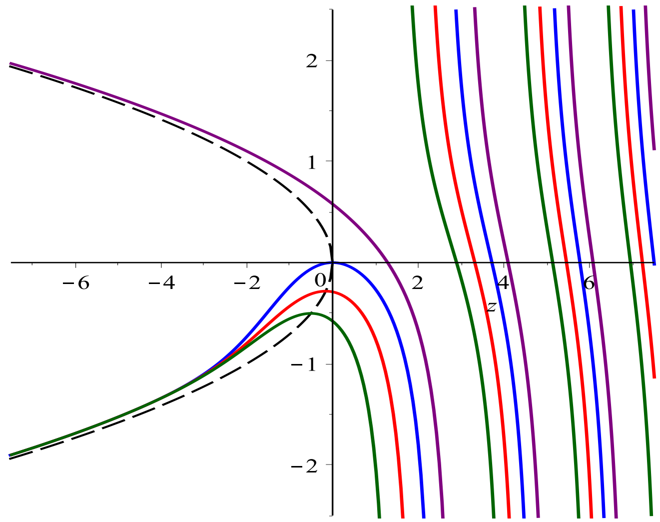
There are solutions $u_\beta(t)$ of (1) such that as $t \to \infty$

$$u_\beta(t) = \begin{cases} \beta t^{-1/2} + \mathcal{O}\left(t^{-2}\right), & \text{as} \quad t \to \infty \\ \beta(-t)^{-1/2} \cos\left\{\tfrac{4}{3}(-t)^{3/2} - \beta\pi\right\} + \mathcal{O}\left(t^{-2}\right), & \text{as} \quad t \to -\infty \end{cases} \tag{2}$$

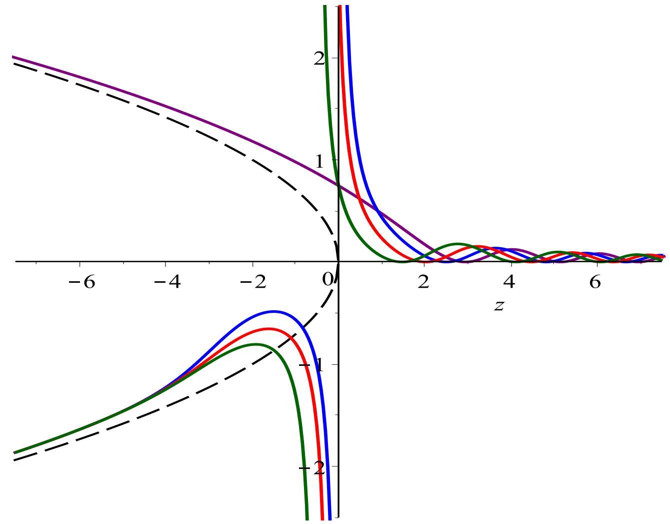
- **Its, Kuijlaars & Östensson [2008]** primarily study $u_1(t)$, i.e. when $n = 2$.
- Letting $\beta = n$ in (2) shows that they are in agreement with the asymptotic expansions for $p_{2n}(z; 0)$.
- **Its, Kuijlaars & Östensson [2009]** conclude that solutions of (1) with asymptotic behaviour (2) are **tronquée solutions**, i.e. have no poles in a sector of the complex plane.

Airy Solutions of S_{II}

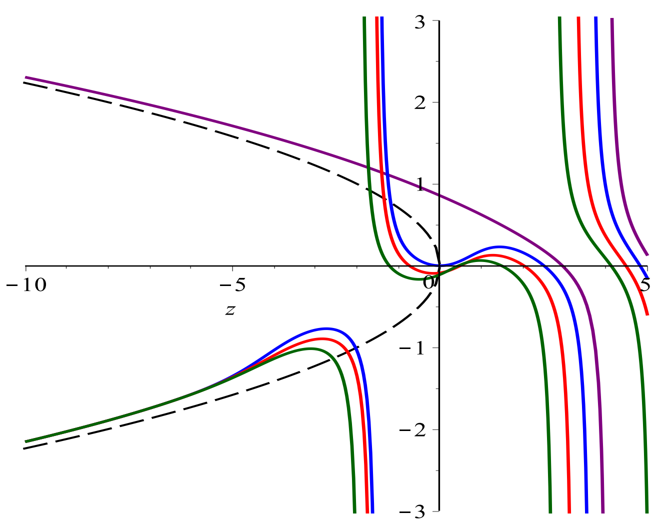
$$\sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$



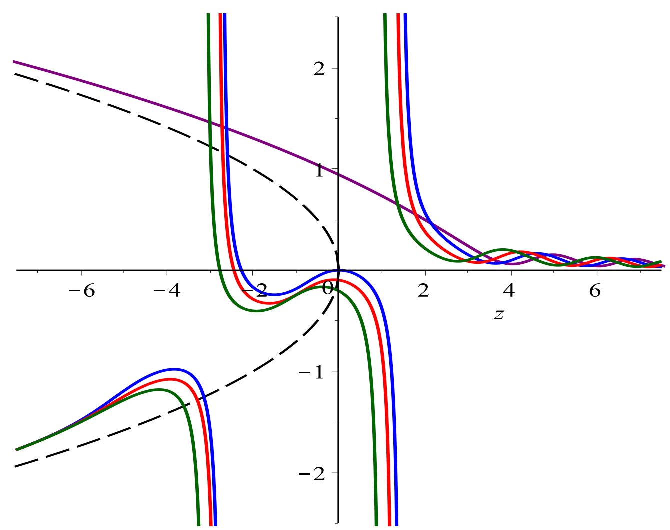
$\boldsymbol{n = 1}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n = 2}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



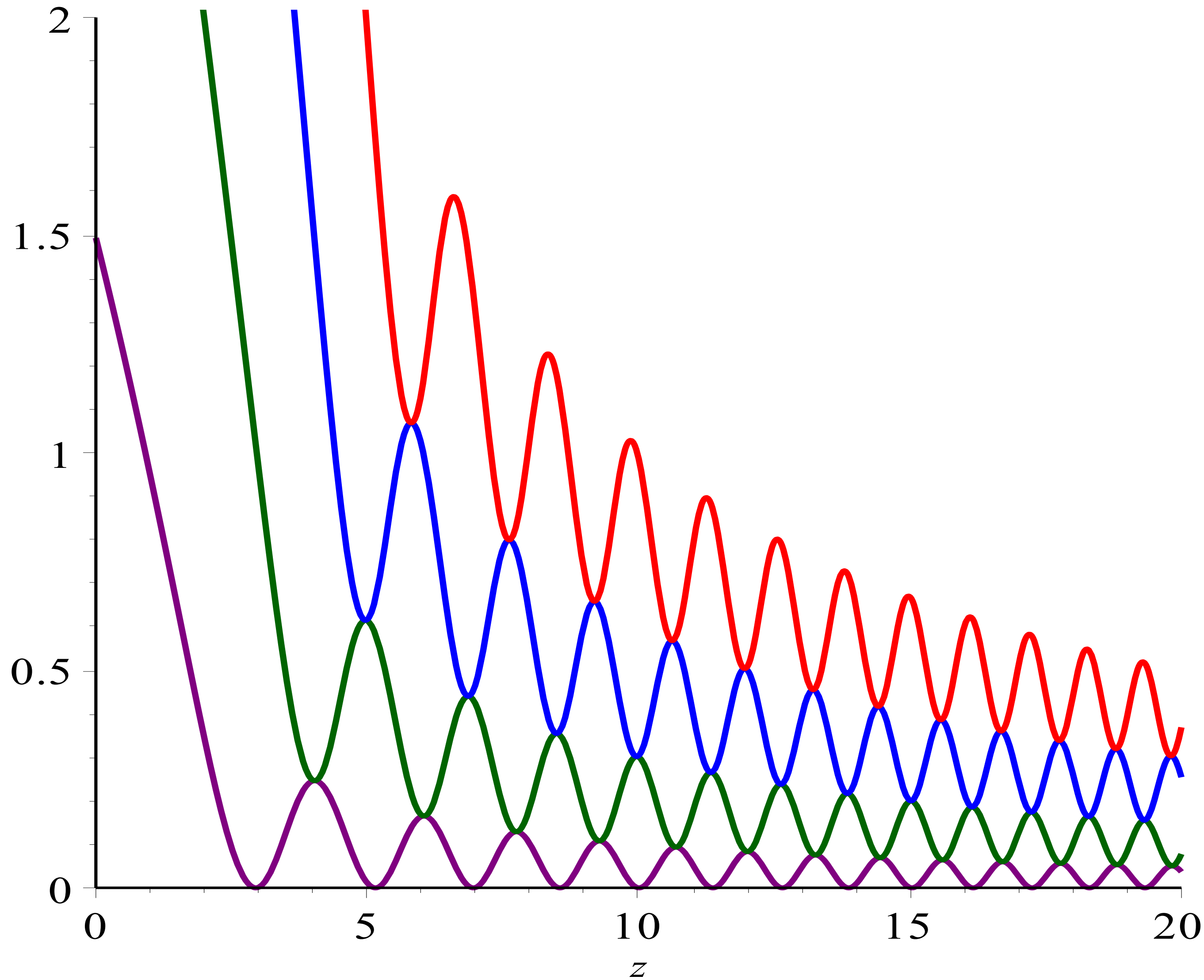
$\boldsymbol{n = 3}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$



$\boldsymbol{n = 4}$, $\vartheta = 0, \frac{1}{3}\pi, \frac{2}{3}\pi, \pi$

Airy Solutions of $\mathrm{S_{II}}$

$$\sigma_n(z;0) = \frac{\mathrm{d}}{\mathrm{d}z} \ln \det \left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}} \mathrm{Ai}(-2^{-1/3}z) \right]_{j,k=0}^{n-1}$$



$n = 2$, $n = 4$, $n = 6$, $n = 8$

University of Kent

Airy Solutions of S_{II}

$$\left(\frac{\mathrm{d}^2\sigma_n}{\mathrm{d}z^2}\right)^2 + 4\left(\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\right)^3 + 2\frac{\mathrm{d}\sigma_n}{\mathrm{d}z}\left(z\frac{\mathrm{d}\sigma_n}{\mathrm{d}z} - \sigma\right) = \tfrac{1}{4}n^2$$

Theorem **(PAC [2016])**

Let $\sigma_n(z;\vartheta)$ *be defined by*

$$\sigma_n(z;\vartheta) = \frac{\mathrm{d}}{\mathrm{d}z}\ln\tau_n(z;\vartheta)$$

with

$$\tau_n(z;\vartheta) = \det\left[\frac{\mathrm{d}^{j+k}}{\mathrm{d}z^{j+k}}\left\{\cos(\vartheta)\,\mathrm{Ai}(-2^{-1/3}\,z) + \sin(\vartheta)\,\mathrm{Bi}(-2^{-1/3}\,z)\right\}\right]_{j,k=0}^{n-1}$$

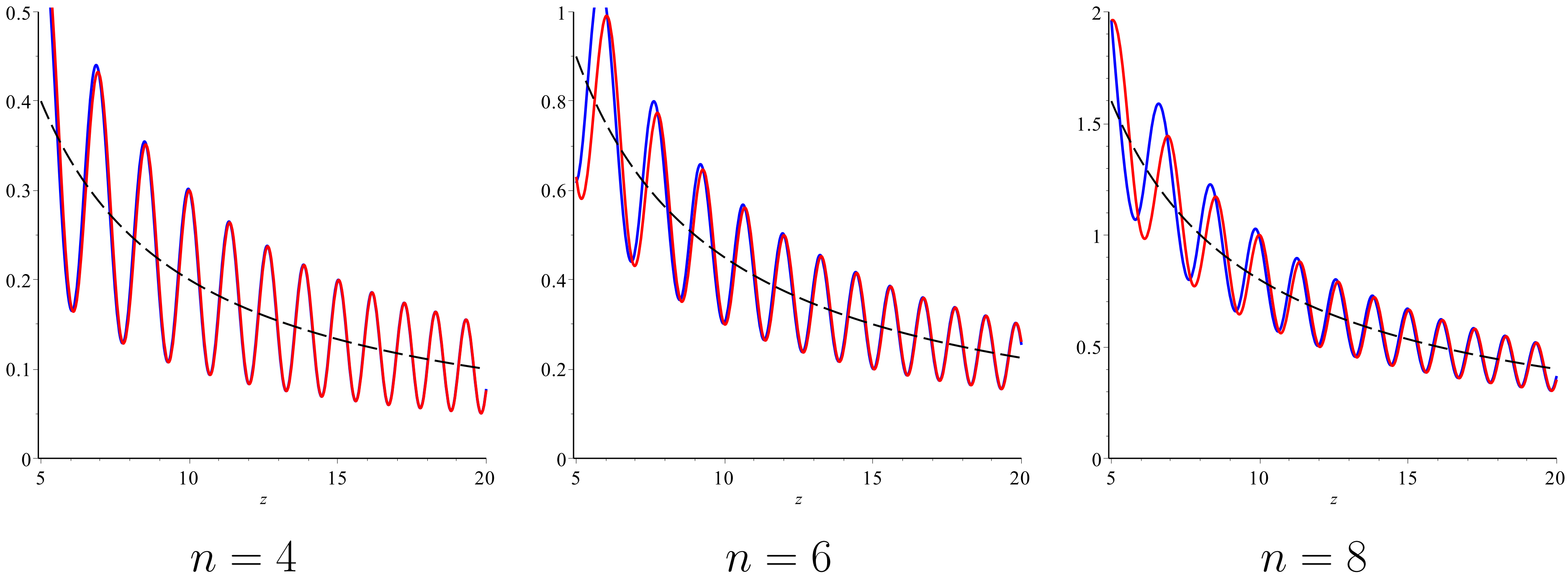
then as $z \to -\infty$

$$\sigma_n(z;\vartheta) = \begin{cases} \dfrac{n(-z)^{1/2}}{\sqrt{2}} - \dfrac{n^2}{4z} - \dfrac{n(4n^2+1)}{16\sqrt{2}\,(-z)^{5/2}} + \mathcal{O}\left(z^{-4}\right), & \textit{if} \quad \vartheta = 0 \\ -\dfrac{n(-z)^{1/2}}{\sqrt{2}} - \dfrac{n^2}{4z} + \dfrac{n(4n^2+1)}{16\sqrt{2}\,(-z)^{5/2}} + \mathcal{O}\left(z^{-4}\right), & \textit{if} \quad \theta \neq 0 \end{cases}$$

and as $z \to \infty$

$$\sigma_{2n}(z;0) = \frac{n}{2z}\left\{n - \sin\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2} - n\pi\right)\right\} + o\left(z^{-1}\right)$$

$$\sigma_n(z;0), \qquad \frac{n^2}{8z} - \frac{n}{4z}\sin\left(\tfrac{4}{3}\sqrt{2}\,z^{3/2} - \tfrac{1}{2}n\pi\right), \qquad \frac{n^2}{8z}$$



Conclusions

- The coefficients in the three-term recurrence relations associated with semi-classical generalizations of orthogonal polynomials can often be expressed in terms of special functions solutions of the Painlevé equations.
- These recursion coefficients can be expressed as Hankel determinants which arise in the solution of the Painlevé equations, in particular the Painlevé σ-equations, the second-order, second-degree equations associated with the Hamiltonian representation of the Painlevé equations.
- These Hankel determinants arise in the special cases of the Painlevé equations when they have solutions in terms of the classical special functions, the "classical solutions" of the Painlevé equations.
- The moments of the semi-classical weights provide the link between the orthogonal polynomials and the associated Painlevé equation.
- These ideas can be extended to orthogonal polynomials in other contexts:
  - **discrete orthogonal polynomials** (**PAC [2013]**);
  - **orthogonal polynomials with discontinuous weights and orthogonal polynomials on the unit circle** (**Smith [2015]**).
- These results illustrate the increasing significance of the Painlevé equations in the field of orthogonal polynomials and special functions.

Thank You!

References

- **P A Clarkson**, "On Airy solutions of the second Painlevé equation", *Stud. Appl. Math.*, **137** (2016) 93–109
- **P A Clarkson & K Jordaan**, "The relationship between semi-classical Laguerre polynomials and the fourth Painlevé equation", *Constr. Approx.*, **39** (2014) 223–254
- **P A Clarkson & K Jordaan**, "Properties of generalized Freud polynomials", arXiv:1606.06026
- **P A Clarkson, K Jordaan & A Kelil**, "A generalized Freud weight", *Stud. Appl. Math.*, **136** (2016) 288–320
- **P A Clarkson, A F Loureiro & W Van Assche**, "Unique positive solution for the alternative discrete Painlevé I equation", *J. Difference Equ. Appl.*, **22** (2016) 656–675