Quantisation of Kadomtsev-Petviashvili equation

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Classical KP

KP: a (2+1)-D integrable equation for $\varphi = \varphi(\sigma, x; t)$.
Let $x, t \in \mathbb{R}$, $\sigma \in \mathbb{R}/2\pi\mathbb{Z} \equiv \mathbb{S}^1$.

$$\varphi_{t\sigma} - \varphi_{xx} - 2\beta(\varphi\varphi_\sigma)_\sigma + \gamma\varphi_{\sigma\sigma\sigma\sigma} = 0$$

Two cases: $\gamma > 0 \sim$ KP-I, $\gamma < 0 \sim$ KP-II.
Poisson bracket:

$$\{\varphi(\sigma, x), \varphi(\tau, y)\} = 2\pi\delta'(\sigma - \tau)\delta(x - y), \quad \sigma, \tau \in \mathbb{S}^1, \quad x, y \in \mathbb{R}.$$

Fix the symplectic sheaf ot the PB:

$$\int_0^{2\pi} \mathrm{d}\sigma\, \varphi(\sigma, x) = 0 \quad \forall x \in \mathbb{R}.$$

Classical KP: Hamiltonians

Local (in x) commuting Hamiltonians

$$H_p = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \, h_p(\sigma, x), \qquad p = 0, 1, 2, \dots$$

$$h_p(\sigma, x) = \frac{1}{2} \left(\partial_\sigma^{-p} \varphi\right) \left(\partial_x^p \varphi\right) + O(\beta) + O(\gamma), \qquad \beta, \gamma \to 0$$

for instance,

$$\begin{aligned} h_0(\sigma, x) &= \frac{1}{2} \varphi^2(\sigma, x), \\ h_1(\sigma, x) &= \frac{1}{2} \left(\partial_\sigma^{-1} \varphi\right) \left(\partial_x \varphi\right), \\ h_2(\sigma, x) &= \frac{1}{2} \left(\partial_\sigma^{-2} \varphi\right) \left(\partial_x^2 \varphi\right) + \frac{\beta}{3} \varphi^3 + \frac{\gamma}{2} (\partial_\sigma \varphi)^2. \end{aligned}$$

$$\{H_p, H_q\} = 0$$

Classical KP: equations of motion

$$\partial_{t_p} = \{\cdot, H_p\}$$

$$\begin{aligned}
\varphi_{t_0} &= \varphi_\sigma, \\
\varphi_{t_1} &= -\varphi_x, \\
\varphi_{t_2} &= \partial_\sigma^{-1}\varphi_{xx} + 2\beta\varphi\varphi_\sigma - \gamma\varphi_{\sigma\sigma\sigma}.
\end{aligned}$$

Galilei transform $x := x + 2vt$, $\sigma := \sigma + vx + v^2 t$
Infinitesimal Galilei boost:

$$B = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x\, x h_0(\sigma, x), \qquad \{\varphi, B\} = x\varphi_\sigma$$

$$\{H_p, B\} = -pH_{p-1}.$$

Quantisation

Correspondence principle $[\cdot,\cdot] \simeq \mathrm{i}\hbar\{\cdot,\cdot\}$

$$[\varphi(\sigma,x),\varphi(\tau,y)] = 2\pi\mathrm{i}\delta'(\sigma-\tau)\delta(x-y), \qquad \varphi^\dagger = \varphi.$$

Fourier components

$$\varphi(\sigma,x) = \sum_{n\in\mathbb{Z}} a_n(x)\,\mathrm{e}^{-\mathrm{i}n\sigma},$$

$$a_n(x) = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi}\,\varphi(\sigma,x)\,\mathrm{e}^{\mathrm{i}n\sigma}, \qquad a_0(x) = 0,$$

form the Heisenberg (oscillator) Lie algebra

$$[a_m(x),a_n(y)] = m\delta_{m+n,0}\delta(x-y), \qquad a_n^\dagger = a_{-n}.$$

Take h.w module $a_n\,|\,0\rangle = 0, \qquad n > 0.$

Bosonic $\Psi\Psi^\dagger$ operators

Let

$$\Psi_n(x) = n^{-1/2} a_n(x), \quad \Psi_n^\dagger(x) = n^{-1/2} a_{-n}(x), \quad n \in \mathbb{N}, \quad x \in \mathbb{R},$$

$$[\Psi_m(x), \Psi_n^\dagger(y)] = \delta_{mn}\delta(x-y), \quad \Psi_m(x)\,|\,0\rangle = 0, \quad m,n \in \mathbb{N},\ x,y \in \mathbb{R}.$$

Normal ordering prescription for $\mathbf{H}_{012}$:

$$\mathbf{H}_0 = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ : \frac{1}{2}\varphi^2(\sigma, x) := \sum_{m\in\mathbb{N}} m \int_{-\infty}^{\infty} \mathrm{d}x\, \Psi_m^\dagger(x)\Psi_m(x),$$

$$\mathbf{H}_1 = \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \ : \frac{1}{2} \left(\partial_\sigma^{-1}\varphi\right)\left(\partial_x\varphi\right) :$$
$$= -\mathrm{i} \sum_{m\in\mathbb{N}} \int_{-\infty}^{\infty} \mathrm{d}x\, \Psi_m^\dagger(x)\, \partial_x \Psi_m(x)$$

Quantum Hamiltonian

$$
\begin{aligned}
\mathbf{H}_2 &= \int_0^{2\pi} \frac{\mathrm{d}\sigma}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \; : \frac{1}{2} \left(\partial_\sigma^{-2}\varphi\right)\left(\partial_x^2\varphi\right) + \frac{\beta}{3}\varphi^3 + \frac{\gamma}{2}(\partial_\sigma\varphi)^2 : \\
&= -\sum_{m\in\mathbb{N}} \frac{1}{m} \int_{-\infty}^{\infty} \mathrm{d}x \, \Psi_m^\dagger(x)\, \partial_x^2 \Psi_m(x) \\
&+ \sum_{m_1,m_2\in\mathbb{N}} \beta_{m_1 m_2} \int_{-\infty}^{\infty} \mathrm{d}x \left[\Psi_{m_1+m_2}^\dagger(x)\, \Psi_{m_1}(x)\, \Psi_{m_2}(x) + \text{h.c.}\right] \\
&+ \sum_{m\in\mathbb{N}} \gamma_m \int_{-\infty}^{\infty} \mathrm{d}x \, \Psi_m^\dagger(x)\Psi_m(x)
\end{aligned}
$$

where

$$
\beta_{m_1 m_2} = \beta_{m_2 m_1} = \beta\sqrt{(m_1+m_2)m_1 m_2}, \qquad \gamma_m = \gamma m^3 .
$$

Features of the model

- Galilean invariance

$$\mathbf{B} = \int_{-\infty}^{\infty} \mathrm{d}x\, x \; : \frac{1}{2} \varphi^2(\sigma, x) := \sum_{m \in \mathbb{N}} m \int_{-\infty}^{\infty} \mathrm{d}x\, x \Psi_m^\dagger(x) \Psi_m(x)$$

$$[\mathbf{H}_0, \mathbf{B}] = 0, \qquad [\mathbf{H}_1, \mathbf{B}] = -\mathrm{i}\mathbf{H}_0, \qquad [\mathbf{H}_2, \mathbf{B}] = -2\mathrm{i}\mathbf{H}_1.$$

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- Unitary transform $\Psi_m \mapsto -\Psi_m$, $\Psi_m^\dagger \mapsto -\Psi_m^\dagger$ equivalent to $\beta \mapsto -\beta$, hence may assume $\beta \geq 0$
- Only $\mathbf{H}_0 = M$ conserved. Particles merge/decay $m_1 + m_2 \leftrightarrow m_1 + m_2$. Cf. Lee model (1954), N-waves model (Manakov 1976; Kulish 1986), continuous Heisenberg magnet (Sklyanin 1988)

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- Coordinate Bethe Ansatz! Factorisation of multiparticle S-matrix into 2-particles ones

Main result: General formula for the Coordinate Bethe Ansatz is conjectured $\forall M$. Verified by computer algebra (Maple) for $M \leq 8$.

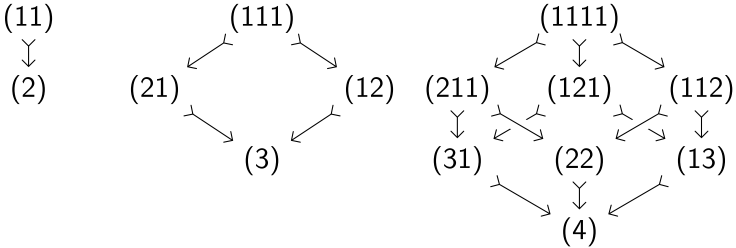
Combinatorics of compositions

A *composition* $\boldsymbol{m}$ of an nonnegative integer $M \in \mathbb{N}$ is defined as a sequence $\boldsymbol{m} = (m_1, \ldots, m_N)$ of $m_i \in \mathbb{N}$ such that $m_1 + \ldots + m_N \equiv |\boldsymbol{m}| = M$. The number $N = \ell(\boldsymbol{m})$ is called *length* of the composition, and $M = |\boldsymbol{m}|$ its *weight*. The number of compositions of M equals 2^{M-1}.

Introduce a partial order $\succ$ on the set of compositions: $\boldsymbol{m} \succ \widetilde{\boldsymbol{m}}$ means that $\ell(\widetilde{\boldsymbol{m}}) = \ell(\boldsymbol{m}) - 1$ and $\widetilde{\boldsymbol{m}}$ can be obtained from $\boldsymbol{m}$ by replacing an adjacent pair (m_i, m_{i+1}) for some $i = 1, \ldots, \ell(\boldsymbol{m}) - 1$ with $m_i + m_{i+1}$, that is

$$\boldsymbol{m} = (m_1, \ldots, m_{i-1}, m_i, m_{i+1}, m_{i+2}, \ldots, m_N),$$
$$\widetilde{\boldsymbol{m}} = (m_1, \ldots, m_{i-1}, m_i + m_{i+1}, m_{i+2}, \ldots, m_N)$$

Compositions hypercube



(a) $M = 2$ (b) $M = 3$ (c) $M = 4$

Figure : Composition hypercubes

Fock space

The mass-M sector $\mathcal{F}_M$ of our Fock space $\mathcal{F}$ is the eigenspace of the mass operator $\mathbf{H}_0$ corresponding to the eigenvalue M. Respectively, F_M splits into the orthogonal sum

$$\mathcal{F}_M = \bigoplus_{\boldsymbol{m}:\ |\boldsymbol{m}|=M} \mathcal{F}_M^{\boldsymbol{m}},$$

where $\mathcal{F}_M^{\boldsymbol{m}}$ is spanned by the vectors

$$|\, f\rangle = \int_{\mathcal{W}_N} \mathrm{d}x_1 \ldots \mathrm{d}x_N\, f_N\begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{x}\end{pmatrix} \prod_{j=1}^{N} \Psi^\dagger_{m_j}(x_j)\,|\,0\rangle \in \mathcal{F}_M^{\boldsymbol{m}},$$

$$\|f\|^2 = \int_{\mathcal{W}_N} \mathrm{d}x_1 \ldots \mathrm{d}x_N \left| f_N\begin{pmatrix}\boldsymbol{m}\\ \boldsymbol{x}\end{pmatrix}\right|^2.$$

where the *Weyl alcove* $\mathcal{W}_N$ is defined as

$$\mathcal{W}_N = \left\{\boldsymbol{x} \in \mathbb{R}^N :\ x_1 < x_2 < \ldots < x_N\right\}.$$

Eigenvalue problem for $\mathbf{H}_2$

"Bulk" differential equation and "jump conditions"

$$\lambda f_N \begin{pmatrix} m_1 & \dots & m_N \\ x_1 & \dots & x_N \end{pmatrix} = \sum_{i=1}^{N} \left(-\frac{1}{m_i} \partial_{x_i}^2 + \gamma_{m_i} \right) f_N \begin{pmatrix} m_1 & \dots & m_N \\ x_1 & \dots & x_N \end{pmatrix}$$
$$+ \sum_{k=1}^{N} \sum_{\substack{n_1, n_2 \in \mathbb{N} \\ n_1 + n_2 = m_k}} \beta_{n_1 n_2} f_{N+1} \begin{pmatrix} m_1 & \dots & m_{k-1} & n_1 & n_2 & m_{k+1} & \dots & m_N \\ x_1 & \dots & x_{k-1} & x_k & x_k & x_{k+1} & \dots & x_N \end{pmatrix}$$

$$\left[(m_i \, \partial_{x_i} - m_{i+1} \, \partial_{x_{i+1}}) f_N \right] \begin{pmatrix} m_1 & \dots & m_{i+1} & m_i & \dots & m_N \\ x_1 & \dots & x_i - 0 & x_i + 0 & \dots & x_N \end{pmatrix}$$
$$+ \left[(m_{i+1} \, \partial_{x_i} - m_i \, \partial_{x_{i+1}}) f_N \right] \begin{pmatrix} m_1 & \dots & m_i & m_{i+1} & \dots & m_N \\ x_1 & \dots & x_i - 0 & x_i + 0 & \dots & x_N \end{pmatrix}$$
$$+ 2 m_i m_{i+1} \beta_{m_i, m_{i+1}} f_{N-1} \begin{pmatrix} m_1 & \dots & m_{i-1} & m_i + m_{i+1} & m_{i+2} & \dots & m_N \\ x_1 & \dots & x_{i-1} & x_i & x_{i+2} & \dots & x_N \end{pmatrix}$$
$$= 0.$$

Equations in sector $M = 2$

Boundary problem:

$$\lambda\, f_2 \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} = (-\partial_{x_1}^2 - \partial_{x_2}^2 + 2\gamma_1) f_2 \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix},$$

$$\lambda\, f_1 \begin{pmatrix} 2 \\ x_1 \end{pmatrix} = \left(-\frac{1}{2}\partial_{x_1}^2 + \gamma_2\right) f_1 \begin{pmatrix} 2 \\ x_1 \end{pmatrix} + \beta_{11}\, f_2 \begin{pmatrix} 1 & 1 \\ x_1 & x_1 \end{pmatrix}.$$

$$2\big[(\partial_{x_1} - \partial_{x_2}) f_2\big] \begin{pmatrix} 1 & 1 \\ x_1 & x_1 \end{pmatrix} + 2\beta_{11}\, f_1 \begin{pmatrix} 2 \\ x_1 \end{pmatrix} = 0$$

Ansatz:

$$f_2 \begin{pmatrix} 1 & 1 \\ x_1 & x_2 \end{pmatrix} = \mathrm{e}^{\mathrm{i}(u_2 x_1 + u_1 x_2)} + S_{21} \mathrm{e}^{\mathrm{i}(u_1 x_1 + u_2 x_2)}, \qquad x_1 < x_2.$$

$$f_1 \begin{pmatrix} 2 \\ x_1 \end{pmatrix} = R \mathrm{e}^{\mathrm{i}(u_1 + u_2) x_1}.$$

Solution in sector $M = 2$

Algebraic equations for λ, S_{21}, R

$$u_1^2 + u_2^2 + 2\gamma_1 = \lambda,$$

$$\left(\frac{1}{2}(u_1 + u_2)^2 + \gamma_2\right) R + \beta_{11}(1 + S_{21}) = \lambda R,$$

$$\mathrm{i}(u_2 - u_1) + \mathrm{i}(u_1 - u_2)S_{21} + \beta_{11} R = 0.$$

Solution:

$$S_{21} = S(u_2 - u_1), \qquad S(u) = -\frac{P(\mathrm{i}u)}{P(-\mathrm{i}u)}$$

where $P(v)$ is the cubic polynomial

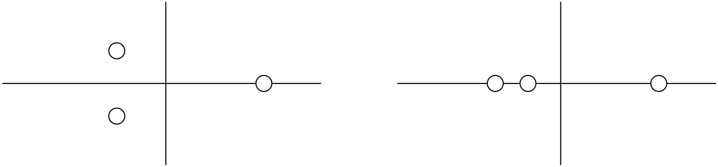
$$P(v) = v^3 + 12\gamma\, v - 4\beta^2.$$

$$R = \frac{4\mathrm{i}\beta_{11} u_{21}}{P(-\mathrm{i}u_{21})} = \frac{4\sqrt{2}\mathrm{i}\beta u_{21}}{P(-\mathrm{i}u_{21})}, \qquad u_{21} \equiv u_2 - u_1.$$

Polynomial $P(v)$

$$P(v) = v^3 + 12\gamma\, v - 4\beta^2$$

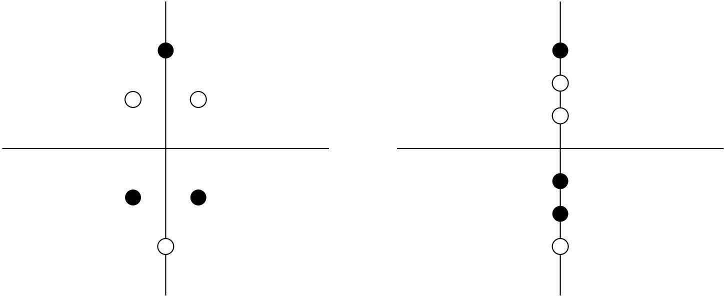
Discriminant $D = -432(16\gamma^3 + \beta^4)$



(a) qKP-I: $D < 0$ (b) qKP-II: $D > 0$

Figure : Zeroes of $P(u)$

Poles/zeroes of $S(u)$



(a) qKP-I: $D < 0$ (b) qKP-II: $D > 0$

Figure : Zeroes and poles of $S(u)$

Bethe Ansatz in sector $\mathcal{F}_M^{(1\dots1)}$

Let $\mathfrak{S}_{[1,M]}$ be the permutation group of $(1,\dots,M)$. Let $\boldsymbol{u} \equiv (u_1,\dots,u_M)$ be the vector of momenta, and $\boldsymbol{v} \equiv \mathrm{i}\boldsymbol{u}$. Define the action of a permutation $\mathfrak{s} = (\mathfrak{s}_1,\dots,\mathfrak{s}_M) \in \mathfrak{S}_{[1,M]}$ on functions of $\boldsymbol{v}$ by substitutions $\mathfrak{s}:\ v_j \mapsto v_{\mathfrak{s}_j}$. Then, for a plane wave

$$\exp(\boldsymbol{v}\cdot\boldsymbol{x}) \equiv \exp\left(v_1x_1 + \ldots + v_Mx_M\right)$$

we have

$$\mathfrak{s}\big(\exp(\boldsymbol{v}\cdot\boldsymbol{x})\big) = \big(\exp(\mathfrak{s}(\boldsymbol{v})\cdot\boldsymbol{x})\big) = \exp\left(v_{\mathfrak{s}_1}x_1 + \ldots + v_{\mathfrak{s}_M}x_M\right).$$

Conjecture 1. The $\mathcal{F}_M^{(1\dots1)}$ component of the eigenfunction of $\mathbf{H}_2$ can be chosen for $\boldsymbol{x} \in \mathcal{W}_{1\dots1}$ as

$$f_M\begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_M \end{pmatrix} = \sum_{\mathfrak{s}\in\mathfrak{S}_{[1,M]}} \operatorname{sgn}(\mathfrak{s}) \left(\prod_{j<k} P(v_{\mathfrak{s}_k} - v_{\mathfrak{s}_j})\right) \mathfrak{s}\big(\exp(\boldsymbol{v}\cdot\boldsymbol{x})\big),$$

$\mathfrak{P}$-operator

Note that the Bethe eigenfunction is antisymmetric in $\boldsymbol{v}$: a special polynomial normalisation invented by to Michel Gaudin (1983) in case of QNLS.

Let

$$\mathbb{P}(\boldsymbol{v}) \equiv \prod_{1 \leq j < k \leq M} P(v_k - v_j)$$

and define the linear operator

$$\mathfrak{P} : \; g(\boldsymbol{v}) \mapsto \sum_{\mathfrak{s} \in \mathfrak{S}_{[1,M]}} \operatorname{sgn}(\mathfrak{s}) \, \mathbb{P}(\mathfrak{s}(\boldsymbol{v})) \, g(\mathfrak{s}(\boldsymbol{v})).$$

Then

$$f_M \begin{pmatrix} 1 & \dots & 1 \\ x_1 & \dots & x_M \end{pmatrix} = \mathfrak{P}\big(\exp(\boldsymbol{v} \cdot \boldsymbol{x})\big).$$

Bethe Ansatz in generic sector

Given a composition $\boldsymbol{m} = (m_1, \ldots, m_N)$, let

$$\boldsymbol{v} = (\overbrace{\underbrace{v_1, \ldots, v_{m_1}}_{=\boldsymbol{w}_1^{\boldsymbol{m}}}}^{m_1}, \overbrace{\underbrace{v_{m_1+1}, \ldots, v_{m_1+m_2}}_{=\boldsymbol{w}_2^{\boldsymbol{m}}}}^{m_2}, \ldots, \overbrace{\underbrace{v_{\ldots}, \ldots, v_{\ldots}}_{=\boldsymbol{w}_N^{\boldsymbol{m}}}}^{m_N}),$$

and let $\langle \boldsymbol{w} \rangle$ denote the sum of the components of a vector $\boldsymbol{w}$, e.g. $\langle \boldsymbol{v} \rangle = v_1 + \ldots + v_M$.

Proposition. The Bethe eigenfunction in the generic subsector $\mathcal{F}_M^{\boldsymbol{m}}$ can be written in the form

$$f_N \begin{pmatrix} m_1 & \ldots & m_N \\ x_1 & \ldots & x_N \end{pmatrix} = \mathfrak{P}\big(Q^{\boldsymbol{m}}(\boldsymbol{v}) \exp(\langle \boldsymbol{w}_1^{\boldsymbol{m}} \rangle x_1 + \ldots + \langle \boldsymbol{w}_N^{\boldsymbol{m}} \rangle x_N\big),$$

where the polynomials $Q^{\boldsymbol{m}}(\boldsymbol{v})$ factorise as

$$Q^{\boldsymbol{m}}(\boldsymbol{v}) = \prod_{k=1}^{N} Q^{(m_k)}(\boldsymbol{w}_k^{\boldsymbol{m}})$$

Equations for $Q^{(M)}(\boldsymbol{v})$

Let $\widetilde{K}(\boldsymbol{v}) = v_1^2 + \ldots + v_M^2 - (v_1 + \ldots + v_M)^2/M$
$V_{m_1 m_2} = m_2(v_1 + \ldots + v_{m_1}) - m_1(v_{m_1+1} + \ldots + v_{m_1+m_2})$
Bulk & jump equations are:

$$\left(\widetilde{K}(\boldsymbol{v}) + (M^3 - M)\gamma\right) Q^{(M)}(v_1, \ldots, v_M)$$
$$+\beta \sum_{\substack{n_1, n_2 \geq 1 \\ n_1 + n_2 = M}} \sqrt{n_1 n_2 M}\, Q^{(n_1)}(v_1, \ldots, v_{n_1})\, Q^{(n_2)}(v_{n_1+1}, \ldots, v_M) \overset{\mathfrak{P}}{\equiv} 0.$$

$$V_{m_1 m_2} Q^{(m_1)}(v_1, \ldots, v_{m_1})\, Q^{(m_2)}(v_{m_1+1}, \ldots, v_{m_1+m_2})$$
$$+ V_{m_2 m_1} Q^{(m_2)}(v_1, \ldots, v_{m_2})\, Q^{(m_1)}(v_{m_2+1}, \ldots, v_{m_1+m_2})$$
$$+ 2 m_1 m_2 \beta_{m_1} \beta_{m_2}\, Q^{(m_1+m_2)}(v_1, \ldots, v_{m_1+m_2}) \overset{\mathfrak{P}}{\equiv} 0.$$

Solution

Let $Q^{(1)}(v) = 1$ and, for $M \geq 2$,

$$Q^{(M)}(\boldsymbol{v}) = \frac{2\sqrt{M}(2\beta)^{1-M}}{M!(M-1)} \sum_{1 \leq i < j \leq M} (-1)^{j-i} \binom{M-1}{j-i-1} (v_i - v_j)^{M-1}$$

Conjecture 2. The above polynomials $Q^{(M)}(v) = 1$ satisfy all the BA consistency equations.

Verified by computer algebra for $M \leq 8$.

Caveat. A direct verification of $\mathfrak{P}$-equivalence, by summation over $M!$ permutations, is unpractical for large M because of the exponential growth. What was verified instead, it is a set of stronger sufficient conditions that have only polynomial computational complexity.

2-reducibility

Recall that

$$P(v) = v^3 + 12\gamma\, v - 4\beta^2.$$

Let $v_{ij} = v_i - v_j$, and $P_{ij} = P(v_i - v_j)$. Assuming $M \geq 2$, we shall say that a polynomial $F(\boldsymbol{v})$ is *2-reducible* and write $F \stackrel{2}{\equiv} 0$ if $F(\boldsymbol{v})$ admits a decomposition (not necessarily unique)

$$F(\boldsymbol{v}) = \sum_{i=1}^{M-1} P_{i,i+1}\, G_i(\boldsymbol{v})$$

with some polynomials $G_i(\boldsymbol{v})$ symmetric under permutation $v_i \leftrightarrow v_{i+1}$ for each i.

Proposition. If $F \stackrel{2}{\equiv} 0$ then $F \stackrel{\mathfrak{P}}{\equiv} 0$.

3-reducibility

Assuming $M \geq 3$, we shall say that a polynomial $F(\boldsymbol{v})$ is *3-reducible* and write $F \overset{3}{\equiv} 0$ if $F(\boldsymbol{v})$ admits a decomposition

$$F(\boldsymbol{v}) = \sum_{i=1}^{M-2} (v_i - 2v_{i+1} + v_{i+2}) \, J_i(\boldsymbol{v})$$

with some polynomials $J_i(\boldsymbol{v})$ symmetric w.r.t. permutations of $\{v_i, v_{i+1}, v_{i+2}\}$.

Proposition. If $F \overset{3}{\equiv} 0$ then $F \overset{\mathfrak{P}}{\equiv} 0$.

Open questions

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- Generalisations: multipole S-matrices?
- Affine Toda field theory

Toda FT

Affine A_{N-1} Toda field theory (Arinshtein, Fateyev, Zamolodchikov 1979)

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{N} \partial_\mu \varphi_i \cdot \partial^\mu \varphi_i - \frac{2M^2}{\beta^2}\sum_{i=1}^{N} \exp\left[\frac{\beta}{\sqrt{2}}(\varphi_i - \varphi_{i+1}\right]$$

2-particle S-matrix

$$S_{11}(\theta) = \frac{\sinh\left(\frac{\theta}{2} + \frac{\mathrm{i}\pi}{N+1}\right)\sinh\left(\frac{\theta}{2} - \frac{\mathrm{i}\pi}{N+1} + \mathrm{i}\frac{b}{2}\right)\sinh\left(\frac{\theta}{2} - \mathrm{i}\frac{b}{2}\right)}{\sinh\left(\frac{\theta}{2} - \frac{\mathrm{i}\pi}{N+1}\right)\sinh\left(\frac{\theta}{2} + \frac{\mathrm{i}\pi}{N+1} - \mathrm{i}\frac{b}{2}\right)\sinh\left(\frac{\theta}{2} + \mathrm{i}\frac{b}{2}\right)}.$$

Upon rescaling

$$\theta = \frac{2\kappa^{-1}\pi u}{N+1} \qquad \text{and} \qquad b = \frac{2\kappa^{-1}\tau\pi}{N+1}$$

then sending $N \to +\infty$ one obtains the rational (nonrelativistic) degeneration

$$\lim_{N\to+\infty} S_{11}(\theta) = \widetilde{S}_{11}(u) = \frac{u^3 + u\left(\kappa^2 + \tau^2 - \kappa\tau\right) - \mathrm{i}\kappa\tau(\kappa - \tau)}{u^3 + u\left(\kappa^2 + \tau^2 - \kappa\tau\right) + \mathrm{i}\kappa\tau(\kappa - \tau)}.$$