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Bi-flat F-manifolds, Painlevé transcendents and complex reflection groups.

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Plan of the talk

1. From Frobenius manifolds to flat and bi-flat F-manifolds.

2. Bi-flat F-manifolds and Painlevé transcendents

3. Bi-flat F-manifolds and complex reflection groups

Frobenius manifolds (Dubrovin)

Definition

A Frobenius manifold $(M, \circ, \eta, e, E)$ *is a manifold equipped with an associative commutative product* $\circ$*, two distinguished vector fields* e *and* E*, a flat pseudo-metric* η*:*

- η *is invariant w.r.t the product:* $\eta_{il} c^l_{jk} = \eta_{jl} c^l_{ik}$.
- *the Levi-Civita connection* ∇ *is compatible with the product:*

$$\nabla_k c^i_{jl} = \nabla_j c^i_{kl}.$$

- e *is the unit of the product and it is flat:* $\nabla e = 0$.
- $\nabla \nabla E = 0$, $[e, E] = e$, $\mathrm{Lie}_E c^i_{jk} = c^i_{jk}$, $\mathrm{Lie}_E \eta = D\eta$.

The product is called *semisimple* if there exist special coordinates s.t. $c^i_{jk} = \delta^i_j \delta^i_k$.

Duality

- a second contravariant flat metric g called the intersection form and defined as $g^{ij} = \eta^{il} c^{j}_{lk} E^{k}$.
- dual product:

$$X * Y := E^{-1} \circ X \circ Y \tag{1}$$

- **Topological field theory**: from the previous axioms it follows that in flat coordinates

$$c^i_{jk} = \eta^{il}\partial_l\partial_j\partial_k F$$

The Frobenius potential satisfies the WDVV equations.

- **Integrable (bi)-Hamiltonian hierarchies**:

$$P^{ij}_{(1)} = \eta^{ij}\partial_x - \eta^{il}\Gamma^j_{lk}u^k_x, \qquad P^{ij}_{(2)} = g^{ij}\partial_x - g^{il}b^j_{lk}u^k_x.$$

- **Painlevé trascendents**: in canonical coordinates the metric η becomes diagonal. It turns out that the the rotation coefficients β_{ij} are symmetric and satisfy the system

$$\partial_k\beta_{ij} = \beta_{ik}\beta_{kj}, \qquad e(\beta_{ij}) = 0, \qquad E(\beta_{ij}) = -\beta_{ij},$$

which is equivalent to a one-parameter subfamily of PVI.

Flat F-manifolds (Manin)

Definition

A flat F-manifold $(M, \circ, \nabla, e)$ is a manifold M equipped with the following data:

1. *a commutative associate product $\circ : TM \times TM \to TM$ with flat unit e.*
2. *a flat torsionless affine connection ∇ compatible with the product:*

$$\nabla_k c^i_{jl} = \nabla_j c^i_{kl}.$$

Oriented associativity equations

In **flat coordinates**

$$c^i_{jk} = \partial_j \partial_k A^i.$$

The vector potential A^i satisfies the equations ($e = \partial_1$):

$$\begin{aligned} \partial_j \partial_l A^i \partial_k \partial_m A^l &= \partial_k \partial_l A^i \partial_k \partial_m A^l \\ \partial_1 \partial_i A^j &= \delta^j_i \end{aligned}$$

Principal hierarchy for F-manifolds with compatible flat connection

Integrable hierarchy:

$$u^i_t = c^i_{jk} X^k u^j_x, \qquad i = 1, ..., n$$

where

$$c^i_{jm} \nabla_k X^m = c^i_{km} \nabla_j X^m, \qquad i, j, k = 1, ..., n. \tag{2}$$

Primary flows: $u^i_{t_{(p,0)}} = c^i_{jk} X^k_{(p,0)} u^j_x, \qquad i = 1, ..., n$

$X^k_{(p,0)}, p = 1, \ldots, n$ is a basis of flat vector fields.

Higher flows: $u^i_{t_{(p,\alpha)}} = c^i_{jk} X^k_{(p,\alpha)} u^j_x, \qquad i = 1, ..., n$

where:

$$\nabla_j X^i_{(p,\alpha)} = c^i_{jk} X^k_{(p,\alpha-1)}, \qquad i, j = 1, ..., n.$$

In flat coordinates the flows of the hierarchy are systems of conservation laws.

Invariant metric

Suppose now that there exists a metric η saisfying the following properties

- $\nabla\eta = 0$
- η is invariant w.r.t the product: $\eta_{il}c^l_{jk} = \eta_{jl}c^l_{ik}$.

then

- Flat F-manifold=Frobenius manifold (without Euler vector field)
- $\eta_{il}A^l = \partial_i F$: oriented associativity equations becomes WDVV associativity equations.
- the principal hierarchy becomes Hamiltonian w.r.t. the Dubrovin-Novikov bracket associated with η.

Lenard-Magri chain without Hamiltonian structures

Classical Lenard-Magri chain

$$P_{(1)}dh_{(p,0)} = 0, \quad P_{(1)}dh_{(p,l+1)} = P_{(2)}dh_{(p,l)}$$

can be also written as

$$\nabla^{(1)}X_{(p,0)} = 0, \qquad \nabla^{(1)}X_{(p,l+1)} = \nabla^{(2)}\left(E \circ X_{(p,l)}\right).$$

"Compatibility": $(d_{\nabla} - d_{\nabla^*})(X\circ) = 0, \quad \forall X$, where d_{∇} is the exterior covariant derivative: $(d_{\nabla}\omega)^l_{i_0\,\dots\,i_k} = \sum_{j=0}^k (-1)^j \nabla_{i_j}\omega^l_{i_0\,i_1\,\dots\,\hat{i}_j\,\dots\,i_k}$.

Bi-flat F-manifolds

Definition

A bi-flat F-manifold $(M, \nabla, \nabla^*, \circ, *, e, E)$ is a manifold M equipped with a pair of flat connections ∇ and ∇^*, a pair of products $\circ$ and $*$ on the tangent spaces T_uM and a pair of vector fields e and E s.t.:

- *E is an Euler vector field: $[e, E] = e$, $\mathrm{Lie}_E c^i_{jk} = c^i_{jk}$.*
- *the product $\circ$ is commutative, associative and with unity e.*
- *the product $*$ is commutative, associative and with unity E. It is defined as: $X * Y = E^{-1} \circ X \circ Y, \forall X, Y$.*
- *∇ is compatible with $\circ$ and ∇^* is compatible with $*$:*
- *$\nabla e = 0$ and $\nabla^* E = 0$,*
- *$(d_{\nabla} - d_{\nabla^*})(X \circ) = 0, \quad \forall X$.*

Classification: the semisimple case

In canonical coordinates

$$c^i_{jk} = \delta^i_j \delta^i_k, \qquad c^{*i}_{jk} = \frac{1}{u^i}\delta^i_j \delta^i_k,$$

$$e = \sum_k \partial_k, \qquad E = \sum_k u^k \partial_k$$

$$\Gamma^{(1)i}_{ij} = \Gamma^{(2)i}_{ij} = \Gamma^i_{ij}, \qquad i \neq j$$

Moreover

$$\begin{aligned}
&\Gamma^{(1)i}_{jk} := 0 \qquad \Gamma^{(2)i}_{jk} := 0 \qquad \forall i \neq j \neq k \neq i\\
&\Gamma^{(1)i}_{jj} := -\Gamma^{(1)i}_{ij}, \qquad \Gamma^{(2)i}_{jj} := -\frac{u^i}{u^j}\Gamma^{(2)i}_{ij} \qquad i \neq j\\
&\Gamma^{(1)i}_{ii} := -\sum_{l \neq i} \Gamma^{(1)i}_{li}, \qquad \Gamma^{(2)i}_{ii} := -\sum_{l \neq i} \frac{u^l}{u^i}\Gamma^{(2)i}_{li} - \frac{1}{u^i},
\end{aligned} \tag{3}$$

Flatness conditions

Let $R_{(l)}$ be the Riemann tensor of the connection $\nabla_{(l)}$, $E_{(1)} = e$ and $E_{(2)} = E$, then the condition $R_{(l)} = 0$ splits in two parts:

1. $[\mathrm{Lie}_{E_{(l)}}, \nabla_{(l)}](T) = 0$, for any tensor field T.
2. geometric version of Tsarev's conditions of integrability: $Z \circ_{(l)} R_{(l)}(W, Y)(X) + W \circ_{(l)} R_{(l)}(Y, Z)(X) + Y \circ_{(l)} R_{(l)}(Z, W)(X) = 0$, for any vector fields X, Y, Z, W.

In canonical coordinates for $\circ$ the first condition reads

$$E_{(l)}(\Gamma^i_{ij}) = -(\partial_j E^j_{(l)})\Gamma^i_{ij}, \qquad i \neq j$$

and the second condition coincides with

$$\partial_j \Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^i_{ik} - \Gamma^i_{ik}\Gamma^k_{kj} - \Gamma^i_{ij}\Gamma^j_{jk} = 0, \quad \text{if } i \neq k \neq j \neq i.$$

As a first step we have to solve the system

$$\begin{aligned} E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3]\Gamma^i_{ij} = 0, \\ E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3]\Gamma^i_{ij} = -\Gamma^i_{ij}, \end{aligned}$$

the solutions of which are given by

$$\Gamma^1_{12} = \frac{F_{12}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^1-u^2}, \quad \Gamma^1_{13} = \frac{F_{13}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^1-u^3}, \quad \Gamma^2_{21} = \frac{F_{21}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^2-u^1},$$

$$\Gamma^2_{23} = \frac{F_{23}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^2-u^3}, \quad \Gamma^3_{31} = \frac{F_{31}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^3-u^1}, \quad \Gamma^3_{32} = \frac{F_{32}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^3-u^2}.$$

where F_{ij}, $i \neq j$ are arbitrary smooth functions.

Imposing Tsarev's conditions and introducing the auxiliary variable $z = \frac{u^2 - u^3}{u^1 - u^2}$, we obtain the system

$$
\begin{aligned}
\frac{dF_{12}}{dz} &= -\frac{(F_{12}(z)F_{23}(z) - F_{12}(z)F_{13}(z))z - F_{12}(z)F_{23}(z) + F_{32}(z)F_{13}(z)}{z(z-1)}, \\
\frac{dF_{21}}{dz} &= \frac{(F_{21}(z)F_{23}(z) - F_{21}(z)F_{13}(z))z + F_{23}(z)F_{31}(z) - F_{23}(z)F_{21}(z)}{z(z-1)}, \\
\frac{dF_{13}}{dz} &= \frac{(F_{12}(z)F_{23}(z) - F_{12}(z)F_{13}(z))z - F_{12}(z)F_{23}(z) + F_{32}(z)F_{13}(z)}{z(z-1)}, \\
\frac{dF_{31}}{dz} &= -\frac{(-F_{31}(z)F_{12}(z) + F_{21}(z)F_{32}(z))z + F_{31}(z)F_{32}(z) - F_{21}(z)F_{32}(z)}{z(z-1)}, \\
\frac{dF_{23}}{dz} &= -\frac{(F_{21}(z)F_{23}(z) - F_{21}(z)F_{13}(z))z + F_{23}(z)F_{31}(z) - F_{23}(z)F_{21}(z)}{z(z-1)}, \\
\frac{dF_{32}}{dz} &= \frac{(-F_{31}(z)F_{12}(z) + F_{21}(z)F_{32}(z))z + F_{31}(z)F_{32}(z) - F_{21}(z)F_{32}(z)}{z(z-1)},
\end{aligned}
\tag{4}
$$

It is straightforward to check that the above system admits three linear first integrals

$$I_1 = F_{12} + F_{13}, \quad (5)$$
$$I_2 = F_{23} + F_{21}, \quad (6)$$
$$I_3 = F_{31} + F_{32}, \quad (7)$$

one quadratic first integral

$$I_4 = F_{31}F_{13} + F_{12}F_{21} + F_{23}F_{32}. \quad (8)$$

We consider also the cubic first integral

$$I_5 = -I_3 I_4 + I_1 I_2 I_3 = F_{21}F_{13}F_{32} + F_{12}F_{23}F_{31} + (I_2 - I_3)F_{13}F_{31} + (I_1 - I_3)F_{23}F_{32}.$$

Theorem

Let $(F_{12}(z), F_{21}(z), F_{13}(z), F_{31}(z), F_{23}(z), F_{32}(z))$ *be a solution of the system* (4) *and* d_1, d_2, d_3, q_1, q_2 *the values of the first integrals* I_1, I_2, I_3, I_4, I_5 *on this solution, then the function* $f(z) = F_{23}F_{32} + zF_{12}F_{21} - \frac{q_1}{2}$ *is a solution of the equation*

$$[z(z-1)f'']^2 = [q_2 - (d_2 - d_3)g_2 - (d_1 - d_3)g_1]^2 - 4f'g_1g_2, \tag{9}$$

where $g_1 = f - zf' + \frac{q_1}{2}$ *and* $g_2 = (z-1)f' - f + \frac{q_1}{2}$. *Furthermore, equation* (9) *reduced to the sigma form of the generic Painlevé VI equation by means of the following transformation*

$$f = -\phi(z) - \frac{1}{4}(d_1 - d_2)^2 z + \frac{1}{4}(d_1 - d_3)(d_1 - d_2).$$

Conversely, given a solution $f(z)$ of the equation

$$[z(z-1)f'']^2 = [q_2 - d_{23}g_2 - d_{13}g_1]^2 - 4f'g_1g_2$$

where $g_1 = f - zf' + \frac{q_1}{2}$ and $g_2 = (z-1)f' - f + \frac{q_1}{2}$, define d_1 as a root of the cubic polynomial

$$\lambda^3 - (2d_{13} - d_{23})\lambda^2 + (d_{13}^2 - d_{13}d_{23} - q_1)\lambda + q_1d_{13} - q_2$$

and $d_2 = d_1 - d_{13} + d_{23}, \quad d_3 = d_1 - d_{13}$, then the solution $\{F_{ij}(z)\}$ is

$$F_{12} = \pm\frac{\mu f'}{\mu d_2 - g_1},\ F_{21} = \pm\left(d_2 - \frac{g_1}{\mu}\right),\ F_{13} = \pm\left(d_1 - \frac{\mu f'}{\mu d_2 - g_1}\right)$$
$$F_{31} = \pm\left(-\mu + d_3\right), \qquad F_{23} = \pm\frac{g_1}{\mu}, \qquad F_{32} = \pm\mu$$

where the choice of the sign in a neighborhood of a point $z_0 \neq 0, 1$ depends on the sign of $f''(z_0)$ and μ satisfies

$$(f' - d_1d_2)\mu^2 + (d_1d_2d_3 + d_1g_1 - d_2g_2 - d_3f')\mu - d_1d_3g_1 + g_1g_2 = 0.$$

Regular non-semisimple case

The manifold M is assumed to be *regular*, which means that for each $p \in M$ the endomorphism $L_p := E_p \circ : T_pM \to T_pM$ has exactly one Jordan block for each distinct eigenvalue.

For three-dimensional manifolds, this gives rise to three cases, corresponding to L_1, L_2 and L_3 given by:

$$L_1 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_2 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_3 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix},$$

(here λ_i with different indices are assumed to be distinct)

Theorem

Regular bi-flat F-manifolds in dimension three such that L_p has three equal eigenvalues and one Jordan block are locally parameterized by solutions of the full Painlevé IV equation.
Regular bi-flat F-manifolds in dimension three such that L_p has two distinct eigenvalues and two Jordan blocks are locally parameterized by solutions of the full Painlevé V equation.

Summarizing

$$L_1 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{corresponds to } PVI$$

$$L_2 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \text{corresponds to } PV$$

$$L_3 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \text{corresponds to } PIV$$

Multiflat structures

Vector field	Associated product	Associated connection
e	$\circ$	∇
E	$\circ_{(1)}$	$\nabla^{(1)}$
$E \circ E$	$\circ_{(2)}$	$\nabla^{(2)}$
$E \circ E \circ E$	$\circ_{(3)}$	$\nabla^{(3)}$
...	...	...

Compatibility

$$(d_{\nabla} - d_{\nabla^{(l)}})(X \circ) = 0 \qquad \forall X. \tag{10}$$

Existence of multiflat structure in the semisimple case

$$\Gamma^{(l)i}_{ij} = \Gamma^i_{ij}, \qquad i \neq j, \quad \forall l.$$

Flatness conditions

$$E_{(l)}(\Gamma^i_{ij}) + (\partial_j E^j_{(l)})\Gamma^i_{ij} = 0, \quad l = 0, \dots N-1 \tag{11}$$

can be written as

$$\hat{E}_{(l)}(\phi_{ij}) := E_{(l)}(\phi_{ij}) - (\partial_j E^j_{(l)}) u^{n+1} \partial_{n+1} \phi_{ij} = 0, \quad l = 0, \dots, N-1. \tag{12}$$

where $\phi_{ij}(u^1, \dots, u^n, u^{n+1})$ is the function defining implicitly Γ^i_{ij}:

$$\phi_{ij}(u^1, \dots, u^n, \Gamma^i_{ij}(u^1, \dots, u^n)) = \text{constant}$$

In this way, determining ϕ_{ij} can be interpreted as the problem of finding invariant functions for the distribution Δ generated by the vector fields $\{\hat{E}_{(l)}\}_{l=0,\dots,N-1}$.

Theorem

Let $\Delta_{(0,\dots,k)}$ be the distribution spanned by the vector fields $\hat{e}, \hat{E}, \dots, \hat{E}_{(k)}$ in the $n+1$-dimensional space with coordinates $(u^1, \dots, u^n, u^{n+1})$. Then:

1. *The distributions $\Delta_{(0,1)}$ and $\Delta_{(0,1,2)}$ are integrable.*
2. *$\Delta_{(0,1,2,3)}$ is not integrable. Furthermore, at the points where $u^i \neq u^k$ $(i \neq k, i, k = 1, ..., n)$ and $u^{n+1} \neq 0$ it is totally non-holonomic, that is the minimal integrable distribution $\bar{\Delta}$ containing $\Delta_{(0,1,2,3)}$ has dimension $n+1$.*

Notice that the extended vector fields $Z_{(l)} := \hat{E}_{(l+1)}$ satisfy the commutation relation

$$[Z_{(l)}, Z_{(m)}] = [\hat{E}_{(l+1)}, \hat{E}_{(m+1)}] = (m-l)\hat{E}_{(m+l+1)} = (m-l)Z_{(m+l)},$$

of the centerless Virasoro algebra.

Three-dimensional tri-flat F-manifolds

First of all we have to solve the systems (for $j = 1, 2, 3$)

$$\begin{aligned}
E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3]\Gamma^i_{ij} = 0,\\
E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3]\Gamma^i_{ij} = -\Gamma^i_{ij},\\
E_{(2)}(\Gamma^i_{ij}) &= [(u^1)^2\partial_1 + (u^2)^2\partial_2 + (u^3)^2\partial_3]\Gamma^i_{ij} = -2u^j\Gamma^i_{ij}.
\end{aligned}$$

The general solution is given by

$$\Gamma^1_{12} = \frac{C_{12}(u^3-u^1)}{(u^2-u^1)(u^2-u^3)},\ \Gamma^1_{13} = \frac{C_{13}(u^1-u^2)}{(u^3-u^1)(u^3-u^2)},\ \Gamma^2_{21} = \frac{C_{21}(u^2-u^3)}{(u^1-u^3)(u^1-u^2)},$$

$$\Gamma^2_{23} = \frac{C_{23}(u^1-u^2)}{(u^3-u^1)(u^3-u^2)},\ \Gamma^3_{31} = \frac{C_{31}(u^2-u^3)}{(u^1-u^3)(u^1-u^2)},\ \Gamma^3_{32} = \frac{C_{32}(u^3-u^1)}{(u^2-u^1)(u^2-u^3)},$$

where $C_{12}, C_{21}, C_{13}, C_{31}, C_{23}, C_{32}$ are arbitrary constants. Imposing Tsarev's condition we obtain immediately the following constraints

$$C_{13} = -C_{12}, \qquad C_{21} = -C_{21}, \qquad C_{32} = -C_{32}, \qquad C_{12}+C_{23}+C_{31} = 1.$$

Four-dimensional tri-flat F-manifolds

First step: we have to solve the system the system (with $j = 1, 2, 3, 4$)

$$
\begin{aligned}
E_{(0)}(\Gamma^i_{ij}) &= [\partial_1 + \partial_2 + \partial_3 + \partial_4]\Gamma^i_{ij} = 0, \\
E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1 + u^2\partial_2 + u^3\partial_3 + u^4\partial_4]\Gamma^i_{ij} = -\Gamma^i_{ij}, \\
E_{(2)}(\Gamma^i_{ij}) &= [(u^1)^2\partial_1 + (u^2)^2\partial_2 + (u^3)^2\partial_3 + (u^4)^2\partial_4]\Gamma^i_{ij} = -2u^j\Gamma^i_{ij}.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
\Gamma^i_{i1} &= F_{i1}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^3-u^2}{(u^1-u^3)(u^1-u^2)}, \quad i = 2, 3, 4, \\
\Gamma^i_{i2} &= F_{i2}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^3-u^1}{(u^2-u^3)(u^2-u^1)}, \quad i = 1, 3, 4, \\
\Gamma^i_{i3} &= F_{i3}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^2-u^1}{(u^3-u^1)(u^3-u^2)}, \quad i = 1, 2, 4, \\
\Gamma^i_{i4} &= F_{i4}\left(\frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}\right)\frac{u^1-u^3}{(u^4-u^1)(u^4-u^3)}, \quad i = 1, 2, 3.
\end{aligned}
$$

The second step seems very difficult. We have to solve a system of 24 equations (Tsarev's conditions) for the 12 unknown functions F_{ij}. This system can be written as a system of ODEs (*two for each unknown function*) in the variable $z = \frac{(u^1-u^2)(u^3-u^4)}{(u^2-u^3)(u^1-u^4)}$ for the unknown functions $F_{ij}(z)$:

$$\frac{dF_{12}}{dz} = -\frac{-F_{12}F_{13} + F_{12}F_{23} + F_{32}F_{13} + F_{12}}{z-1} = -\frac{-F_{42}F_{14} + F_{12}F_{14} - F_{12}F_{24}}{z},$$

$$\frac{dF_{13}}{dz} = \frac{F_{12}F_{23} - F_{12}F_{13} + F_{32}F_{13} - F_{13}}{z} = \frac{-F_{14}F_{13} + F_{14}F_{43} + F_{34}F_{13}}{z},$$

$$\frac{dF_{14}}{dz} = -\frac{-F_{42}F_{14} + F_{12}F_{14} - F_{12}F_{24}}{z} = -\frac{(F_{34}F_{13} + F_{14}F_{43} - F_{14}F_{13})z + F_{14}}{z(z-1)},$$

$$\frac{dF_{21}}{dz} = -\frac{F_{23}F_{21} - F_{13}F_{21} - F_{23}F_{31} + F_{21}}{z-1} = -\frac{-F_{24}F_{21} + F_{24}F_{41} + F_{14}F_{21}}{z},$$

$$\frac{dF_{23}}{dz} = -\frac{-F_{13}F_{21} + F_{23}F_{21} - F_{23}F_{31} - F_{23}}{(z-1)z} = \frac{F_{23}F_{34} - F_{23}F_{24} + F_{43}F_{24}}{z},$$

$$\frac{dF_{24}}{dz} = \frac{F_{14}F_{21} - F_{24}F_{21} + F_{24}F_{41} - F_{24}z}{(z-1)z} = -\frac{z(F_{34}F_{23} - F_{24}F_{23} + F_{24}F_{43}) + F_{24}}{(z-1)z},$$

$$
\begin{aligned}
\frac{dF_{31}}{dz} &= -\frac{-F_{31}F_{14}+F_{31}F_{34}-F_{41}F_{34}}{z} = \frac{F_{31}F_{12}+F_{21}F_{32}-F_{31}F_{32}+F_{31}}{z},\\
\frac{dF_{32}}{dz} &= \frac{F_{31}F_{12}+F_{21}F_{32}-F_{31}F_{32}-F_{32}}{(z-1)z} = \frac{F_{34}F_{42}-F_{34}F_{32}+F_{24}F_{32}}{z},\\
\frac{dF_{34}}{dz} &= -\frac{F_{31}F_{34}-F_{41}F_{34}-F_{31}F_{14}+F_{34}z}{(z-1)z} = \frac{F_{34}F_{42}-F_{34}F_{32}+F_{24}F_{32}}{z},\\
\frac{dF_{41}}{dz} &= \frac{F_{41}F_{12}+F_{21}F_{42}-F_{41}F_{42}+F_{41}}{z} = -\frac{F_{31}F_{43}+F_{41}F_{13}-F_{41}F_{43}-F_{41}}{z-1},\\
\frac{dF_{42}}{dz} &= \frac{F_{41}F_{12}+F_{21}F_{42}-F_{41}F_{42}-F_{42}}{(z-1)z} = -\frac{F_{42}F_{23}-F_{42}F_{43}+F_{32}F_{43}+F_{42}}{z-1},\\
\frac{dF_{43}}{dz} &= \frac{F_{31}F_{43}-F_{41}F_{43}+F_{41}F_{13}+F_{43}}{(z-1)z} = \frac{F_{42}F_{23}-F_{42}F_{43}+F_{32}F_{43}-F_{43}}{z}.
\end{aligned}
$$

Comparing the right and sides of the above equations we obtain some constraints on the functions F_{ij}. We have the following relations

$$\begin{aligned}
F_{14}(z) - F_{12}(z) &= C_1, \\
zF_{13}(z) + (z-1)F_{12}(z) &= C_1, \\
F_{32}(z) - F_{34}(z) &= C_2, \\
(z-1)F_{34}(z) - F_{31}(z) &= C_2, \\
-zF_{43}(z) - (z-1)F_{42}(z) &= C_3, \\
\frac{F_{41}(z)}{z} - \frac{(z-1)}{z}F_{42}(z) &= C_3, \\
\frac{zF_{23}(z)}{z-1} + \frac{F_{21}(z)}{z-1} &= C_7, \\
(z-1)F_{24}(z) - F_{21}(z) &= C_7.
\end{aligned}$$

Since for each unknown we have two equations, we have still to impose that such equations coincide. In general this seems a very complicate task. However, assuming $C_1 = 0$ we obtain the following additional constraints

$$\begin{aligned} C_7 &= C_2 + C_3 - 2, \\ F_{42}(z) &= \frac{(1 - C_3)z + F_{34}(z)(z-1) - C_2}{z - 1}, \\ F_{21}(z) &= F_{34}(z)(z-1) + 1 - C_2, \\ F_{34}(z) &= C_3 + F_{12}(z) - 1. \end{aligned}$$

After this, all the equations of the original system reduce to the first order equation

$$\frac{dF_{12}(z)}{dz} = -\frac{F_{12}(z)[(F_{12}(z) + C_3 - 1)(1 - z) + C_2]}{z(z-1)} \tag{13}$$

whose general solution is given by

$$F_{12}(z) = \frac{C_9 z^{C_2}(z-1)^{-C_2}}{C_8 C_9 z^{C_9} + \mathrm{hypergeom}([C_2, C_9], [1 + C_9], \frac{1}{z})} \tag{14}$$

where $C_9 = 1 - C_3$ and C_8 is an additional integration constant.

Multiflat structures in the non semisimple regular case

$$c_{ij}^{k} = \delta_{i+j-1}^{k},$$
$$E_{(0)} = e = \partial_{u^1},$$
$$E_{(l+1)} = E^l = (u^1)^l \partial_{u^1} + l u^2 (u^1)^{l-1} \partial_{u^2} + \left(l u^3 (u^1)^{l-1} + \frac{1}{2}(l^2 - l)(u^2)^2 (u^1)^{l-2} \right) \partial_{u^3},$$
$$\Gamma_{11}^{(l+1)1} = -\frac{l}{u^1}, \quad \Gamma_{11}^{(l+1)2} = \frac{l u^2 (l a^2 + l a + a + 2)}{(a+2)(u^1)^2}$$
$$\Gamma_{11}^{(l+1)3} = \frac{l((2la^2 + 2la + a + 2)u^1 u^3 - (la^2 + 2la + a + 2)(u^2)^2 + (lab + 2lb)u^1 u^2)}{(a+2)(u^1)^3}$$
$$\Gamma_{12}^{(l+1)1} = \Gamma_{21}^{(l+1)1} = 0, \quad \Gamma_{12}^{(l+1)2} = \Gamma_{21}^{(l+1)2} = -\frac{l(a^2 + 2a + 2)}{(u^1)(a+2)}, \quad \Gamma_{23}^{(l+1)3} = \Gamma_{32}^{(l+1)3} = \frac{a}{u^2}$$
$$\Gamma_{12}^{(l+1)3} = \Gamma_{21}^{(l+1)3} = \frac{l((la^2 + a^2 + 2la + 4a + 4)(u^2)^2 - 2a^2 u^1 u^3 - (2ab + 4b)u^1 u^2)}{2u^2 (a+2)(u^1)^2},$$
$$\Gamma_{13}^{(l+1)1} = \Gamma_{31}^{(l+1)1} = \Gamma_{13}^{(l+1)2} = \Gamma_{31}^{(l+1)2} = \Gamma_{22}^{(l+1)1} = 0, \quad \Gamma_{13}^{(l+1)3} = \Gamma_{31}^{(l+1)3} = -\frac{l(a+1)}{u^1},$$
$$\Gamma_{22}^{(l+1)3} = -\frac{((la^2 + 3la + 2l)(u^2)^2 - (ab - 2b)u^1 u^2 + 2au^1 u^3)}{(a+2)u^1 (u^2)^2}, \quad \Gamma_{22}^{(l+1)2} = \frac{a(a+1)}{u^2(a+2)}$$
$$\Gamma_{23}^{(l+1)1} = \Gamma_{32}^{(l+1)1} = \Gamma_{23}^{(l+1)2} = \Gamma_{32}^{(l+1)2} = \Gamma_{33}^{(l+1)1} = \Gamma_{33}^{(l+1)2} = \Gamma_{33}^{(l+1)3} = 0,$$

Multiflat structures in the non semisimple regular case

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$$\Gamma_{11}^{(l+1)3} = \frac{l((2la^2 + 2la + a + 2)u^1 u^3 - (la^2 + 2la + a + 2)(u^2)^2 + (lab + 2lb)u^1 u^2)}{(a+2)(u^1)^3}$$
$$\Gamma_{12}^{(l+1)1} = \Gamma_{21}^{(l+1)1} = 0, \quad \Gamma_{12}^{(l+1)2} = \Gamma_{21}^{(l+1)2} = -\frac{l(a^2 + 2a + 2)}{(u^1)(a+2)}, \quad \Gamma_{23}^{(l+1)3} = \Gamma_{32}^{(l+1)3} = \frac{a}{u^2}$$
$$\Gamma_{12}^{(l+1)3} = \Gamma_{21}^{(l+1)3} = \frac{l((la^2 + a^2 + 2la + 4a + 4)(u^2)^2 - 2a^2 u^1 u^3 - (2ab + 4b)u^1 u^2)}{2u^2 (a+2)(u^1)^2},$$
$$\Gamma_{13}^{(l+1)1} = \Gamma_{31}^{(l+1)1} = \Gamma_{13}^{(l+1)2} = \Gamma_{31}^{(l+1)2} = \Gamma_{22}^{(l+1)1} = 0, \quad \Gamma_{13}^{(l+1)3} = \Gamma_{31}^{(l+1)3} = -\frac{l(a+1)}{u^1},$$
$$\Gamma_{22}^{(l+1)3} = -\frac{((la^2 + 3la + 2l)(u^2)^2 - (ab - 2b)u^1 u^2 + 2au^1 u^3)}{(a+2)u^1 (u^2)^2}, \quad \Gamma_{22}^{(l+1)2} = \frac{a(a+1)}{u^2 (a+2)}$$
$$\Gamma_{23}^{(l+1)1} = \Gamma_{32}^{(l+1)1} = \Gamma_{23}^{(l+1)2} = \Gamma_{32}^{(l+1)2} = \Gamma_{33}^{(l+1)1} = \Gamma_{33}^{(l+1)2} = \Gamma_{33}^{(l+1)3} = 0,$$

Complex reflection groups and bi-flat F-manifolds

In the cases G_4 G_5, G_6, G_8, G_9, G_{10}, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}, G_{23}, G_{24}, G_{25}, G_{26}, G_{27} the dual product and the dual connection:

$$c^{*i}_{jk}(p) = -\Gamma^{*i}_{jk}(p) = h^{im}\left\{\frac{1}{N}\sum_{s=1}^{M}\frac{\kappa_s}{||\alpha_s||^2}\frac{(\alpha_s)_j\,(\alpha_k)_p\,(\bar{\alpha}_s)_m}{\alpha_s(p)}\right\}, \qquad (15)$$

where

- α_s *are constant covectors in* $\mathbb{C}^n$*;*
- h^{im} *are the components of the inverse of a suitable Hermitian metric. In all the cases a part from one (G_{27}) h is the standard Hermitian metric;*
- *N is a normalizing factor chosen in such a way that* $c^{*i}_{jk}E^k = \delta^i_j$*;*
- *M is the number of distinct factors of* $\det\frac{\partial u^i}{\partial p^j}$ *(the number of reflecting hyperplanes);*
- κ_s *is the order of the (pseudo)-reflection defined by the hyperplane* $\ker(\alpha_s)$*.*

Moreover, the flat coordinates of ∇ *(generalized Saito flat coordinates) are flat basic invariants* $(u_1, ..., u_n)$ *satisfying the following condition*

$$\frac{\partial^2 u^i}{\partial p^j \partial p^k} = (d_i - 1) c^{*s}_{jk} \frac{\partial u^i}{\partial p^s}.$$

1. A. Arsie and P. Lorenzoni, *F*-manifolds with eventual identities, bidifferential calculus and twisted Lenard-Magri chains. IMRN (2012)
2. A. Arsie and P. Lorenzoni, From Darboux-Egorov system to bi-flat *F*-manifolds, Journal of Geometry and Physics (2013).
3. P. Lorenzoni, Darboux-Egorov system, bi-flat *F*-manifolds and Painlevé VI, IMRN (2014).
4. A. Arsie and P. Lorenzoni, *F*-manifolds, multi-flat structures and Painlevé transcendents, arXiv:1501.06435.
5. A. Arsie and P. Lorenzoni, Complex reflection groups, logarithmic connections and bi-flat F-manifolds, arXiv:1604.04446.

Painlevé equations

$$
\begin{aligned}
\frac{d^2w}{dz^2} &= 6w^2 + z \\
\frac{d^2w}{dz^2} &= 2w^3 + zw + \alpha \\
\frac{d^2w}{dz^2} &= \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w} \\
\frac{d^2w}{dz^2} &= \frac{1}{2w}\left(\frac{dw}{dz}\right)^2 + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w} \\
\frac{d^2w}{dz^2} &= \left(\frac{1}{2w} + \frac{1}{w-1}\right)\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\left(\frac{dw}{dz}\right) + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \\
&\quad + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1} \\
\frac{d^2w}{dz^2} &= \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)\left(\frac{dw}{dz}\right) + \\
&\quad + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)
\end{aligned}
$$

σ-form of Painlevé equations

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2z\frac{d\sigma}{dz} - 2\sigma = 0$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^3 + 2\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} - \sigma\right) = \frac{1}{4}\left(\alpha + \frac{1}{2}\right)^2$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 + \left[4\left(\frac{d\sigma}{dz}\right)^2 - 1\right]\left(z\frac{d\sigma}{dz} - \sigma\right) + \lambda_0\lambda_1\frac{d\sigma}{dz} = \frac{1}{4}\left(\lambda_0^2 + \lambda_1^2\right)$$

$$\left(\frac{d^2\sigma}{dz^2}\right)^2 - 4\left(z\frac{d\sigma}{dz} - \sigma\right)^2 + 4\frac{d\sigma}{dz}\left(z\frac{d\sigma}{dz} + 2\theta_0\right)\left(z\frac{d\sigma}{dz} + 2\theta_\infty\right) = 0$$

$$\left(z\frac{d^2\sigma}{dz^2}\right)^2 - 2\left[2\left(\frac{d\sigma}{dz}\right)^2 - z\frac{d\sigma}{dz} + \sigma\right]^2 + 4\prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \kappa_j\right) = 0$$

$$\frac{d\sigma}{dz}\left(z(z-1)\frac{d^2\sigma}{dz^2}\right)^2 + \left[\frac{d\sigma}{dz}\left(2\sigma - (2z-1)\frac{d\sigma}{dz}\right) + \beta_1\beta_2\beta_3\beta_4\right]^2 = \prod_{j=1}^{4}\left(\frac{d\sigma}{dz} + \beta_j\right)$$