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Nonanticipative functional calculus and controlled rough paths

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Notations

$D([0,T],\mathbb{R}^d)$ space of cadlag functions (right continuous with left limits).
$C^\alpha([0,T],\mathbb{R}^d)$ $\alpha-$Holder functions
For a path $\omega \in D([0,T],\mathbb{R}^d)$, denote by

- $\omega(t) \in \mathbb{R}^d$ the value of ω at t
- $\omega_t = \omega(t \wedge .)$: path stopped at t
- $\omega_{t-} = \omega \quad 1_{[0,t[} + \omega(t-) \quad 1_{[t,T]}$

For a process X we denote

- $X(t)$ its value and
- $X_t = X(t \wedge .)$ its path stopped at t.

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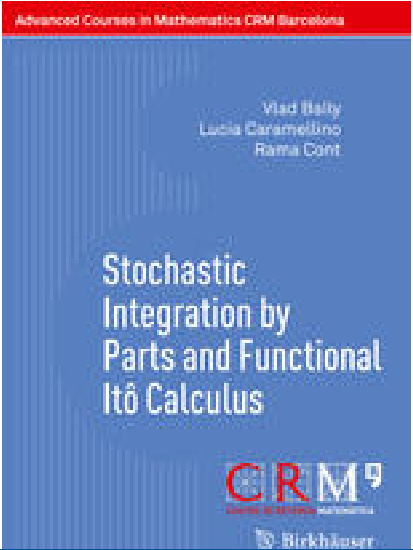
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A pathwise approach of the Ito formula

Consider a continuous $\mathbb{R}^d$-valued process X and $f \in C^2(\mathbb{R}^d, \mathbb{R})$. The main idea in the oroof of the Ito formula is to consider a sequence of partitions $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ of $[0, T]$ with step size decreasing to zero and expand increments of $f(X(t))$ along the partition using a 2nd order Taylor expansion:

$$f(X(t)) - f(X(0)) = \sum_{\pi_n} f(X(t_{i+1}^n)) - f(X(t_i^n))$$

$$= \sum_{\pi_n} \nabla f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))$$

$$+\frac{1}{2}{}^t(X(t_{i+1}^n) - X(t_i^n)) \nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n)) + r(X(t_{i+1}^n), X(t_i^n))$$

Summing over π_n we get

$$f(X(t)) - f(X(0)) = S_1(\pi_n, f) + S_2(\pi_n, f) + R(\pi_n, f)$$

- By uniform continuity of $r(x,y) = f(y) - f(x) - \nabla f(x).(y-x) - 0.5^t(y-x)\nabla^2 f(x)(y-x)$,

$$r(x,y) \le \varphi(||x-y||)||x-y||^2$$

  with $\varphi(u) \to 0$ as $u \to 0$ so $R(\pi_n, f) = \sum_{\pi_n} r(X(t_{i+1}^n), X(t_i^n)) \to 0$ pointwise if $\sum_{\pi_n} \|X(t_{i+1}^n) - X(t_i^n)\|^2$ bounded.
- Under this condition the (left) Riemann sum $S_1(\pi_n, f) = \sum_{\pi_n} \nabla f(X(t_i^n)).(X(t_{i+1}^n\text{-X}(t_i^n))$ converges **if and only if** the 'quadratic Riemann sum'

$$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} {}^t(X(t_{i+1}^n) - X(t_i^n))\nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))$$

  converges.

Quadratic Riemann sums

For d=1: given a path of X, pointwise convergence of 'quadratic Riemann sums'

$$S_2(\pi_n, f) = \frac{1}{2} \sum_{\pi_n} \nabla^2 f(X(t_i^n)).(X(t_{i+1}^n) - X(t_i^n))^2$$

along the path for *every* $f \in C^2(\mathbb{R}^d, \mathbb{R})$ is exactly equivalent to the weak convergence of the sequence of discrete measures

$$\mu_n = \sum_{t_j \in \pi^n} (X(t_{j+1}^n) - X(t_j^n))^2 \delta_{t_j}$$

where δ_t denotes a point mass at t. This is a joint property of X and (π_n).
This motivated Föllmer (1981)'s definition of 'pathwise quadratic variation along a sequence of partitions.

Definition (Quadratic variation along a sequence of partitions)

Let $\pi_n = (0 = t_0^n < t_1^n .. < t_{N(\pi_n)}^n = T)$ be a sequence of partitions of $[0, T]$ with step $|\pi_n|$ decreasing to zero. A càdlàg function $x \in D([0, T], \mathbb{R})$ is said to have finite quadratic variation along the sequence of partitions $(\pi_n)_{n \geq 1}$ if

(i) the sequence of discrete measures

$$\sum_{t_j \in \pi^n} (x(t_{j+1}^n) - x(t_j^n))^2 \delta_{t_j} \quad \overset{n \to \infty}{\Rightarrow} \quad \mu(dt) = d[\omega]_\pi$$

converges weakly;

(ii) $[x]_\pi^c$ defined by $[x]_\pi^c(t) = \mu([0, t]) - \sum_{0 < s \leq t} |\Delta x(s)|^2$ is continuous and increasing.

We denote $Q_\pi([0, T], \mathbb{R})$ the set of functions with the above properties.

Characterization in continuous case

Proposition (C. & Das (2017))

Let $x \in C^0([0,T],\mathbb{R})$ and define

$$[x]_{\pi_n}(t) = \sum_{t_j \in \pi^n} (\omega(t^n_{j+1} \wedge t) - \omega(t^n_j \wedge t))^2$$

The following properties are equivalent:

1. *x has finite quadratic variation along the sequence of partitions $(\pi_n)_{n\geq 1}$.*
2. *The sequence $[x]_{\pi_n}$ converges uniformly on $[0,T]$ to a continuous function $[x]_\pi$.*
3. *The sequence $[x]_{\pi_n}$ converges pointwise on $[0,T]$ to a continuous function $[x]_\pi$.*

Definition (Pathwise quadratic variation: multidimensional case)

$x \in Q_\pi([0,T],\mathbb{R}^d)$ if, for all $1 \le i,j \le d$, $x^i, x^i + x^j$ in $Q_\pi([0,T],\mathbb{R})$.
$[x]_\pi$ is a positive symmetric $d \times d$ matrix:

$$[x]_\pi(t) = \lim_{n\to\infty} \sum_{t_i^n \le t} (x(t_{i+1}^n) - x(t_i^n)) \cdot {}^t(x(t_{i+1}^n) - x(t_i^n)) < +\infty,$$

with elements given by

$$([x]_\pi)_{i,j}(t) = \frac{1}{2}\left([x^i + x^j]_\pi(t) - [x^i]_\pi(t) - [x^j]_\pi(t)\right)$$
$$= [x^i, x^j]_\pi^c(t) + \sum_{0<s\le t} \Delta x^i(s)\Delta x^j(s), \quad i,j = 1,\ldots,d$$

Föllmer's 'pathwise Ito formula'

Proposition (Föllmer, 1981)

$\forall f \in C^2(\mathbb{R}^d, \mathbb{R}), \forall \omega \in Q_\pi([0,T], \mathbb{R}^d)$, *the non-anticipative Riemann sums along* π

$$\sum_{\pi_n} \nabla f(\omega(t_i^n)).(\omega(t_{i+1}^n) - \omega(t_i^n)) \overset{n\to\infty}{\to} \int_0^T \nabla f(\omega(t)).d^\pi\omega$$

converge pointwise and

$$f(\omega(t)) - f(\omega(0)) = \int_0^t \nabla f(\omega).d^\pi\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi^c > + \sum_{s\le t} f(\omega(s)) - f(\omega(s-)) - \nabla f(\omega(s-)).\Delta\omega(s)$$

Dependence on the partition

Consider now two sequences of partitions π, τ and a continuous path $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap Q_\tau([0,T],\mathbb{R}^d)$.
Since $\forall f \in C^2(\mathbb{R}^d)$,

$$\begin{aligned} f(\omega(t)) - f(\omega(0)) &= \int_0^t \nabla f(\omega).d^\pi\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\pi > \\ &= \int_0^t \nabla f(\omega).d^\tau\omega + \frac{1}{2}\int_0^t < \nabla^2 f(\omega), d[\omega]_\tau > \qquad (1) \end{aligned}$$

the pathwise integrals are equal if and only if $[\omega]_\pi = [\omega]_\tau$.
But the pathwise quadratic variation **does** depend on the sequence of partition...

Quadratic variation along a sequence of partitions

This notion of 'pathwise quadratic variation along a sequence of partitions' depends on the chosen sequence of partitions:

Proposition ((Friedman))

Let $\omega \in C^0([0,T],\mathbb{R}^d)$. There exists a sequence of partitions (π_n) such that $[\omega]_\pi = 0$.

Proof: We construct recursively partitions π_n such that

$$|\pi_n| \leq \frac{1}{n} \qquad \text{and} \qquad \sum_{\pi_n} |\omega(t^n_{k+1}) - \omega(t^n_k)|^2 \leq \frac{1}{n}.$$

Assume we have constructed π_n with this property. Adding to π_n the points $k/(n+1), k = 1..n$ we obtain a partition $\sigma_n = (s^n_i, i = 0..M_n)$ with $|\sigma_n| \leq 1/(n+1)$.

For $i = 0..(M_n - 1)$, we further refine $[s_i^n, s_{i+1}^n]$ as follows. Let $J(i)$ be an integer with

$$J(i) \geq (n+1)M_n|\omega(s_{i+1}^n) - \omega(s_i^n)|^2,$$

$$\tau_{i,k+1}^n = \inf\{t \geq \tau_{i,k}^n, \quad \omega(t) = \omega(s_i^n) + \frac{k\left(\omega(s_{i+1}^n) - \omega(s_i^n)\right)}{J(i)}\}.$$

Then points $(\tau_{i,k}^n, k = 1..J(i))$ defines a partition of $[s_i^n, s_{i+1}^n]$ with

$$|\tau_{i,k+1}^n - \tau_{i,k}^n| \leq \frac{1}{n+1} \quad \text{and} \quad |\omega(\tau_{i,k+1}^n) - \omega(\tau_{i,k}^n)| = \frac{|\omega(s_{i+1}^n) - \omega(s_i^n)|}{J(i)}$$

$$\text{so} \quad \sum_{k=1}^{J(i)} |\omega(\tau_{i,k+1}^n) - \omega(\tau_{i,k}^n)|^2 \leq J(i)\frac{|\omega(s_{i+1}^n) - \omega(s_i^n)|^2}{J(i)^2} = \frac{1}{(n+1)M_n}.$$

Sorting $(\tau_{i,k}^n, i = 0..M_n, k = 1..J(i))$ gives $\pi_{n+1} = (t_j^{n+1})$ such that

$$|\pi_{n+1}| \leq \frac{1}{n+1}, \qquad \sum_{\pi_{n+1}} |\omega(t_{i+1}^n) - \omega(t_i^n)|^2 \leq \frac{1}{n+1}.$$

Definition (Well-balanced sequence of partitions)

Let $\underline{\pi_n} = \inf_{i=0..N(\pi_n)-1} |t^n_{i+1} - t^n_i|$.
The sequence of partitions $(\pi_n)_{n\geq 1}$ well-balanced if

$$\exists c > 0, \qquad \forall n \geq 1, \quad \frac{|\pi_n|}{\underline{\pi_n}} \leq c. \tag{2}$$

Theorem (R.C. & P. Das, 2016)

Let $\alpha > 0$, $f \in C^\alpha([0,T],\mathbb{R}^d)$ and $\tau = (\tau^n)_{n\geq 1}$ and $\sigma = (\sigma^n)_{n\geq 1}$ two well-balanced partition sequences such that

$$f \in Q_\tau([0,T],\mathbb{R}^d) \cap Q_\sigma([0,T],\mathbb{R}^d) \quad \text{and} \qquad [f]_\sigma > 0, \quad [f]_\tau > 0.$$

Then: $\forall t \in [0,T], \qquad [f]_\sigma(t) = [f]_\tau(t)$

Non-anticipative Functionals

Denote $\omega_t = \omega(t \wedge .)$ the *past* i.e. the path stopped at t.

Definition (Non-anticipative Functionals)

A *causal*, or *non-anticipative functional* is a functional $F : [0, T] \times D([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$ whose value only depends on the past:

$$\forall \omega \in \Omega, \quad \forall t \in [0, T], \qquad F(t, \omega) = F(t, \omega_t). \tag{3}$$

Causal functional= map on the space Λ_T^d of stopped paths, defined as the quotient space:

$$\Lambda_T^d := \left([0, T] \times D([0, T], \mathbb{R}^d)\right) \Big/ \sim$$

where $(t, x) \sim (t', x') \leftrightarrow t = t', x_t = x'_t$. Λ_T^d is equipped with a metric

$$d_\infty((t, x), (t', x')) = \sup_{u \in [0, T]} |x(u \wedge t) - x'(u \wedge t')| + |t - t'|.$$

Functionals of piecewise constant paths

A piecewise-constant path $\omega = \sum_{k=1}^{n} x_k 1_{[t_k, t_{k+1}[}$ is obtained by

- "horizontal stretchings" from t_k to t_{k+1}, followed by
- addition of a jump at each discontinuity point: $\omega_{t_{k+1}} = \omega_{t_k} + (x_{k+1} - x_k) 1_{t_{k+1}}$

Key idea: The evolution of a non-anticipative functional along ω may be decomposed into its variations with respect to two types of operations:

- "horizontal extension" of the path from t_k to t_{k+1}

$$F(t_{k+1}, \omega_{t_k}) - F(t_k, \omega_{t_k})$$

- 'vertical step' at partition points: addition of a jump at t_{k+1}

$$F(t_{k+1}, \omega_{t_{k+1}}) - F(t_{k+1}, \omega_{t_k})$$

If one can control the behavior of F under these two types of path perturbations, then one can follow/reconstitute $F(t, \omega)$.

Definition (Horizontal and vertical derivatives)

A non-anticipative functional F is said to be:

- horizontally differentiable at $(t,\omega) \in \Lambda_T^d$ if the finite limit exists

$$\mathcal{D}F(t,\omega) := \lim_{h \to 0+} \frac{F(t+h,\omega_t) - F(t,\omega_t)}{h}.$$

- vertically differentiable at $(t,\omega) \in \Lambda_T^d$ if the map

$$\mathbb{R}^d \to \mathbb{R}, \;\; e \mapsto F(t,\omega(t \wedge .) + e1_{[t,T]})$$

is differentiable at 0; its gradient at 0 is denoted by $\nabla_\omega F(t,\omega)$.

Note that $\mathcal{D}F(t,\omega)$ is **not** the partial derivative in t:

$$\mathcal{D}F(t,\omega) \neq \partial_t F(t,\omega) = \lim_{h \to 0} \frac{F(t+h,\omega) - F(t,\omega)}{h}.$$

Smooth functionals

Definition ($\mathbb{C}_b^{1,2}(\Lambda_T^d)$ functionals)

We denote by $\mathbb{C}_b^{1,2}(\Lambda_T^d)$ the set of non-anticipative functionals $F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$, such that

- F is horizontally differentiable with $\mathcal{D}F$ continuous at fixed times,
- F is twice vertically differentiable with $\nabla_\omega^j F \in \mathbb{C}_l^{0,0}(\Lambda_T^d)$ for $j = 1, 2$;
- $\mathcal{D}F, \nabla_\omega F, \nabla_\omega^2 F \in \mathbb{B}(\Lambda_T^d)$.

Examples of smooth functionals

Example (Cylindrical functionals)

For $g \in C^0(\mathbb{R}^{d\times n}), h \in C^k(\mathbb{R}^d)$ with $h(0) = 0$. Then

$$F(t,\omega) = h\left(\omega(t) - \omega(t_n-)\right) \quad 1_{t\geq t_n} \quad g(\omega(t_1-), \omega(t_2-)..., \omega(t_n-))$$

is in $\mathbb{C}_b^{1,k}$ and

$$\mathcal{D}_t F(\omega) = 0, \quad \text{and} \qquad \forall j = 1..k,$$

$$\nabla_\omega^j F(t,\omega) = h^{(j)}\left(\omega(t) - \omega(t_n-)\right) 1_{t\geq t_n} g\left(\omega(t_1-), \omega(t_2-)..., \omega(t_n-)\right)$$

$\mathbb{S}(\Lambda_T, \pi_n) :=$ space of simple predictable cylindrical functionals piecewise constant along π_n, $\mathbb{S}(\Lambda_T, \pi) := \cup_{n>1} \mathbb{S}(\Lambda_T, \pi_n)$x

Examples of smooth functionals

Example (Integral functionals)

For $g \in C_0(\mathbb{R}^d)$, $Y(t) = \int_0^t g(X(u))\rho(u)du = F(t, X_t)$ where

$$F(t,\omega) = \int_0^t g(\omega(u))\rho(u)du \tag{4}$$

$F \in \mathbb{C}_b^{1,\infty}$, with:

$$\mathcal{D}_t F(\omega) = g(\omega(t))\rho(t) \qquad \nabla_\omega^j F(t,\omega) = 0 \tag{5}$$

Conditional expectations as smooth functionals

Let $\sigma \in \mathbb{C}^{0,0}(\mathcal{W}_T)$ be such that

$$X(t) = X(0)\exp\left(\int_0^t \sigma(u)dW(u) - \frac{1}{2}\int_0^t \sigma^2(u).du\right) \qquad (*)$$

is a martingale, i.e. $E(X(T)) = 1$ and denote by $\mathbb{Q}^\sigma$ the law of (*).

Proposition (Cont & Riga 2015)

Let $h : (D([0,T],R), \|.\|_\infty) \mapsto \mathbb{R}$ be $\mathbb{Q}^\sigma$-integrable and Lipschitz. Assume that for $(t,\omega) \in \mathcal{W}_T$, the map

$$g^h(.;t,\omega) : e \in \mathbb{R}^d \to \quad g^h(e) = h\left(\omega + e1_{[t,T]}\right), \qquad (6)$$

is twice differentiable at 0, with derivatives bounded uniformly in $(t,\omega) \in \mathcal{W}_T$ in a neighborhood of 0. Then, there exists $F \in \mathbb{C}_b^{0,2}(\mathcal{W}_T)$ such that $F(t,X_t) = E^{\mathbb{Q}^\sigma}[H|\mathcal{F}_t^X]$ $\mathbb{Q}^\sigma$−a.s.

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Weak Euler schemes as smooth functionals

Let $\sigma : (\Lambda_T, d_\infty) \to \mathbb{R}^{d\times d}$ be a Lipschitz map. Then

$$ {}_nX(t_{j+1},\omega) = {}_nX(t_j,\omega) + \sigma(t_j, {}_nX_{t_j}(\omega)) \cdot (\omega(t_{j+1}-) - \omega(t_j-)) . \qquad (7) $$

defines a non-anticipative functional ${}_nX$ which approximates

$$ X(t) = X(0) + \int_0^t \sigma(u, X_u) dW(u) \qquad (8) $$

For a Lipschitz functional $g : (D([0,T],\mathbb{R}^d), \|.\|_\infty) \to \mathbb{R}$, consider the 'weak Euler approximation' of $\mathbb{E}\left[g(X_T)|\mathcal{F}_t^W\right]$:

$$ F_n(t,\omega) = \mathbb{E}\left[g\left({}_nX_T(W_T)\right)|\mathcal{F}_t^W\right](\omega). \qquad (9) $$

(R Cont- Yi Lu, SPA 2016): $F_n \in \mathbb{C}_b^{1,\infty}(\mathcal{W}_T)$.

Controlled Hölder rough paths

Definition (Controlled rough path (Gubinelli 2004))

Let $X \in C^{\alpha}([0,T],V)$. $Y \in C^{\alpha}([0,T],W)$ is a controlled rough path controlled by X if there exists $Y' \in C^{\alpha}([0,T],\mathcal{L}(V,W))$ such that

$$R(s,t) = Y(t) - Y(s) - Y'_s.(X(t) - X(s)), \qquad T \geq t \geq s \geq 0$$

satisfies $\|R\|_{\nu} < \infty$. for some $\nu > \alpha$.

X is called the *control* or *reference path*.
$R(s,t)$ can be thought of as the remainder in a first order Taylor expansion.
Any Y' satisfying this property is called a 'Gubinelli derivative' for Y.

Regular functionals as controlled rough paths

If $F \in \mathcal{R}(\Lambda_T^d), \omega \in C^{\nu}([0,T],\mathbb{R}^d)$ then $t \mapsto (F(t,\omega), \nabla_\omega F(t,\omega)$ is a rough path controlled by ω in the sense of Gubinelli (2004):

Proposition

Let $\omega \in C^{\nu}([0,T],\mathbb{R}^d)$ for some $\nu \in (1/3, 1/2]$ and $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d, \mathbb{R}^n)$ with $\nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d, \mathbb{R}^{n\times d})$ and $F \in Lip(\Lambda_T^d, \|\cdot\|_\infty)$. Define

$$R_{s,t}^F(\omega) := F(s,\omega_s) - F(t,\omega_t) - \nabla_\omega F(t,\omega_t)(\omega(s)-\omega(t)). \tag{10}$$

Then there exists a constant $C_{F,T,\omega}$, increasing in T and $\|\omega\|_\nu$, such that

$$|R_{s,t}^F(\omega)| \le C_{F,T,\omega}|s-t|^{2\nu(1+\nu)}$$

Controlled rough paths as regular functionals

Conversely, a **family** of controlled rough paths indexed by the reference path is none other than a .. vertically differentiable functional, whose 'Gubinelli derivative' is none other than the Dupire/ vertical derivative:

Proposition (Ananova & Cont, 2017)

Let $F \in \mathbb{C}^{0,0}(\mathcal{W}_T^d, \mathbb{R})$, $G \in \mathbb{C}^{0,0}(\mathcal{W}_T^d, \mathbb{R}^d)$ be non-anticipative functionals. Assume that for any $\omega \in C^\nu([0,T])$, $\nu \in (0,1)$ the pair $(F(\cdot,\omega), G(\cdot,\omega))$ is a controlled rough path with respect to ω s.t. $\exists C_{F,T,\omega} > 0$

$$|F(s,\omega_s) - F(t,\omega_t) - G(t,\omega_t)(\omega(s)-\omega(t))| \leq C|s-t| + C|s-t|^{\nu(1+\nu)},$$

where the constants depend only on T, F and $\|\omega\|_\nu$. Then $F \in \mathbb{C}^{0,1}(\mathcal{W}_T^d, \mathbb{R})$ and $\nabla_\omega F(t,\omega) = G(t,\omega)$.

Chain rule for functionals

Proposition

Let $G \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$ and $F \in \mathbb{C}_b^{1,1}(\Lambda_T^1)$ be non anticipative functionals and $H(\omega, t) := F(G(t, \omega), t)$. Then $H \in C_b^{1,1}(\Lambda_T^d)$ and

$$\mathcal{D}H(t, \omega_t) = \mathcal{D}F(t, G(t, \omega_t)) + \nabla F(t, G(t, \omega_t))\mathcal{D}G(t, \omega_t),$$

$$\nabla H(t, \omega_t) = \nabla F(t, G(t, \omega_t))\nabla G(t, \omega_t).$$

Moreover , if $G, F \in C_b^{1,2}$ then $H \in C_b^{1,2}$ l and

$$\nabla^2 H(t, \omega_t) = \nabla F(t, G(t, \omega_t))\nabla G(t, \omega_t)\,{}^t\nabla G(t, \omega_t) + \nabla F(t, G(t, \omega_t))\nabla^2 G(t, \omega_t) \tag{11}$$

This result implies the stability of the concept controlled rough paths under smooth functionals:

Proposition (Change of variable formula for controlled rough paths)

Let $(X, X') \in \mathcal{D}^{2\nu}_{\omega}([0,T], \mathbb{R}^d)$ *be a controlled rough path with control* $\omega \in C^{\nu}([0,T], \mathbb{R}^d)$. *Then for any* $F \in \mathbb{C}^{1,1}_b(\Lambda^1_T)$,

$$(F(t,X), \nabla_\omega F(t,X).X') \in \mathcal{D}^{2\nu}_X([0,T], \mathbb{R}^d)$$

is a controlled rough path with control X.

Similar transformation rules exist in the theory of controlled rough paths (see Friz-Hairer Ch .4) but here the derivation is much simpler.

The concept of controlled rough path does not come with a natural approximation theory. Our representation yields such an approximation theory.
Let $(X, X') \in \mathcal{D}^{2\nu}_{\omega}([0,T],\mathbb{R}^d)$ be a controlled rough path with control $\omega \in C^{\nu}([0,T],\mathbb{R}^d)$ and $F \in \mathbb{C}^{1,1}_b(\Lambda^1_T)$ a functional such that $(X, X') = (F(.,\omega), \nabla_\omega F(.,\omega))$.
Then if $X_n = F_n(.,\omega)$ is a sequence of (piecewise) smooth approximations of X then a natural approximaation for (X, X') is $(X_n, \nabla_\omega F_n(.,\omega))$
Example (C.-Lu, 2016): numerical approximations of martingale representations.

Theorem (Change of variable formula (Cont- Fournié ,2010))

Let $\omega \in Q^{\pi}\left([0,T],\mathbb{R}^d\right)$ *such that* $\sup_{t\in[0,T]\setminus\pi^n} |\Delta\omega(t)| \to 0$ *and denote* $\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n-)\mathbf{1}_{[t_i^n,t_{i+1}^n)} + \omega(T)\mathbf{1}_{\{T\}}$. *Then for any* $F \in \mathbb{C}_b^{1,2}(\Lambda_T^d)$, *the limit*

$$\int_0^T \nabla_\omega F(t,\omega_{t-})\cdot d^\pi\omega = \lim_{n\to\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n-}^{n,\Delta\omega(t_i^n)})(\omega(t_{i+1}^n)-\omega(t_i^n))$$

exists, and

$$F(T,\omega) = F(0,\omega) + \int_0^T \nabla_\omega F(t,\omega_{t-})\cdot d^\pi\omega + \int_0^T \mathcal{D}F(t,\omega_{t-})dt$$
$$+ \int_0^T \frac{1}{2} tr\left(\nabla_\omega^2 F(t,\omega_{t-}) d[\omega]_\pi^c(t)\right) + \sum_{s\in[0,T]} \left(F(s,\omega_s) - F(s,\omega_{s-}) - \nabla_\omega F(s,\omega_{s-})\cdot\Delta\omega(s)\right). \quad \text{(ChV)}$$

Functional Ito formula

Applied to a semimartingale, these results lead to a functional extension of the Ito formula:

Theorem (Functional Ito formula (Dupire 09, C.& Fournié 2009))

Let X be a continuous semimartingale and $F \in \mathbb{C}^{1,2}_{loc}([0, T[)$. For any $t \in [0, T[$,

$$F(t, X_t) - F_0(X_0) = \int_0^t \mathcal{D}_u F(X_u) du + \int_0^t \nabla_\omega F_u(X_u).dX(u) + \int_0^t \frac{1}{2} \mathrm{tr}\left({}^t\nabla^2_\omega F_u(X_u)\ d[X](u)\right) \quad a.s.$$

In particular, $Y(t) = F(t, X_t)$ is a semimartingale.

Brownian martingales as harmonic functionals

Theorem (R.C. & D Fournié 2010)

Let $\mathbb{P}$ be the Wiener measure on the canonical space, $H : (D([0,T], R^d), \|.\|_\infty) \mapsto \mathbb{R}$ a $\mathbb{P}$-integrable functional. If there exists $F \in \mathbb{C}^{1,2}_{loc}(\mathcal{W}_T)$ such that

$$M(t) = F(t, W_.) = E^{\mathbb{P}}[H(W_.)|\mathcal{F}_t^W] \ \mathbb{P} - a.s.$$

then

$$\forall (t,\omega) \in \mathcal{W}_T, \qquad \mathcal{D}F(t,\omega) + \frac{1}{2}\mathrm{tr}(\nabla_\omega^2 F)(t,\omega) = 0$$

and

$$M(t) = F(t, W_.) = M(0) + \int_0^t \nabla_\omega F(s, W) dW(s).$$

These result allows to construct $\int_0^{\cdot} \nabla_\omega F$ as a pointwise limit of non-anticipative 'Riemann sums':

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) \cdot d^\pi \omega = \lim_{n \to \infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n -}^{n, \Delta\omega(t_i^n)})(\omega(t_{i+1}^n) - \omega(t_i^n))$$

Remark

$$F \in \mathcal{R}(\Lambda_T^d),\ \omega \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d) \Rightarrow \nabla_\omega F(t, \omega) \in C^{\frac{1}{2}-}([0,T], \mathbb{R}^d).$$

The pathwise integral is a strict extension of the Young integral.

Does this integral verify any continuity/ stability property?
Does it share any of the other 'nice' properties of the Ito integral?

Assumptions on F

Assumption (Lipschitz continuity of F)

$F \in Lip(\Lambda_T^d, \|\cdot\|_\infty)$: $\exists K > 0, \quad \forall \omega, \omega' \in D([0,T], \mathbb{R}^d)$,

$$|F(t,\omega) - F(t,\omega')| \le K \|\omega_t - \omega'_t\|_\infty$$

Assumption (Regularity of F)

$F \in \mathbb{C}^{1,2}(\Lambda_T) \quad \text{and} \quad \nabla_\omega F \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$.

Denote $$C^{\frac{1}{2}-}([0,T],\mathbb{R}^d) = \bigcap_{\nu<1/2} C^{\nu}([0,T],\mathbb{R}^d)$$

Theorem (Pathwise Isometry formula, A. Ananova, R. C. 2016)

Under the above assumptions on $F \in \mathbb{C}^{1,2}(\Lambda_T)$, for any path $\omega \in Q_\pi([0,T],\mathbb{R}) \cap C^{1/2-}([0,T],\mathbb{R}^d)$ and any sequence of partitions $\pi = (\pi_n)_{n\geq 1}$ satisfying $osc(F(.,\omega),\pi_n) \to_{n\to+\infty} 0$ we have

$$[F(t,\omega)]^{\pi}(t) =$$

$$\left[\int_0^{\cdot} \nabla_\omega F(s,\omega).d^{\pi}\omega\right]^{\pi}(t) = \int_0^t \langle {}^t\nabla_\omega F(s,\omega).\nabla_\omega F(s,\omega), d[\omega]^{\pi}(s)\rangle. \quad \text{(Isometry)}$$

As a consequence:

Proposition

Let $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{1/2-}([0,T],\mathbb{R}^d)$ such that $\frac{d[\omega]}{dt} := a(t) > 0$ is right-continuous. Then the path $t \mapsto F(t,\omega)$ has a zero quadratic variation along the partition π if and only if $\nabla_\omega F(t,\omega) = 0$, $\forall t \in [0,T]$.

Proof of the pathwise isometry formula

For simplicity set $d = 1$. Let

$$\mathcal{R}^{F,\omega}_{s,t} := F(s,\omega) - F(t,\omega) - \nabla_\omega F(t,\omega)(\omega(s) - \omega(t)).$$

First we prove that

$$|\mathcal{R}^{F,\omega}_{s,t}| \lesssim_{F,T,\|\omega\|_\nu} |s-t|^{\nu(1+\nu)}, \forall \nu < \frac{1}{2}. \quad (*)$$

For that, we will use the following formula for $\mathcal{R}^{F,\lambda}_{s,t}$ for Lipschitz continuous paths λ:

$$\mathcal{R}^{F,\lambda}_{t,s} = \int_t^s \mathcal{D}F(u,\lambda)du + \int_t^s \mathcal{D}\nabla^i_\omega F(r,\lambda)(\lambda^i(s) - \lambda^i(r))dr + \int_t^s \nabla^{ij}_\omega F(r,\lambda)\dot{\lambda}^j(r)(\lambda^i(s) - \lambda^i(r))dr. \quad (12)$$

(12) follows from the following result

Lemma (R. C. 2012)

Assume $G \in C_b^{1,1}(\Lambda_T)$ and λ is a continuous path with finite variation on $[t, s]$, then

$$G(s, \lambda) - G(t, \lambda) = \int_t^s \mathcal{D}G(u, \lambda)du + \int_t^s \nabla_\omega G(u, \lambda)d\lambda(u),$$

where the second integration is in the Riemann-Stieltjes sense.

Thanks to (12), Lipschitz property of F and since $\omega \in C^\nu, \forall \nu < \frac{1}{2}$, we can construct a Lipschitz continuous approximation ω^N to ω such that

$$|\mathcal{R}_{t,s}^{F,\omega^N}| \lesssim |s-t| + N^{1-\nu}|s-t|^{2\nu}, \text{ and } |\mathcal{R}_{t,s}^{F,\omega^N} - \mathcal{R}_{t,s}^{F,\omega}| \lesssim N^{-\nu}|s-t|^\nu.$$

Thus

$$|\mathcal{R}_{t,s}^{F,\omega}| \lesssim |s-t| + N^{1-\nu}|s-t|^{2\nu} + N^{-\nu}|s-t|^\nu.$$

We conclude the proof of $(*)$ by choosing $N \approx |s-t|^{-\nu}$.

For the proof of the Theorem, note that from the assumptions on the partitions π_n, we have

$$M_n := \max_i \left| \mathcal{R}^{F,\omega}_{t_i^n, t_{i+1}^n} \right| \to 0.$$

Next we choose ν close to $\frac{1}{2}$ so that $\nu^2 + \nu > \frac{1}{2}$, then from $(*)$

$$\sum_i \left| \mathcal{R}^{F,\omega}_{t_i^n, t_{i+1}^n} \right|^2 \leq C M_n^{2 - \frac{1}{\nu^2+\nu}} \sum_i |t_{i+1}^n - t_i^n| \leq C T M_n^{2 - \frac{1}{\nu^2+\nu}} \to 0.$$

Thus, since

$$\left|\left(F(t^n_{i+1},\omega) - F(t^n_i,\omega_{t^n_i})\right)^2 - \nabla_\omega F(t^n_i,\omega)^2(\omega(t^n_{i+1}) - \omega(t^n_i))^2\right| \leq |\mathcal{R}^{F,\omega}_{t^n_i,t^n_{i+1}}|^2 + C_F|\mathcal{R}^{F,\omega}_{t^n_i,t^n_{i+1}}||\omega_{t^n_i,t^n_{i+1}}|,$$

using the triangle and Cauchy-Schwarz inequalities, we get

$$\left|\sum_i \left(F(t^n_{i+1},\omega) - F(t^n_i,\omega_{t^n_i})\right)^2 - \sum_i \nabla_\omega F(t^n_i,\omega)^2(\omega(t^n_{i+1}) - \omega(t^n_i))^2\right| \leq \sum_i |\mathcal{R}^{F,\omega}_{t^n_i,t^n_{i+1}}|^2 + C_F\sqrt{\sum_i |\mathcal{R}^{F,\omega}_{t^n_i,t^n_{i+1}}|^2}\sqrt{\sum_i |\omega_{t^n_i,t^n_{i+1}}|^2} \to 0.$$

The result of the Theorem now follows from the fact

$$\sum_i \nabla_\omega F(t_i^n,\omega)^2(\omega(t_{i+1}^n)-\omega(t_i^n))^2 \to \int_0^T \nabla_\omega F(s,\omega)^2 d[\omega]^\pi(s)$$

which is a consequence of the weak convergence of

$$\sum_{t_i\in\pi^n} (\omega(t_{i+1}^n)-\omega(t_i^n))^2\delta_{t_j} \rightharpoonup d[\omega]^\pi$$

and the strong convergence of

$$\sum_{t_i^n\le t} \nabla_\omega F(t_i^n,\omega)^2\mathbf{1}_{[t_i^n,t_{i+1}^n)} \to \nabla_\omega F(t,\omega)^2.$$

Relation with Ito isometry

Let $\mathbb{P}$ be a martingale measure on $C^0([0,T],\mathbb{R})$ under which the canonical process X is a square integrable martingale. Then the integral $\int_0^t \nabla_\omega F(t,\omega).d^\pi\omega$ is a version of the Ito integral $\int_0^t \nabla_\omega F(t,X)dX$ and integrating the pathwise isometry formula with respect to $\mathbb{P}$ yields the well-known Ito isometry formula :

$$E\left([\int_0^. \nabla_\omega F(t,X)dX](t)\right) = E\left(\int_0^t |\nabla_\omega F(t,X)|^2 d[X]\right).$$

So our pathwise isometry formula uncovers a pathwise relation which underlies the Ito isometry property.

Theorem (Properties of the pathwise integral (R.C. 2012))

1. *Quadratic covariation formula: for* $\phi, \psi \in \mathbb{V}(\Lambda_T^d)$, *the limit*

$$[I_\omega(\phi), I_\omega(\psi)]_\pi(T) :=$$

$$\lim_{n\to\infty} \sum_{\pi_n} \left(I_\omega(\phi)(t_{k+1}^n) - I_\omega(\phi(t_k^n)\right)\left(I_\omega(\psi)(t_{k+1}^n) - I_\omega(\psi)(t_k^n)\right)$$

$$\text{exists and} \quad [I_\omega(\phi), I_\omega(\psi)]_\pi(T) = \int_0^T < \psi^t \phi(t, \omega_{t-}), d[\omega] > .$$

2. *Associativity: Let* $\phi \in V(\Lambda_T^d), \psi \in V(\Lambda_T^1)$ *and* $x \in D([0,T], \mathbb{R})$ *defined by* $x(t) = \int_0^t \phi(u, \omega_{u-}).d^\pi \omega$. *Then*

$$\int_0^T \psi(t, x_{t-}).d^\pi x = \int_0^T \psi(t, (\int_0^\cdot \phi(u, \omega_{u-}).d^\pi \omega))\phi(t, \omega_{t-})d^\pi \omega.$$

Regular functionals

Assumption (Horizontal local Lipschitz property)

A functional $G\colon \Lambda_T^d \to \mathbb{R}$ *is said to satisfy horizontal local Lipschitz property, if:* $\forall \omega \in D([0,T],\mathbb{R}^d), \exists\ C>0,\ \eta>0, \forall h \geq 0, \forall t \leq T-h,$

$$\|\omega_t - \omega'_t\|_\infty < \eta, \Rightarrow |G(t+h,\omega'_t) - G(t,\omega'_t)| \leq Ch.$$

Definition (Regular Functionals)

$\mathcal{R}(\Lambda_T^d)$ = set of functionals $F \in \mathbb{C}^{1,2}(\Lambda_T^d,)$ with $\nabla_\omega^k F, \in \mathbb{C}_b^{1,1}(\Lambda_T^d)$, $k=\overline{1,2}$, $F, \mathcal{D}F, \nabla^3 F \in Lip(\Lambda_T^d, \|\cdot\|_\infty)$ and $\nabla_\omega^3 F$ horizontally locally Lipschitz.

Example: cylindrical non-anticipative functionals are regular.

Uniqueness and pathwise nature of integral

Proposition (Föllmer integral as a limit of Riemann sums)

Let $F \in \mathcal{R}(\Lambda_T^d)$ *and* $\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{1/2-}([0,T],\mathbb{R}^d)$. *Then*

$$\int_0^T \nabla_\omega F(u,\omega) d^\pi \omega = \lim_{n \to +\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega) \cdot \left(\omega(t_{i+1}^n) - \omega(t_i^n)\right).$$

In particular, if $\nabla F(t,\omega) = \nabla G(t,\omega)$ *for* $F, G \in \mathcal{R}(\Lambda_T^d)$ *then*

$$\forall t \in [0,T], \quad \int_0^t \nabla_\omega F(u,\omega) d^\pi \omega = \int_0^t \nabla_\omega G(u,\omega) d^\pi \omega.$$

Lemma

Under the assumptions of the previous result, for consecutive endpoints $t < s \in \pi^n$, we have

$$F(s,\omega_s) - F(t,\omega_t) = \int_t^s \mathcal{D}_t F(u,\omega_u)du + \nabla_\omega F(t,\omega_t)\,(\omega(s)-\omega(t))$$

$$+\frac{1}{2}\langle \nabla_\omega^2 F(t,\omega_t), (\omega(s)-\omega(t)) \otimes (\omega(s)-\omega(t))\rangle + O(|s-t|^{3\nu^2+\nu}).$$

The pathwise integral as a continuous map

Definition ($a-$harmonic functionals)

Let $a\colon [0,T] \to S^d_+$ be a continuous function taking values in positive-definite symmetric matrices. $F \in \mathcal{H}_a(\Lambda_T)$ if

$$\forall (t,\omega) \in \Lambda_T, \qquad \mathcal{D}F(t,\omega_t) + \frac{1}{2}\langle \nabla^2_\omega F(t,\omega_t), a(t)\rangle = 0.$$

Let $\bar{\omega} \in Q_\pi([0,T],\mathbb{R}^d) \cap C^p([0,T],\mathbb{R}^d)$, $d[\bar{\omega}]_\pi/dt = a$. Then

$$\forall F \in \mathcal{H}_a(\Lambda_T), \qquad F(t,\bar{\omega}) = F(0,\bar{\omega}) + \int_0^t \nabla_\omega F(u,\bar{\omega}).d^\pi\bar{\omega}.$$

so by the isometry formula

$$[F(.,\bar{\omega})]_\pi(t) = \int_0^t {}^t\nabla_\omega F(u,\bar{\omega}).a(u)\nabla_\omega F(u,\bar{\omega})du = \|\nabla_\omega F(.,\bar{\omega})\|^2_{L^2([0,T],a)} < \infty.$$

Continuity of the pathwise integral

Let $\bar{\omega} \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{1/2-}([0,T],\mathbb{R}^d)$, $d[\bar{\omega}]_\pi/dt = a > 0$.

$$\mathcal{H}_a(\bar{\omega}) := \left\{ F(\cdot,\bar{\omega}.) \,\middle|\, F \in \mathcal{H}_a(\Lambda_T) \right\} \subset Q_\pi([0,T],\mathbb{R}),$$
$$\mathbb{V}_a(\bar{\omega}) := \left\{ \nabla_\omega F(\cdot,\bar{\omega}.) \,\middle|\, F \in \mathcal{H}_a(\Lambda_T) \right\} \subset L^2([0,T],a).$$

Proposition (Pathwise integral as an injective isometry)

The pathwise integral

$$I_{\bar{\omega}}(\phi) = \lim_{n\to\infty} \sum_{\pi_n} \phi(t_k^n).(\bar{\omega}(t_{k+1}^n) - \bar{\omega}(t_k^n))$$

defines an injective isometry

$$I_{\bar{\omega}} : \ \left(\mathbb{V}_a(\bar{\omega}), \|\cdot\|_{L^2([0,T],a)}\right) \to \left(\mathcal{H}_a(\bar{\omega}), \|\cdot\|_\pi\right)$$

A pathwise 'Doob-Meyer' decomposition

Given $\bar\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{\frac{1}{2}-}[0,T],\mathbb{R}^d$ with strictly increasing quadratic variation along π:

$$\bar\omega \in Q_\pi([0,T],\mathbb{R}^d) \cap C^{1/2-}([0,T],\mathbb{R}^d) \quad \text{with} \quad \frac{d[\bar\omega]_\pi}{dt} > 0 \; dt - a.e. \tag{13}$$

and consider the set of regular transformations of $\bar\omega$:

$$\mathcal{R}(\bar\omega) := \left\{ F(\cdot,\bar\omega) \,\middle|\, F \in \mathcal{R}(\Lambda_T) \right\} \subset Q_\pi([0,T],\mathbb{R}).$$

Proposition (Rough-smooth decomposition of paths)

Any path $\omega \in \mathcal{R}(\bar\omega)$ has a unique decomposition

$$\omega(t) = \omega(0) + \int_0^t \phi . d^\pi \bar\omega + s(t)$$

where $\phi \in \mathbb{V}_a(\bar\omega)$ and $[s]_\pi = 0$.

- This result may be viewed as a pathwise analogue of the well-known decomposition of a continuous semimartingale as the sum of a local martingale and a process with finite variation.
- Similar results were obtained using rough path techniques Hairer-Pillai (2013) using a uniform Hölder roughness condition on the path and by by Hu & Tindel (2013) for fractional Brownian motion.
- Our setting is closer to the original semimartingale decomposition: the components are distinguished based on (pathwise) quadratic variation.

As in Cass-Litterer-Hairer-Tindel (2012) and Hairer-Pillai (2013), we obtain a **'Norris Lemma'** for this decomposition under a roughness condition on the reference path $\bar{\omega}$:

Theorem (Stability of rough-smooth decomposition)

Let $\bar{\omega} \in C^{1/2-}([0,T]) \cap Q_\pi^{++}([0,T],\mathbb{R}^d)$ *such that*
$\exists \theta < 1,\ L_\theta(\bar{\omega}) > 0\ \forall t \in [0,T], \epsilon \in (0, T/2], v \in \mathbb{R}^d,$

$$\exists s \in [0,T], \qquad |t-s| \le \epsilon \text{ and } |v \cdot (\bar{\omega}(s) - \bar{\omega}(t))| > L_\theta(\bar{\omega})\epsilon^\theta.$$

There exists $p, q > 0$ *such that for any* $\omega \in \mathcal{R}(\bar{\omega})$ *with rough-smooth decomposition*

$$\omega(t) = \omega(0) + \int_0^t \phi_\omega . d^\pi \bar{\omega} + s_\omega(t), \quad \phi \in \mathbb{V}_a(\bar{\omega}), \quad [s]_\pi = 0.$$

$$\text{we have} \qquad \|\phi_\omega\|_\infty + \|s_\omega\|_\infty \le CM^p \|\omega\|_\infty^q.$$

where $M(\omega) := 1 + L_\theta(\bar{\omega})^{-1} + \|\phi'\|_\nu + \|R^\phi\|_{2\nu} + \|\bar{\omega}\|_\nu + \|d[\bar{\omega}]/dt\|_\infty + \|s\|_\nu.$

$Q_\pi([0,T],\mathbb{R})$ is not a vector space and, given two paths $(\omega_1,\omega_2)\in Q_\pi([0,T],\mathbb{R})$ the quadratic covariation along π cannot be defined in general.

By contrast, the space

$$\mathcal{R}(\bar\omega) := \left\{ F(\cdot,\bar\omega) \,\middle|\, F\in\mathcal{R}(\Lambda_T)\right\} \subset Q_\pi([0,T],\mathbb{R}).$$

is a vector space of paths with finite quadratic variation along π. Moreover, for any pair of elements $(\omega_1,\omega_2)\in\mathcal{U}(\bar\omega)^2$, the quadratic covariation along π is well defined; if $\omega_i=\int_0 \phi_i.d^\pi\overline{\omega}+s_i$ is the rough-smooth decomposition of ω_i the quadratic covariation is given by

$$[\omega_1,\omega_2]_\pi(t)=\int_0^t <\phi_1^t\phi_2, d[\overline{\omega}]> .$$

This bilinear form on $\mathcal{R}(\bar\omega)$ allows to define a weak pathwise functional derivative (R.C.-Yi Lu, 2017) and extend the formulas above to a larger class of functionals.

A regularity structure on path space

Let $X \in C^{1/2-} \cap Q_\pi([0,T],\mathbb{R}^d)$ with $[X]_\pi$ strictly increasing.
Define $A = \{-\frac{1}{2}-, 0-, 0, \frac{1}{2}-, 1\}$,

$$T_0 =< 1 >, T_{1/2-} =< X^1, ..., X^d >, T_{-1/2} =< d^\pi X^1, ..., d^\pi X^d >$$

$$T_{0-} =< d[X]^{i,j}, i,j = 1..d >$$

The bijective regular functionals $G \in \mathcal{R}(\mathcal{W}_\mathcal{T})$ with $\nabla_\omega F \in GL(d,\mathbb{R})$ then define a group of transformations which acts on

$$\mathcal{R}(X) = \{F(.,X), F \in \mathcal{R}(\mathcal{W}_\mathcal{T})\} \subset Q_\pi([0,T],\mathbb{R}^d)$$

Expansion at (t,ω) :
$T_{(t,x)}F(s,) = F(t,x) + (s-t)\mathcal{D}F + \nabla_\omega F(t,x).(y-x) + 1/2 < \nabla^2_\omega F(t,x).(y-x)\otimes(y-x) >$ The functional chain rule then allows to transpose a functional expansion $T_y F$ at any $y = Y(X) \in \mathcal{R}(X)$ to an expansion $T_z F = \Gamma_{y,z}(T_y)F$ at $z = Z(X) \in \mathcal{R}(X)$.

A regularity structure on path space

The group of transformations $G = \{\Gamma_{y,z}\}$ then allows to define a *regularity structure* (Hairer 2014) on the space of regular functionals of an irregular path X:

Proposition (C. 2017)

Let $X \in C^{1/2-}([0,T]) \cap Q_\pi([0,T],\mathbb{R}^d)$ with $[X]_\pi$ strictly increasing. (T, A_X, Γ) defines a regularity structure over the space of paths $\mathcal{R}(X)$. A realization of this regularity structure is given by the L^2 closure of regular functionals of X and their $(1,2)-$jets given by the horizontal and (1st, 2nd) vertical derivatives.

In addition to this regularity structure, we also have an additional structure on $\mathcal{R}(X)$ given by the quadratic form $[.]_\pi$.

Summary

Non-anticipative functional calculus for paths with finite quadratic variation which gives a

- Global formulation and calculus for controlled rough paths.
- Pathwise analog of the Ito isometry: pathwise integral with respect to paths of finite quadratic variation which satisfies a pathwise isometry property
- Pathwise analog of the semimartingale decomposition for functionals of an irregular path with strictly increasing quadratic variation
- Regularity structure for functionals defined on typical sample paths of semimartingales.

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