Rough paths with jumps and semimartingales

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Motivation

• Important aspect in stochastic modelling: jumps

• Rough paths à la Lyons very successful in dealing with **continuous** stochastic processes (e.g. F-Victoir book 2010)

- continuous semimartingales
- continuous Gaussian processes (ρ-variation of covariance ....)
- Markov processses (with generator $L = \partial_i \left(a^{ij} (x) \partial_j \right)$ ...)

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• Even without jumps: advantages of p-variation over Hölder in applications?

Background I: "general" Young integration

- Given $X, Y \in D^p$, i.e. càdlàg and of finite p-variation, $p < 2$
- **Theorem (Young)** Write π for finite partition of $[0, T]$. Then

$$\exists \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} Y_s^- X_{s,t} =: \int_0^T Y^- dX$$

- Same limit if Y_s^- replaced by Y_s.
- Local expansion with precise error estimate (Young inequality)
- Convergence in RRS sense does not require càdlàg. But if either is càdlàg, as above, get MRS convergence.
- If $X \in BV$ and both càdlàg, limit agrees with RS integral $\int_{(0,T]}$
- NB: We insist on left-point evaluation of integrand. If evaluate integrand at arbitrary $r \in [s, t]$ get, assuming no common jumps, only RRS convergence. (Nice survey by Dudley–Norvaisa '98.)

Background II: general semimartingales

- Let X càdlàg semimartingale, with $X(\omega) \in D^{2^+}$, let Y càdlàg adapted
- **Theorem:** The Itô integral $\int Y^- dX$ has the presentation

$$\int_0^T Y^- dX = \lim_n \sum_{[s,t] \in \pi_n} Y_s^- X_{s,t}$$

- Actually same result if Y_s^- replaced by Y_s.

• **Theorem (Lépingle's BDG)** Let M be a local càdlàg martingale. Then, for $p = 2^+$ and φ convex, moderate

$$E\varphi\left(\|M\|_{p\text{-var};[0,\infty)}\right) \asymp E\varphi\left([M]_\infty^{1/2}\right)$$

• **Theorem** For nice coefficients vector fields: existence, uniqueness for differential equations driven by general semimartingales.

cont'd: semimartingale stability

• **Theorem (Kurtz-Protter, 91)** Let $X, (X^n)_{n\geq 1}, H, (H^n)_{n\geq 1}$ be càdlàg adapted processes (with respect to some filtrations $\mathcal{F}^n$). Suppose $(H^n, X^n)_{n\geq 1}$ converges in law (resp. in probability) to (H, X) in the Skorokhod topology as $n \to \infty$, and that $(X^n)_{n\geq 1}$ is a sequence of càdlàg semimartingales satisfying **UCV**. Then X is a semimartingale (with respect to some filtration $\mathcal{F}$) and $(H^n, X^n, \int_0^\cdot H^n_{t-} dX^n_t)$ converge in law (resp. in probability) to $(H, X, \int_0^\cdot H_{t-} dX_t)$ in the Skorokhod topology as $n \to \infty$.

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• Also applies to differential equations driven by X^n under **UCV** and yields flexible limit theorems for Itô SDEs. (All this before rough paths.)

• With a extra work, also gives Wong-Zakai result for *continuous* driving semimartingales. But: UCV typically fails for homogenization problems.

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• **Reminder on UCV:** A sequence of semimartingales (X^n) satisfies uniformly controlled variation (UCV) condition if (in essence) one has decompositions $X^n = M^n + A^n$ with uniform control on $[M^n]$ and the BV-norm of A^n. (Various localizations built in the actual definition).

cont'd: Marcus canoncial SDEs

• Idea/intuition: transform $X \in D^p([0,T])$ into a $\tilde{X} \in C^{p\text{-var}}([0,\tilde{T}])$, by "stretching" time when

$$\Delta_s X \neq 0$$

and replace the jumps by straight line connecting X_{s-} with X_s

• In case of càdlàg semimartingales, implement as

$$\int_0^T f(X) \diamond dX \quad := \int_0^T f(X_{t-})\, dX_t + \frac{1}{2}\int_0^T Df(X_{t-})\, d[X,X]_t^c + \sum_{t \in (0,T]} \Delta_t X \left\{ \int_0^1 f(X_{t-} + \theta \Delta_t X) - f(X_{t-}) \right\} d\theta$$

• Also works for differential equations: $dY = f(Y) \diamond dX$ etc ("Marcus canonical SDE", a.k.a. "geometric solutions")

• **Theorem (Kurtz, Pardoux, Protter '95)** Wong-Zakai theorem for such canonical SDEs, with general driving semimartingales

Background III: (continuous) rough paths

- Consider $p \in [2,3)$ and $\mathbf{X} = (X, \mathbb{X}) \in \mathbf{C}^{p\text{-var}}$ (not necessarily geometric)
- For nice F, set $Y = F(X)$ and $Y' = DF(X)$ have

$$\sup_{\pi} \sum_{[s,t]\in\pi} |R_{s,t}|^{p/2} < \infty \text{ where } R_{s,t} := Y_{s,t} - Y'_s X_{s,t}$$

- **Theorem (Lyons '98)** The following rough integral is well-defined

$$\exists \lim_{|\pi|\to 0} \sum_{[s,t]\in\pi} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t} =: \int_0^T Y d\mathbf{X}$$

Moreover, for nice vector fields have existence, uniqueness for differential equations (RDEs) driven by continuous rough paths. Locally Lipschitz continuity of solution map in p-variation rough path metrics, also $p \geq 3$.

- Non-geometric case $p \geq 3$: branched rough paths (Gubinelli)

Background V: the work of D. Williams ('01)

- Marcus canonical (a.k.a. geometric) solutions in the Young regime.
- Forward solution, in sense of Young integral equation, similar to Itô.
- Then considers X multidimensional Lévy, constructs stochastic area $\mathbb{A}$ such that

$$\sup_{\pi} \sum_{[s,t]\in\pi} |\mathbb{A}_{s,t}|^{p/2} < \infty \text{ a.s.}$$

- Transforms $(X(\omega), \mathbb{A}(\omega))$ on $[0,T]$ into $\tilde{\mathbf{X}}(\omega) \in \mathbf{C}^{p\text{-var}}[0, \tilde{T}(\omega)]$

- Solves RDE $d\tilde{Y} = f(\tilde{Y})d\tilde{\mathbf{X}}(\omega)$. Undoing time-stretching *defines* the solution to

$$dY = f(Y)\, d(X, \mathbb{A})$$

- So-constructed solution coincides with Marcus canoncial SDE solution.

- Williams' work begs the definition of a "general" rough path
- **Definition (F-Shekhar '15):** Fix $p \in [2,3)$. We call $\mathbf{X} = (X, \mathbb{X})$ a *general (a.k.a. càdlàg) rough path*, in symbols $\mathbf{X} \in \mathbf{D}^p$, if

$$t \mapsto \mathbf{X}_{0,t} \text{ càdlàg } \; \& \; X \in D^p \; \& \; \sup_{\pi} \sum_{[s,t] \in \pi} |\mathbb{X}_{s,t}|^{p/2} < \infty$$

- In Marcus resp. Williams setting insert in straight lines $\implies$

$$\lim_{s \uparrow t} \mathbb{X}_{s,t} =: \Delta_t \mathbb{X} = \underbrace{\Delta_t \mathbb{A}}_{=0} + \frac{1}{2} (\Delta_t X)^{\otimes 2}$$

- **Definition cont'd:** Say $\mathbf{X} \in \mathbf{D}^p$ is <u>g</u>*eometric* if values in $G^2(\mathbb{R}^d)$ and <u>*M*</u>*arcus-like* if in addition

$$\log \Delta_t \mathbf{X} = (\Delta_t X, \Delta_t \mathbb{A}) \in \mathbb{R}^d \oplus \{0\} \subset \mathfrak{g}^2(\mathbb{R}^d)$$

- Non-linear rough path spaces: $\mathbf{D}^p_{\mathrm{M}} \subset \mathbf{D}^p_{\mathrm{g}} \subset \mathbf{D}^p$.
- Extension to all $p < \infty$.

Elementary examples: geometric càdlàg rough path

- Consider a pure jump path: $\Delta_i X := \Delta_{\tau_i} X \neq 0$ for times (τ_i)

  Lifts "canonically" to a Marcus-like rough path $\mathbf{X} \in \mathbf{D}^p_{\mathrm{M}}$ (any $p > 0$)

$$\mathbf{X}_{0,t} = \otimes_{i:\tau_i \leq t} \exp \Delta_i X = \left(X_{0,t}, \sum_{i<j} \Delta_i X \otimes \Delta_j X + \frac{1}{2} \sum_i (\Delta_i X)^{\otimes 2} \right)$$

- Pure area jump path: $X \equiv 0$ and $\Delta_i \mathbb{X} = \Delta_i \mathbb{A} \neq 0$ at times τ_i

$$\mathbf{X}_t = \otimes_{i:\tau_i \leq t} \exp \Delta_i \mathbb{A} = \exp \left(\sum_{i:\tau_i \leq t} \Delta_i \mathbb{A} \right)$$

  (Note that, for any $p > 0$, have $\mathbf{X} \in \mathbf{D}^p_{\mathrm{g}}$ but $\notin \mathbf{D}^p_{\mathrm{M}}$.)

Elementary examples: non-geometric càdlàg rough paths

- Consider $X \in D^1[0,T]$, i.e. càdlàg BV, with $X_0 = 0$. Enhanced with

$$\mathbb{X}_{0,t} = \int_{(0,t]} X^- \otimes dX \ ,$$

obtain a (non-geometric) rough path $(X, \mathbb{X}) = \mathbf{X} \in \mathbf{D}^1$ (but $\notin \mathbf{D}^1_{\mathrm{g}}$),

$$\mathbb{X}_{s,t} = \int_{(s,t]} (X_{r-} - X_s) \otimes dX_r$$

- Similar for $p \in [1,2)$ using Young integration
- Càdlàg not so important, cf. earlier remarks on Young integration

Semimartingales as càdlàg rough paths

- Assume X is d-dimensional semimartingale lifted to $\mathbf{X}^* = (X, \mathbb{X}^*)$ where $*$ distinguishes between Itô- and Marcus lift:

$$\begin{aligned} \mathbb{X}^{\mathrm{I}}_{s,t} &:= \int_s^t X^-_{s,r} \otimes dX_r \ (\underline{\mathrm{I}}\text{tô}) \\ \mathbb{X}^{\mathrm{M}}_{s,t} &:= \int_s^t X_{s,r} \diamond \otimes dX_r \ (\underline{\mathrm{M}}\text{arcus}) \end{aligned}$$

- **Theorem (F-Chevyrev '17)** A general multidimensional semimartingale, enhanced with (Marcus resp. Itô) interated integals yields a random, càdlàg (geometric resp. branched) p-rough path: $\forall p \in (2,3)$,

$$\mathbf{X}^{\mathrm{M}}(\omega) \in \mathbf{D}^p_{\mathrm{M}} \quad \text{and} \quad \mathbf{X}^{\mathrm{I}}(\omega) \in \mathbf{D}^p \quad \text{a.s.}$$

If X is a càdlàg local martingale, then "rough" BDG holds:

$$E\varphi\left(|||\mathbf{X}^*|||_{p\text{-var};[0,\infty)} \right) \asymp E\varphi\left(|[M]_\infty|^{1/2} \right)$$

- Unifies previous works of Lépingle, Williams, Coutin-Lejay, F-Victoir ...

Semimartingales as rough paths, idea of proof

• Showing a.s. $\mathbf{X}^{\mathrm{I}}(\omega) \in \mathbf{D}^p$ is equivalent to $\mathbf{X}^{\mathrm{M}}(\omega) \in \mathbf{D}^p$. Indeed, the difference

$$(\mathbb{X}^{\mathrm{M}}_{s,t})^{i,j} - (\mathbb{X}^{\mathrm{I}}_{s,t})^{i,j} = [X^i, X^j]^c_{s,t} + \sum_{r \in (s,t]} \Delta_r X^i \Delta_r X^j$$

. is a.s. BV, hence of finite $p/2$-variation for $p > 2$, as required.

• Key steps in the argument:
(1) establish BDG for Marcus lift in uniform norm.
(2) Control moments of p-rough norm with a certain interpolation estimate. Define greedy (stopping times) partition $(\tau^\delta_j)_{j=0}^\infty$ based on "$\geq \delta$-oscillations". Define $\nu(\delta) := \inf\{j \geq 0 \mid \tau^\delta_j = \infty\}$ and exploit

$$||\mathbf{X}||^p_{p\text{-var}} \lesssim \sum_k 2^{-pk} \nu(2^{-k}).$$

(3) Adaption of Lépingle's arguments. □

"General" rough integration with jumps

Theorem (F-Shekhar '15) Consider a càdlàg rough path $\mathbf{X} = (X, \mathbb{X}) \in \mathbf{D}^p$ for $p \in [2, 3)$. Set

$$Y_t = F(X_t), \; Y'_t = DF(X_t)$$

(have notion of càdlàg controlled rough path ...) Set also

$$\tilde{\mathbb{X}}_{s,t} = \mathbb{X}_{s,t} + \Delta_s X \otimes X_{s,t}.$$

Then the following rough integral is well-defined

$$\int_0^T Y_- d\mathbf{X} := \lim_{|\pi| \to 0} \sum_{[s,t] \in \pi} Y_{s-} X_{s,t} + Y'_{s-} \tilde{\mathbb{X}}_{s,t} \sim \sum_{[s,t] \in \pi} Y_s X_{s,t} + Y'_s \mathbb{X}_{s,t}$$

(and gives rise to càdlàg controlled rough path ...)

Rough vs. Itô stochastic integration

As before, $Y_t = F(X_t)$, $Y'_t = DF(X_t)$.

• **Theorem (Chevyrev-F):** Let $X = X(\omega)$ be general (a.k.a. càdlàg) semimartingale with Itô lift $\mathbf{X}^{\mathrm{I}}(\omega) \in \mathbf{D}^p$. Then

$$\int_0^T Y_{s-} dX_s = \int_0^T Y_{s-} d\mathbf{X}_s^{\mathrm{I}} \text{ a.s}$$

Extends to adapted càdlàg controlled rough path (Y, Y'). Implies that forward solution to RDEs, in sense of rough integral equation, with random driving rough path $\mathbf{X}^{\mathrm{I}}(\omega)$ are classical Itô SDE solutions.

- Unifies previous works where X is Brownian motion, continuous semimartingale, Levy process etc ...

RDEs with jumps: basic observations in BV case ($p = 1$)

• Differential equations without solution: let $X_t = 1_{\{t \geq 1\}}$ on $[0, 2]$, and consider the following equation,

$$Y_t = 1 + \int_{(0,t]} Y_s dX_s \text{ (in LS sense)}$$

Evaluation at time 1 gives $Y_1 = 1 + Y_1$. Contradiction.

• Better to study the forward/Itô equation: $Y_t = 1 + \int_{(0,t]} Y_s^- dX_s$, with explicit ("Doléans-Dade") solution

$$Y_t = \exp(X_t) \Pi_{0<s\leq t} \exp^{-\Delta_s X} (1 + \Delta_s X) .$$

• Or geometric solution for Marcus equation: $dY_t = Y_t \diamond dX_t$,

given by $\qquad Y_t = \exp(X_t) .$

RDEs with jumps: basic observations cont'd

• Let Y be either (forward / geometric) solution from the previous slide. Easy to see that (local Lipschitz) continuity of the solution map ("Itô map")

$$\|Y - \bar{Y}\|_{1-var} \lesssim \|X - \bar{X}\|_{1-var}$$

• In case of geometric solution could use ∞-norm but this is an artefact of the scalar situation here.

• Time-change commutes with solution map. Hence trivial to rephrase in Skorokhod J1 variant of 1-variation distance. Applies e.g. to piecewise constant approximations of driver.

• But: if X^{ϵ} denotes a mollification of a (discontinuous) càdlàg BV path X, it cannot possibly converge in J1-type Skorokhod metric. Fortunately, Skorokhod M1-type works.

Rough path metrics

Recall rough path p-variation metric

$$\rho_{p\text{-var};[s,t]}(\mathbf{x},\bar{\mathbf{x}}) = \max_{1\le k\le N} \sup_{\mathcal{D}\subset[s,t]} \left(\sum_{t_i\in D} \left| \mathbf{x}^k_{t_i,t_{i+1}} - \bar{\mathbf{x}}^k_{t_i,t_{i+1}} \right|^{p/k} \right)^{k/p}. \quad (1)$$

Introduce Skorokhod-J1 variant given by

$$\sigma_p(\mathbf{x},\bar{\mathbf{x}}) = \inf_{\lambda\in\Lambda} \max\{|\lambda|, \rho_p(\mathbf{x}\circ\lambda,\bar{\mathbf{x}})\}$$

and Skorokhod-M1 variant given by

$$\alpha_{p\text{-var}}(\mathbf{x},\bar{\mathbf{x}}) = \lim_{\delta\to 0} \sigma_{p\text{-var}}(\mathbf{x}^{\phi,\delta},\bar{\mathbf{x}}^{\bar{\phi},\delta}) \quad (2)$$

where $\mathbf{x}^{\phi,\delta}$ involves δ-stretching time such as to connect all jumps with via *path-function* ϕ. (Example: log-linear connector, think: Marcus)

Limit theorem for canonical RDEs

Definition (Canonical RDE)

Let $\mathbf{x}$ be a geometric càdlàg rough path and ϕ a path-function. For nice vector fields V, let $\tilde{y}$ be the unique solution to the continuous RDE

$$d\tilde{y}_t = V(\tilde{y}_t)d\mathbf{x}_t^{\phi}, \;\; \tilde{y}_0 = y_0 \in \mathbb{R}^e.$$

We define $y := \tilde{y} \circ \tau_{\mathbf{x}} \in D^{p\text{-var}}([0, T], \mathbb{R}^e)$ as solution to the canonical RDE

$$dy_t = V(y_t) \diamond d(\mathbf{x}_t, \phi), \;\; y_0 \in \mathbb{R}^e, \tag{3}$$

where $\tau_{\mathbf{x}}$ "undoes the time-stretching" used to define $\mathbf{x}_t^{\phi}$.
In the particular case that ϕ is the log-linear path function, we call it Marcus canonical RDE and write

$$dy_t = V(y_t) \diamond d\mathbf{x}_t, \;\; y_0 \in \mathbb{R}^e.$$

Limit theorem for canonical RDEs cont'd

• **Theorem (Chevyrev-F '17)** For nice vector fields V, the afore-mentioned canoncial RDE has a unique solution and the solution map

$$(y_0, (\mathbf{x}, \phi)) \mapsto y$$

is locally Lipschitz (w.r.t. $\alpha_{p\text{-var}}$ in driving rough path /path-function). In particular,

$$\lim_{n\to\infty} |y_0^n - y_0| + \alpha_{p\text{-var}}(\mathbf{x}^n, \mathbf{x}) = 0 \tag{4}$$

implies that

$$\sup_n ||y^n||_{p\text{-var}} < \infty, \quad \text{and} \quad \lim_{n\to\infty} y_t^n = y_t \ \text{ for all continuity points } t \text{ of } \mathbf{x}.$$

• Get flow of diffeomorphisms and corresponding continuity results.

• Solution to Marcus canonical RDE $dy_t = V(y_t) \diamond d\mathbf{x}_t$ with random driving rough path $\mathbf{X}^{\mathrm{M}}(\omega)$ yields Marcus canonical SDE solution.

Application to Marcus canonical SDEs

• **Theorem (Chevyrev-F '17)** Let $(X^n)_{n\geq 1}$ be a sequence of semimartingales such that X^n converges in law (resp. in probability) to a semimartingale X in the Skorokhod topology. Suppose moreover that $(X^n)_{n\geq 1}$ satisfies the Kurtz-Protter UCV condition. Then the lifted processes $(\mathbf{X}^n)_{n\geq 1}$ converge in law (resp. in probability) to the lifted process $\mathbf{X}$ in the Skorokhod space $D([0,T], G^2(\mathbb{R}^d))$, and for every $p > 2$, $(||\mathbf{X}^n||_{p\text{-var}})_{n\geq 1}$ is a tight collection of real random variables. (Here all lifts are in Marcus sense, but similar result for Itô lift!)

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• By interpolation, for every $p' > p$ and for ϕ a nice pathfunction, one sees $(\mathbf{X}^n, \phi) \to (\mathbf{X}, \phi)$ in law (resp. in probability) w.r.t. metric $\alpha_{p'\text{-var}}$.

• This applies e.g. to $\mathbf{X}^n :=$ mesh $(1/n)$-piecewise constant approximation of $\mathbf{X}$ and ϕ log-linear, gives Wong-Zakai result for Marcus canonical SDEs driven by general semimartingales à la Kurtz-Protter-Pardoux ('95).

Canonical theory, comments

• Feature of the canonical/Marcus-type approach: can rely on body of results of continuous (rough path) theory

• Recover classical results (such as e.g. Kurtz-Protter-Pardoux) as benchmark, but can easily go beyond. Example: construct geometric càdlàg rough path with arbitrary $p > 1$ directly via suitable Lie-group valued Levy processes (F-Shekhar) or semimartingales ... similar in non-geometric setting (Butcher groups instead of free nilpotent groups)

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• Methodology well-adapted to (also non-linear) SPDEs with càdlàg rough noise because it can be played back - whenever available - to a (rough)pathwise SPDE theory with continuous drivers (think: Lions-Souganidis stochastic viscosity theory, Lions-Perthame-Souganidis stochastic conservation laws with rough flux ... also works of F-Gassiat-Lions-Souganidis, Gess-Souganidis, Hofmanova)

Limit theorem for forward RDEs

• Recall definition of rough integral against càdlàg rough path in terms of compensated RS-sums. Gives meaning to the "forward" rough differential equation

$$Y_t = y_0 + \int_0^t V(Y_s^-) d\mathbf{X}_s. \tag{5}$$

• **Theorem (F-Zhang '17):** For nice vector fields V, this RDE has a unique solution and the solution map

$$(y_0, \mathbf{x}) \mapsto Y$$

is locally Lipschitz, w.r.t. Skorokhod J1 type rough path metirc $\sigma_{p\text{-var}}$. Moreover, have local expansion of solution with estimates.

• Forward solution to RDEs, in sense of rough integral equation, with random driving rough path $\mathbf{X}^{\mathrm{I}}(\omega)$ are classical Itô SDE solutions.

Forward/Itô theory, comments (deterministic)

• Cannot rely on continuous (rough path) theory, redo everything! Behind the scene: a sewing lemma with non-regular controls. Might be useful elsewhere (e.g. rough conservation law context, Deya et al.)

• In contrast to canonical (a.k.a. geometric) theory: local estimates!

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• In contrast to canonical (a.k.a. geometric) theory: local estimates!
• Effectively an Itô-version of Lyons' famous limit theorem. Applies e.g. to $\mathbf{X}^n :=$ mesh $(1/n)$-piecewise constant approximation of $\mathbf{X}$ and so provides an immediate proof of the convergence of (higher-order) Euler schemes to rough differential equations. (Continuous case due to Davie when $p < 3$.)

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• Extension to branched (càdlàg) rough path drivers possible
• Every piecewise constant path can be lifted to branched rough path. Branched DE driven by such object reduces to discrete recurrence relation. Existence and uniqueness trivial in the case.

Forward/Itô theory, comments (deterministic)

• Cannot rely on continuous (rough path) theory, redo everything! Behind the scene: a sewing lemma with non-regular controls. Might be useful elsewhere (e.g. rough conservation law context, Deya et al.)
• In contrast to canonical (a.k.a. geometric) theory: local estimates!
• Effectively an Itô-version of Lyons' famous limit theorem. Applies e.g. to $\mathbf{X}^n :=$ mesh $(1/n)$-piecewise constant approximation of $\mathbf{X}$ and so provides an immediate proof of the convergence of (higher-order) Euler schemes to rough differential equations. (Continuous case due to Davie when $p < 3$.)
• Extension to branched (càdlàg) rough path drivers possible
• Every piecewise constant path can be lifted to branched rough path. Branched DE driven by such object reduces to discrete recurrence relation. Existence and uniqueness trivial in the case.
• However, have a entire scale of p-variation metrics for branched rough path capable of controlling the output in the uniform metric. Applies potentially to quantify stability of (very) deep neuronal networks (joint with C. Bayer). Practicality to be seen ...

Forward RDEs with random drivers

Back to a stochastic setting, $p \in [2,3)$ for simplicity.

Theorem (F-Zhang '17) Given random rough paths $\mathbf{X}^n \to \mathbf{X}$ weakly (or in probability) under the uniform (or Skorokhod) metric, with $\{|||\mathbf{X}^n|||_{p\text{-var};[0,T]} : n \geq 1\}$ tight for some $p < 3$. Assume

$$dY^n = F(Y^n_-) d\mathbf{X}^n$$

and similarly for Y, driven by $d\mathbf{X}$, with the same initial value y_0. Then the random forward RDE solution Y^n converges weakly (or in probability) to Y in uniform (or Skorokhod) sense. Moreover,

$$\{\|Y^n\|_{p,[0,T]}(\omega) : n \geq 1\}$$

is tight and one also has the weakly (or in probability) convergence in p'-variation (or Skorokhod) metric for any $p' > p$.

Random forward solutions, comments (stochastic)

• Recall UCV implies tightness in (rough path) 2^+-variation for Itô-lifted semimartingales.

• Recover precisely the Kurtz-Protter result for convergence of SDEs with UCV approximate semimartingale drivers (benchmark!)

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• Since limit driving rough path is continuous, can do the same with discrete Hölder rough path (Kelly '16; see also Erhard-Hairer's discretisations of regularity structures '17), need $6+$ moments.
• Could be helped with Besov (rough path / model) spaces with integrability $< \infty$, but best to use p-variation (as long as $\dim(t) = 1$, case of rough paths, and certainly in presence of jumps). Should be useful to reduce moment assumptions in the (homogenization) works of (Kelly-)Melbourne '16

My coauthors

This talk based on work with

- **Atul Shekhar** (KTH Stockholm), joint paper *[General Rough integration, Levy Rough paths and a Levy–Kintchine type formula]* to appear in Ann. Prob.
- **Ilya Chevyrev** (Oxford), joint paper *[Canonical RDEs and general semimartingales as rough paths]* on arXiv
- **Huilin Zhang** (Shandong University and TU Berlin), paper *[Differential Equations driven by Rough Paths with Jumps]*, soon on arXiv

Thank you for your attention!