Pseudodifferential operator and Lévy processes

Hausenblas Erika
(joint work with Razafimandimby and Fernando)

Montanuniversität Leben, Austria

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Outline

Definition

An $\mathbb{R}$–valued stochastic process $L = \{L(t) : 0 \leq t < \infty\}$ is a Lévy process over $(\Omega; \mathcal{F}; \mathbb{P})$ iff

- $L(0) = 0$;
- L has independent and stationary increments;
- L is stochastically continuous, i.e. for any $f \in C_b(\mathbb{R}^d)$ the function $t \mapsto \mathbb{E}f(L(t))$ is continuous on $\mathbb{R}_0^+$;
- L has a.s. paths;

Lévy Hincin Formula:

$$\mathbb{E}e^{iL(t)\xi} = e^{\psi(\xi)t},$$

where

$$\psi(\xi) = ib\xi - \xi^T\xi + \int_{\mathbb{R}^d} \left(e^{iy^T\xi} - 1\right) \nu(dy).$$

Nonlinear Filtering - the Lévy case

Given:

- Two independent Brownian motions V and W and two independent Lévy processes L_1 and L_2;
- A signal process $X = \{X(t) : 0 \leq t < \infty\}$;
$$dX(t) = b(X(t))\,dt + \sigma(X(t))\,dW(t) + dL_1(t).$$
- An observable process $Y = \{Y(t) : 0 \leq t < \infty\}$
$$dY(t) = h(X(t))\,dt + dV(t) + dL_2(t).$$

Task:

Let $\phi : \mathbb{R} \to \mathbb{R}$ be a nice function. Given the path of Y up to time t, give an estimate of $\phi(X(t))$.

Nonlinear Filtering - the Lévy case

Definition

Let A_0 be the infinitesimal generator of the Markovian semigroup of X, i.e.

$$A_0 f(x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2} f(x) + b(x)\frac{\partial}{\partial x} f(x) + \int_{\mathbb{R}} (f(x+y) - f(y) - f'(x)y)\, \nu(dy), \quad f \in C^{(2)}(\mathbb{R}).$$

Theorem

Now, under appropriate assumptions one can show that ρ is a solution to the so called Zakai–equation, i.e. we have $\mathbb{Q}$–a.s. for all $t \geq 0$

$$\rho_t(f) = \pi_0(f) + \int_0^t \rho_s(A_0 f)\, ds + \int_0^t \rho_s(\phi h)\, dY_s.$$

Motivation

- Weak error estimates;

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- Support Theorems, Feller property;

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- Nonlocal operators in engineering: Polymers, synthetic materials, etc..

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- Weak error estimates;
- Support Theorems, Feller property;
- Nonlocal operators in engineering: Polymers, synthetic materials, etc..

The talk is also related to Bally's and Litter's talk; It is also related to work of Krylov/Kim who applied pde techniques, and related to a work of Dong, Peszat and Xu.

Outline

Lévy's symbol

Let $B = \{B(t) : t \geq 0\}$ be a Brownian motion. Then the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ defined by

$$\mathcal{P}_t f(x) := \mathbb{E}\left[f(B_t + x)\right]$$

has as infinitesimal generator

$$A_0\, f := \lim_{h\to 0} \frac{1}{h}\left(\mathcal{P}_h - \mathcal{P}_0\right)\, f = \frac{\partial^2}{\partial x^2} f, \quad f \in C_2^{(2)}(\mathbb{R}).$$

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Note, one can write also

$$(A_0 f)(x) = \int_{\mathbb{R}^d} e^{ix^T\xi} \xi^T \xi\, \hat{f}(\xi)\, d\xi.$$

Lévy's symbol

Let $L = \{L_t : t \geq 0\}$ be a Lévy process with Lévy measure. Then the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ defined by

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$$A_0\, f := \lim_{h\to 0} \frac{1}{h} \left(\mathcal{P}_h - \mathcal{P}_0\right)\, f$$

given by

$$(A_0 u)(x) = \int_{\mathbb{R}} e^{ix^T\xi} \psi(\xi) \hat{u}(\xi)\, d\xi \quad u \in C_c^\infty(\mathbb{R}^d),$$

where the symbol ψ is defined by

$$\psi(\xi) := -\lim_{t\downarrow 0} \frac{1}{t} \ln\left(\mathbb{E}\left[e^{iL(t)^T\xi}\right]\right), \quad \xi \in \mathbb{R}^d.$$

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Observe that

$$\psi(\xi) = \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\xi^T y} - 1\right) \nu(dy), \quad \xi \in \mathbb{R}^d.$$

Lévy's symbol

Definition

we call a symbol ψ is of type (ω, θ), $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, iff

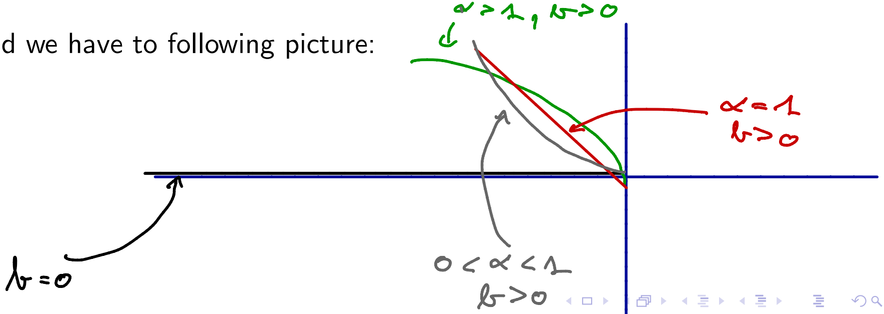
$$-\mathfrak{Rg}(\psi) \subset \mathbb{C} \setminus \omega + \Sigma_{\theta+\frac{\pi}{2}}.$$

Lévy's symbol and pseudo differential operators

Take an α–stable Lévy process with drift b. Then

$$\psi(\xi) = |\xi|^\alpha + ib\xi, \quad \xi \in \mathbb{R},$$

and we have to following picture:



Lévy's symbol

Definition

Let L be a Lévy process with symbol ψ. Then the upper index β^+ and lower index β^- of order k are defined by

$$\beta^+(\psi) := \inf_{\substack{\lambda>0 \\ j\le k}} \left\{ \limsup_{|\xi|\to\infty} (1+|\xi|^2)^{\frac{j}{2}} \frac{|\partial_\xi^j \psi(\xi)|}{|\xi|^\lambda} = 0 \right\}$$

and

$$\beta^-(\psi) := \inf_{\substack{\lambda>0 \\ j\le k}} \left\{ \liminf_{|\xi|\to\infty} (1+|\xi|^2)^{\frac{j}{2}} \frac{|\partial_\xi^j \psi(\xi)|}{|\xi|^\lambda} = 0 \right\}.$$

Lévy's symbol

Relation to the Blumenthal Getoor index:

Let L be a Lévy process with symbol ψ. Then the generalized Blumenthal Getoor index is related to

$$\alpha := \inf_{\alpha>0} \left\{ \lim_{z\to 0} \frac{\nu((z,\infty))}{z^\alpha} < \infty \right\}.$$

(plus negative part)

Lévy's symbol

- Fix $\alpha \in (0,2)$. L be a symmetric α–stable process without drift. Then
$$\psi(\xi) = |\xi|^\alpha,$$
- L be a tempered α–stable process, $\alpha < 1$, then $\nu(A) = \int_{A\cap\mathbb{R}^+\setminus 0} |z|^{-\alpha-1} e^{-\rho\,|z|}\,dz$. and
$$\psi(\xi) \sim \Gamma(-\alpha)\cdot(\rho - i\xi)^\alpha - \rho^\alpha.$$
- L be a tempered α–stable process, then $\nu(A) = \int_{A\setminus 0} |z|^{-\alpha-1} e^{-\rho\,|z|}\,dz$. and
$$\psi(\xi) \sim \Gamma(-\alpha)\,C(\rho)\,|\xi|^\alpha.$$
- L be the Meixner process, then for $m \in \mathbb{R}$, $\delta, a > 0$, $b \in (-\pi,\pi)$.
$$\psi_{m,\delta,a,b}(\xi) = -im\xi + 2\delta\left(\log\cosh\left(\frac{a\xi - ib}{2}\right) - \log\cos\left(\frac{b}{2}\right)\right), \quad \xi \in \mathbb{R},$$
Here the upper and lower index is 1.
- L be the normal inverse Gaussian process; Then
$$\psi_{NIG}(\xi) = -im\xi + \delta\left(\sqrt{a^2 - (b+i\xi)} - \sqrt{a^2 - b^2}\right), \quad \xi \in \mathbb{R},$$
where $m \in \mathbb{R}$, $\delta > 0$, $0 < |b| < a$. The upper and lower index is 1.

Hoh's-Jacob's symbols

Let $B = \{B(t) : t \geq 0\}$ be a Brownian motion and $X = \{X(t,x) : t \geq 0, x \in \mathbb{R}^d\}$ be a solution to a SDE given by

$$dX(t,x) = b(X(t,x))\,dt + \sigma(X(t,x))\,dB(t), \quad X(0,x) = x.$$

Then the Markovian semigroup $(\mathcal{P}_t)_{t\geq 0}$ defined by

$$\mathcal{P}_t f(x) := \mathbb{E}\left[f(X(t,x))\right]$$

has as infinitesimal generator for $f \in C_2^{(2)}(\mathbb{R})$

$$A_0\, f := \lim_{h\to 0} \tfrac{1}{h} \left(\mathcal{P}_h - \mathcal{P}_0\right)\, f = b(x)\nabla f(x) + \sigma(x)\sigma^T(x)\tfrac{\partial^2}{\partial x^2} f.$$

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Note, one can write also

$$(A_0 f)(x) = \int_{\mathbb{R}^d} e^{ix^T\xi}\left(ib(x)\xi + \xi^T a(x)\xi\right)\,\hat{f}(\xi)\,d\xi.$$

with $a(x) = \sigma^T(x)\sigma(x)$.

Hoh's-Jacob's symbols

Let $B = \{L(t) : t \geq 0\}$ be a Lévy process and $X = \{X(t,x) : t \geq 0\}$ be a solution to a SDE given by

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$$A_0\, f := \lim_{h\to 0} \tfrac{1}{h} \left(\mathcal{P}_h - \mathcal{P}_0\right)\, f$$

given by

$$(A_0)(x)f = \int_{\mathbb{R}^d} e^{-ix^T\xi} \left(ib(x)\xi + a(x,\xi)\right) \hat{f}(\xi)\, d\xi,$$

with $a(x,\xi) = \phi(\sigma(x)^T\xi)$.

Outline

Pseudo differential operators

Definition

Let $m \in \mathbb{R}$, and ρ, δ two real numbers such that $0 \le \rho \le 1$ and $0 \le \delta \le 1$. Let $S^m_{\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x, \xi)$ is infinitely often differentiable, i.,e. $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$;
- for any two multi-indices α and β there exists $C_{\alpha,\beta}$ such that

$$\left| \partial_x^\alpha \partial_\xi^\beta a(x, \xi) \right| \le C_{\alpha,\beta} \langle |\xi| \rangle^{m - \rho|\beta|} \langle |x| \rangle^{\delta\alpha}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Definition

Let $a(x, \xi)$ be a symbol. Then, the to $a(x, \xi)$ corresponding operator $a(x, D)$ defined by

$$(a(x, D_x)u)(x) := \int_{\mathbb{R}^d} e^{ix^T \xi} a(x, \xi)\, \hat{u}(\xi)\, d\xi \quad u \in \mathcal{S}(\mathbb{R}^d)$$

is called a pseudodifferential operator.

Pseudo differential operators

Literatur:

- Niel Jacob's three Volumes
- Hoh's habilitation
- Applebaum's book on Lévy processes
- Schilling and Böttcher: Lévy matters III
- Elias Stein: Lectures on pseudodifferential operators
- Treves: pseudodifferential operators and Fourier integrals operators (1980)
- Shubin: pseudodifferential operators and spectral theory (1985/2001)
- Rodinio: Global pseudodifferential operators (2010)
- Abels: Pseudodifferential operators and singular integral operators (2012)

Pseudo differential operators

It is straightforward to show that

$$a(x, D_x) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d).$$

Pseudo differential operators

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$$a(x, D_x) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d).$$

Fix $1 \leq p \leq \infty$. When does it holds that

$$a(x, D_x) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$$

is bounded?

Kernel Representation

The operator can also be represented by a kernel of the form

$$a(x, D_x)f(x) = \int_{\mathbb{R}^d} k(x, x-y)f(y)\, dy, \quad x \in \mathbb{R}^d,$$

where the kernel is given by the inverse Fourier transform

$$k(x, z) = \mathcal{F}_{\xi \to z}\left[a(x, \xi)\right](z).$$

One important estimate is given by

$$|k(x, z)| \le \left|\partial_\xi^\alpha p(x, \xi)\right| \, |z|^{-\alpha}.$$

The Young inequality for convolution gives

$$| \int_{\mathbb{R}^d} k(x, x-y)f(y)\, dy|_{L^q} \le \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} |k(x, z)| \, dz \, |f|_{L^q}.$$

By this estimate one can calculate bounds of the operator between Lebesgue spaces, like

$$|a(x, D_x)f|_{L^q} \le |a|_{\mathcal{A}^0_{\gamma,0;1,0}} |f|_{L^q},$$

for $\gamma \ge d+1$.

Pseudo differential operators

Definition

Let $m \in \mathbb{R}$. Let $\mathcal{A}^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$ be the set of all functions $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, where

- $a(x,\xi)$ is k_1–times differentiable in ξ and k_2 times differentiable in x;
- and for any two multi-indices α and β with $|\alpha| \le k_1$ and $|\beta| \le k_2$, there exists a positive constant $C_{\alpha,\beta} > 0$ depending only on α and β such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \le C_{\alpha,\beta} \langle|\xi|\rangle^{m-\rho|\beta|} \langle|x|\rangle^{\delta|\alpha|}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d.$$

Pseudo differential operators

Semi–norm in $\mathcal{A}^m_{k_1,k_2;\rho,\delta}(\mathbb{R}^d,\mathbb{R}^d)$:

$$\|a\|_{\mathcal{A}_{k_1,k_2;\delta,\rho}{}^m} := \sup_{|\alpha|\le k_1, |\beta|\le k_2} \sup_{(x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)\right| \langle|\xi|\rangle^{\rho|\beta|-m} \langle|x|\rangle^{-\delta|\alpha|}.$$

Pseudo differential operators

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$\Rightarrow$

From the calculations before it follows

$$|a(x,D_x)g|_{L^p} \le C|a|_{\mathcal{A}^0_{d+1,0;1,0}}|f|_{L^p}.$$

Hoh's-Jacob's symbol

Let $\psi : \mathbb{R} \to \mathbb{C}$ a Lévy symbol with Blumenthal Getoor index m of order $d+1$. Then

$$\partial_\xi^\alpha \psi(\sigma(x)^T \xi) = \sigma(x)^\alpha \psi^{(\alpha)}(\xi) =< |\xi| >^{m-\alpha} .$$

Pseudo differential operators

Fix $m \in \mathbb{R}$, $1 \leq p, r \leq \infty$. When does it holds

$$a(x, D_x) : B^{m+\kappa}_{p,r}(\mathbb{R}^d) \to B^{m}_{p,r}(\mathbb{R}^d).$$

for some $\kappa \in \mathbb{R}$?

Besov spaces

Choose a function $\psi \in \mathcal{S}(\mathbb{R}^d)$ such that $0 \le \psi(x) \le 1$, $x \in \mathbb{R}^d$ and

$$\psi(x) = \begin{cases} 1, & \text{if} \quad |x| \le 1, \\ 0 & \text{if} \quad |x| \ge \frac{3}{2}. \end{cases}$$

Then put

$$\begin{cases} \phi_0(x) &= \psi(x),\ x \in \mathbb{R}^d, \\ \phi_1(x) &= \psi(\frac{x}{2}) - \psi(x),\ x \in \mathbb{R}^d, \\ \phi_j(x) &= \phi_1(2^{-j+1}x),\ x \in \mathbb{R}^d, \quad j = 2, 3, \ldots . \end{cases}$$

Definition

Let $s \in \mathbb{R}$, $0 < p \le \infty$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. If $0 < q < \infty$ we put

$$|f|_{B^s_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{sjq} \left|\mathcal{F}^{-1}\left[\phi_j \mathcal{F} f\right]\right|^q_{L^p} \right)^{\frac{1}{q}} = \left\| \left(2^{sj} \left|\mathcal{F}^{-1}\left[\phi_j \mathcal{F} f\right]\right|_{L^p} \right)_{j\in\mathbb{N}} \right\|_{l^q}.$$

with the usual modifications for $p = \infty$.

Pseudo differential operators

Theorem

(see Abels) Fix $m \in \mathbb{R}$, $1 \leq p, r \leq \infty$ and $a \in S^{\kappa}_{1,0}(\mathbb{R}^d, \mathbb{R}^d)$. Then

$$a(x, D_x) : B^{m+\kappa}_{p,r}(\mathbb{R}^d) \to B^{m}_{p,r}(\mathbb{R}^d)$$

is a bounded operator.

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is a bounded operator.

Remark

Tracing step by step of the proof of Theorem ~~6.19~~ before one can see that the following inequality holds for any $k \geq d + 1$

$$|a(x, D_x)f|_{B^{m+\kappa}_{p,r}} \leq |a|_{\mathcal{A}^{\kappa}_{k,0;\delta,0}} |f|_{B^{m}_{p,r}}$$

Pseudo differential operators

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is a bounded operator.

Remark

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$$\left| a(x, D_x) f \right|_{B^{m+\kappa}_{p,r}} \leq \left| a \right|_{\mathcal{A}^{\kappa}_{k,0;\delta,0}} \left| f \right|_{B^{m}_{p,r}}$$

Remark

If no dependence in x, paraproducts (see Bahouri, Chemin and Danchin). If a dependence in x is given, it is more sophisticated but the same idea.

Pseudo differential operators

- Decomposition into a sum of operators

$$a(x, D_x) = a_0(x, D_x) + \sum_{j=1}^{\infty} a_j(x, D_x)$$

where $a_j(x,\xi) = a(x,\xi)\phi_j(\xi)$.

Pseudo differential operators

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  where $a_j(x,\xi) = a(x,\xi)\phi_j(\xi)$.
- Evaluating

$$\phi_k(\xi)a_j(x,\xi) = \phi_k(\xi)a(x,\xi)\phi_j(\xi).$$

  If a is independent of x, then $\phi_k(\xi)a_j(x,\xi) = \phi_k(\xi)a(x,\xi)\phi_j(\xi) = 0$ for $k \neq j-1, j, j+1$.

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- Evaluating

$$\phi_k(\xi)a_j(x,\xi) = \phi_k(\xi)a(x,\xi)\phi_j(\xi).$$

If a is independent of x, then $\phi_k(\xi)a_j(x,\xi) = \phi_k(\xi)a(x,\xi)\phi_j(\xi) = 0$ for $k \neq j-1, j, j+1$.

- $\Rightarrow$ the to $a(x, D_x)$ adjoint operator comes in.

Oscillatory Integral

Let $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Then, let us define

$$\mathrm{Os} - \iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta := \lim_{\epsilon \to 0} \iint \chi(\epsilon y, \epsilon \eta)\, e^{-iy\eta} a(y,\eta)\, dy\, d\eta$$

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Theorem

Let $a \in \mathcal{A}^m_{d+1+m,d+1;1,0}(\mathbb{R}^d, \mathbb{R}^d)$, and let $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Then

$$\mathrm{Os} - \iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta$$

exists and

$$\left| \mathrm{Os} - \iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta \right| \le C_{m,\alpha} |a|_{\mathcal{A}^m_{d+1+m,d+1;1,0}}.$$

Composition of pseudo differential operators

Theorem

Let $a \in S^m_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\chi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$ with $\chi(0,0) = 1$. Then

$$\mathrm{Os} - \iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta := \lim_{\epsilon\to 0} \iint \chi(\epsilon y, \epsilon\eta)\, e^{-iy\eta} a(y,\eta)\, dy\, d\eta$$

exists and

$$\left| \mathrm{Os} - \iint e^{-iy\eta} a(y,\eta)\, dy\, d\eta \right| \le C_{m,\alpha} |a|_{\mathcal{A}^m_{d+1+m,d+1}}.$$

Theorem

- The Fubini Theorem holds for oscillatory integral;
- The Leibniz rule for composition operators works;

Composition of pseudo differential operators

Attention

$$p(x,\xi)q(x,\xi) \neq p(x,\xi)q(x,\xi)$$

Composition of pseudo differential operators

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$$p(x,\xi)q(x,\xi) \neq p(x,\xi)q(x,\xi)$$

Some calculations

$$\begin{aligned}
[p(x,D_x)q(x,D_x)]\, f(x) &= p(x,D_x)\int_{\mathbb{R}^d} e^{-ix^T\xi} q(x,\xi)\hat{f}(\xi)\,d\xi \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-ix^T\xi} p(x,\eta) e^{iy^T\eta} e^{-iy^T\xi} q(y,\xi)\hat{f}(\xi)\,d\xi\,dy\,d\eta \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{i(x-z)\,\xi} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-iy^T\eta} p(x,\xi+\eta)q(x+y,\xi)\,dy\,d\eta\, f(z)\,dz\,d\xi.
\end{aligned}$$

This gives

$$(p\sharp q)(x,\xi) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e^{-iy^T\eta} p(x,\xi+\eta)q(x+y,\xi)\,dy\,d\eta.$$

Composition of pseudo differential operators

The symbol of the composition:

$$(a_1 \sharp a_2)(x,\xi) = \sum_\alpha \frac{1}{\alpha!} \left(\partial_\xi^\alpha a_1(x,\xi) \right) \left(\partial_x^\alpha a_2(x,\xi) \right).$$

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Approximation and its reminder term:

$$(a_1 \sharp a_2)(x,\xi) - \sum_{|\alpha|\le N} \frac{1}{\alpha!} \left(\partial_\xi^\alpha a_1(x,\xi) \right) \left(\partial_x^\alpha a_2(x,\xi) \right)$$

$$= (N+1) \sum_{|\alpha|=N+1} \frac{1}{\alpha!} \text{ Os–} \iint e^{-iy\eta} r_\alpha(x,\xi,y,\eta)\, dy\, d\eta,$$

with

$$r_\alpha(x,\xi,y,\eta)$$

$$= \int_0^1 \left[\partial_{\xi'}^\alpha p_1(x',\xi') \mid_{\substack{\xi'=\xi+\theta\eta \\ x'=x}} \partial_{x'}^\alpha p_2(x',\xi') \mid_{\substack{\xi'=\xi \\ x'=x+y}} (1-\theta)^N \right] d\theta.$$

Elliptic Pseudodifferential operators

Definition

A symbol $a \in S^m_{\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d)$ is called globally *elliptic* in the class $\mathrm{Hyp}^{m,m_0}_{\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d)$, if for some $R > 0$,

$$\langle |\xi| \rangle^{m_0} \lesssim |a(x,\xi)|, \quad |\xi| \geq R,\, x \in \mathbb{R}^d.$$

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Norm:

$$|a|_{\mathrm{Hyp}^{m_0}_{k_1,k_2;\delta,\rho}} = \sup_{\substack{|\alpha|\leq k_1\\ |\beta|\leq k_2}} \sup_x \sup_{|\xi|\geq R} \left| \partial^\alpha_\xi \partial^\beta_x \left[\frac{1}{a(x,\xi)} \right] \right| \langle|\xi|\rangle^{-m_0+|\alpha|\delta} \langle|x|\rangle^{-\rho|\beta|}.$$

Elliptic Pseudodifferential operators

Problem

Given f, when the equation

$$a(x, D_x)u = f,$$

has a solution.

Corollary

Let $a \in \text{Hyp}^{m,m_0}_{\rho,\delta}(\mathbb{R}^d, \mathbb{R}^d)$ be an elliptic symbol. Then there exists some $q \in S^{-m}_{1,0}(\mathbb{R}^d, \mathbb{R}^d)$ such that

$$q(x, D_x)a(x, D_x) = I + r(x, D_x), \quad a(x, D_x)q(x, D_x) = I + r'(x, D_x),$$

with $r, r' \in S^{-1}_{1,0}(\mathbb{R}^d, \mathbb{R}^d)$.

Elliptic Pseudodifferential operators

From the Commutator estimate it follows:

$$r(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha \frac{1}{p(x,\xi)} \partial_x^\alpha p(x,\xi)$$

Reminder term of the first Taylor approximation:

$$r_\alpha(x,\xi,y,\eta) = \int_0^1 \left[\partial_{\xi'}^\alpha 1/p_1(x',\xi') \mid_{\substack{\xi'=\xi+\theta\eta \\ x'=x}} \partial_{x'}^\alpha p_1(x',\xi') \mid_{\substack{\xi'=\xi \\ x'=x+y}} \right] d\theta.$$

Elliptic Pseudodifferential operators

Theorem

(E.H. and Pani, 2017) Let
$a \in \mathcal{A}^{-1}_{2d+3,d+2;1,0}(\mathbb{R}^d \times \mathbb{R}^d) \cap \text{Hyp}^{\kappa}_{d+1,0;1,0}(\mathbb{R}^d, \mathbb{R}^d)$. Then, the operator $a(x, D_x)$ is invertible and we have

$$|u|_{B^{m+\kappa}_{p,r}} \leq C\, |a|_{\mathrm{Hyp}^{\kappa}_{d+1,0;1,0}}\big(1 + |a|_{\mathcal{A}^{0}_{2d+3,d+2;1,0}}\big)|f|_{B^{m}_{p,r}}, \quad f \in B^{m}_{p,r}(\mathbb{R}^d).$$

Elliptic Pseudodifferential operators

Idea

- Decomposition in high and low mode, i.e. $a(x,\xi) = \chi_R(\xi)a(x,\xi) + (1-\chi_R(\xi))a(x,\xi)$;
- Problem to solve $a_1(x,D_x)u = f$ with Ansatz $q(x,\xi) := 1/a_1(x,\xi)$;
- We know

$$q(x,D_x)f = q(x,D_x)a_1(x,D_x)u = [I + r_R(x,D_x)]\, u$$

  - $|q(x,D_x)f|_{B^{m+\kappa}_{p,q}} \le |q(x,D_x)|_{\mathcal{A}^{\kappa}_{d+1,0;1,0}} |f|_{B^m_{p,q}}$;
  - the norm of r_R tends to zero as $R \to \infty$ and it is smoothing.

Analytic semigroups

Definition

Let X be a Banach space and let A be the generator of a degenerate analytic C_0-semigroup on X. We say that A is *of type* (ω, θ, K), where $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $K > 0$, if $\omega + \Sigma_{\frac{\pi}{2}+\theta} \subseteq \rho(A)$ and

$$|\lambda - \omega| \|(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq K \quad \text{for all } \lambda \in \omega + \Sigma_{\frac{\pi}{2}+\theta}.$$

The semigroup of pseudo differential operators

Theorem (see Pazy)

A linear unbounded operator A of a strongly continuous semigroup generates an analytic semigroup in E, if there exists a constant $C > 0$ such that for every $\sigma > 0$, $\tau \neq 0$, we have

$$\|R(\sigma + i\tau; A)\|_{L(E;E)} \leq \frac{C}{|\tau|}.$$

Here $R(\lambda; A)$ denotes the resolvent, i.e. the inverse of $\lambda + A$.

The semigroup of pseudo differential operators

Theorem

(E.H. and Pani) Let $a \in \mathcal{A}^m_{2d+2,d+1;1,0}(\mathbb{R}^d, \mathbb{R}^d) \cap \mathrm{Hyp}^m_{2d+2,d+1;1,0}(\mathbb{R}^d, \mathbb{R}^d)$. Then $a(x, D_x)$ generates an analytic semigroup in $B^s_{p,q}(\mathbb{R}^d)$.

Remark

$p, q = \infty$ does not work, since $\mathcal{S}(\mathbb{R}^d)$ is not dense in $B^s_{p,q}(\mathbb{R}^d)$.

The semigroup of pseudo differential operators

Aim

Let B be a pseudodifferential operator with symbol $b(x,\xi)$. Estimates of

$$|B\mathcal{P}_t f|_{B^m_{p,q}} \lesssim t^{?}|f|_{B^r_{p,q}}$$

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Example

Transition density $p(t,x,y) = \mathcal{P}_t\delta_x$;

$$|\mathcal{P}_t f|_{B^n_{\infty,\infty}} \lesssim t^?|f|_{B^d_{\infty,\infty}}$$

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Problem

The expression $e^{-t\phi(\xi)}$ is nice defined, but in case the symbol depends on x, we were not able to give any meaning to the symbol $e^{-t\phi(\xi)}$.

The semigroup of pseudo differential operators

Representation of the semigroup:

The symbol of the semigroup can be written as follows

$$\mathcal{P}(t,x,\xi) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda : a(x,\xi))\, d\lambda$$

where Γ is the path composed from the two rays $\rho e^{i\delta}$ and $\rho e^{-i\delta}$, $0 < \rho < \infty$ and $\frac{\pi}{2} < \delta < \pi 2\delta$. Γ is oriented so that $\Im\lambda$ increases along Γ.

Analytic semigroups

Proposition

(see Cox and E.H.) Let X be a Banach space. Let A_0 be the generator of a degenerate analytic C_0-semigroup T on X and let B be a possibly unbounded operator acting on X. Suppose A_0 is of type (ω,θ,K) for some $\omega\in\mathbb{R},\theta\in(0,\frac{\pi}{2})$ and $K>0$. Suppose there exist an $\epsilon\in[0,1)$ and a constant $C(A_0,B)$ such that for all $\lambda\in\omega+\Sigma_{\frac{\pi}{2}+\theta}$ one has:

$$\|(\lambda-A_0)^{-1}B\|_{L(X,X)} \quad\le\quad C(A_0,B)|\lambda-\omega|^{\varepsilon-1}.$$

Then for all $t>0$ we have:

$$\|T_0(t)B\|_{L(X,X)} \quad\le\quad 2\Gamma(\varepsilon)[\sin\theta]^{-1}C(A_0,B)\,e^{\omega t}\,t^{-\varepsilon}.$$

Hoh's-Jacob's symbol and pseudo differential operators

Corollary

(P. Razafimandimby, E.H., Pani 2017)

- *Given an infinitesimal operators of a Lévy process A_0 with symbol ψ such that*
    - *ψ is of type (ω, θ);*
    - *has lower index α^- of order $[d/2]$*
- *An pseudodifferential operator B with symbol φ such that φ has upper index β^+ of order $[d/2]$.*

Then

$$\|\mathcal{P}_t Bx\|_{H^s(\mathbb{R})} \leq \frac{C}{\sin\theta}\, t^{-\frac{\beta^+}{\alpha^-}} \|x\|_{H^s(\mathbb{R})}, \quad x \in L^2(\mathbb{R}). \tag{1}$$

where $\mathcal{P} = (\mathcal{P}(t))_{t\geq 0}$ is the Markovian semigroup associated to the Lévy process with infinitesimal generator A_0.

Outline

Strong Feller Property

Theorem

If L is a Lévy process with Blumenthal–Getoor index $0<\delta<2$ *of order* $2d+3$ *and* $\sigma\in C_b^{d+1}(\mathbb{R}^d)$ *is bounded away from zero, then the process defined by*

$$dX(t,x)=\sigma(X(t,x))dL(t);\quad X(0,x)=x,$$

is strong Feller. In particular, we have for any $\gamma<\delta$ *and* $\rho<2\gamma/\delta$

$$|\mathcal{P}_t u|_{C_0^\gamma(\mathbb{R}^d)}=|\mathcal{P}_t u|_{B^\gamma_{\infty,\infty}(\mathbb{R}^d)}\le\frac{K_d}{t^\rho}\,|u|_{L^\infty(\mathbb{R}^d)}\,. \tag{2}$$

Weak error Estimate - one dimension

- Fix a truncation parameter $0 < \epsilon < 1$;
- $\nu_\epsilon[B) := \nu(B \cap (-\infty, \epsilon) \cap (\epsilon, \infty))$, $B \in \mathcal{B}(\mathbb{R})$;
- $\tilde{L}_\epsilon$ be the Lévy process with Lévy measure ν_ϵ;
- W_ϵ be the Wiener process with variance Σ given by

$$\Sigma(\epsilon) = \int_{[-\epsilon,\epsilon]^d} \langle y, y \rangle \nu(dy)$$

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Then

$$\hat{L} = \tilde{L}_\epsilon + W_\epsilon$$

has the following symbol for the Markovian generator

$$\psi_\epsilon(\xi) = \int_{\mathbb{R}} (e^{i\xi y} - 1)\nu_\epsilon(dy) - \Sigma(\epsilon)\xi^2$$

Weak error Estimate

Theorem

Let us assume $\sigma \in C_b^{d+1}(\mathbb{R}^d)$. *If* σ *is bounded away from zero, then for* $\alpha \in (1,2)$, $r_1, r_2 \in (0,1)$ *such that* $r_1 + r_2 > 1$ *and* $2r_1 > r_2$ *with* $\delta_1 = \frac{\alpha r_2}{2}$ *and* $\delta_2 = \alpha(r_1 - \frac{r_2}{2})$,

$$\left\| \mathcal{P}_t - \hat{\mathcal{P}}_t^\epsilon \right\|_{L(B^{-\delta_2}_{\infty,\infty}, B^{\delta_1}_{\infty,\infty})} \le C t^{\alpha(r_1+r_2)-1} \, \epsilon^{(2-\alpha)}.$$

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Idea:

$$\begin{aligned} & R(\lambda : \psi(\sigma(x)\xi)) - R(\lambda : \psi_\epsilon(\sigma(x)\xi)) \\ = \quad & R(\lambda : \psi(\sigma(x)\xi)) \underbrace{[\psi(\sigma(x)\xi - \psi_\epsilon(\sigma(x)\xi]}_{\epsilon^{2-\alpha}} R(\lambda : \psi_\epsilon(\sigma(x)\xi)). \end{aligned}$$

Outline

Remaining work for the future

- Regularity in x - can this regularity be relaxed?
- Relation to criteria coming from Malliavin calculus

Thank you for the attention ☺