Solving high-dimensional nonlinear partial differential equations and high-dimensional nonlinear backward stochastic differential equations using deep learning

What can data science contribute to stochastic analysis?

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Joint works with

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Organized by Tom Cass (Imperial), Dan Crisan (Imperial), & David Applebaum (Sheffield)

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Introduction

Consider $T > 0$, $d \in \mathbb{N}$, $f\colon \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, $g\colon \mathbb{R}^d \to \mathbb{R}$, $u \in C([0,T] \times \mathbb{R}^d, \mathbb{R})$ such that $u(T,x) = g(x)$, $u|_{[0,T)\times\mathbb{R}^d} \in C^{1,2}([0,T) \times \mathbb{R}^d, \mathbb{R})$, and

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}(\Delta_x u)(t,x) + f\big(u(t,x), (\nabla_x u)(t,x)\big) = 0. \qquad \text{(PDE)}$$

for $t \in [0,T)$, $x \in \mathbb{R}^d$. **Goal:** Solve (PDE) approximatively.

Applications: Pricing of financial derivatives, portfolio optimization, operations research

Approximations methods such as finite element methods, finite differences, sparse grids, regression methods suffer under the curse of dimensionality.

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Deep BSDE solver

Consider probability space $(\Omega, \mathcal{F}, \mathbb{P})$, Brownian motion $W\colon [0,T]\times\Omega \to \mathbb{R}^d$, normal filtration $\mathbb{F}$ generated by W, continuous and adapted $Y\colon [0,T]\times\Omega\to\mathbb{R}$ and $Z\colon [0,T]\times\Omega\to\mathbb{R}^d$ such that $\forall\, t\in[0,T]$ $\mathbb{P}$-a.s.:

$$Y_t = g(\xi + W_T) + \int_t^T f(Y_s, Z_s)\,ds - \int_t^T \langle Z_s, dW_s\rangle_{\mathbb{R}^d}. \qquad \text{(BSDE)}$$

Under suitable assumptions (Pardoux & Peng 1990 ...) it holds $\forall\, t\in[0,T]$ $\mathbb{P}$-a.s.:

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$$Y_{t_{n+1}} \approx$$
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(Optimization problem)

We suggest that

$$\Lambda_1 \approx u(0, \xi).$$

Consider stochastic gradient descent-type approximations $\Theta = (\Theta^{(1)}, \ldots, \Theta^{(\rho)}) \colon \mathbb{N}_0 \times \Omega \to \mathbb{R}^\rho$ associated to (Optimization problem). We suggest for sufficiently large $N, \rho, m \in \mathbb{N}$ that

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Numerical simulations

Implementations in PYTHON using TENSORFLOW
on a MACPOOK PRO 2.9 GHz (INTEL i5, 16 GB RAM)

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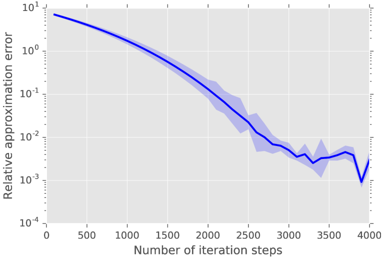
100-dimensional Allen-Cahn equation

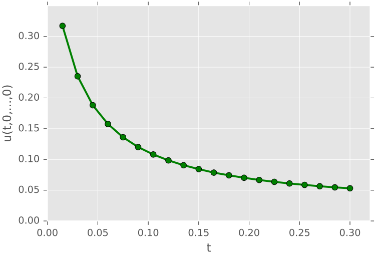
Consider

$$\frac{\partial u}{\partial t}(t,x) = (\Delta_x u)(t,x) + u(t,x) - [u(t,x)]^3 \qquad \text{(Allen-Cahn)}$$

with $u(0,x) = \frac{1}{(2+0.4\|x\|^2)}$ for $t \in [0, \frac{3}{10}]$, $x \in \mathbb{R}^{100}$.



(a) Relative L^1-error for $u(\frac{3}{10}, 0) \approx 0.0528$



(b) Approximative plot of $u(t,0)$, $0 \leq t \leq \frac{3}{10}$

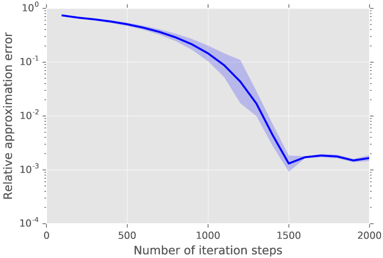
Deep BSDE solver ($N = 20$, $\gamma = \frac{5}{10000}$): L^1**-error:** 0.3%, **Runtime:** 647 **seconds**.

100-dimensional Hamiltonian-Jacobi-Bellmann equation

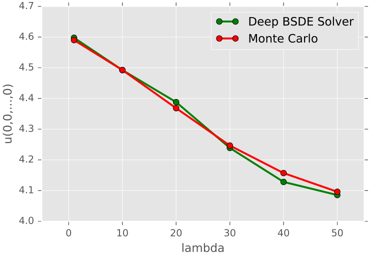
Consider

$$\frac{\partial u}{\partial t}(t,x) + (\Delta u_x)(t,x) - \lambda \|(\nabla_x u)(t,x)\|^2 = 0 \qquad \text{(HJB)}$$

with $u(1,x) = \frac{2}{(1+\|x\|^2)}$, $\lambda \geq 0$ for $t \in [0,1]$, $x \in \mathbb{R}^{100}$.



(a) Relative L^1-error when $\lambda = 1$



(b) Optimal cost against different λ

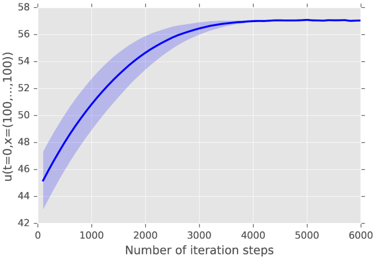
Deep BSDE solver ($N = 20$, $\gamma = \frac{1}{100}$): L^1**-error:** 0.17%, **Runtime:** 330 **seconds**.

100-dimensional nonlinear derivative pricing model

Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF:

$$\frac{\partial u}{\partial t}(t,x) + \bar{\mu}\,\langle x, (\nabla_x u)(t,x)\rangle_{\mathbb{R}^d} + \frac{\bar{\sigma}^2}{2}\sum_{i=1}^{d} |x_i|^2 \frac{\partial^2 u}{\partial x_i^2}(t,x) - Q(u(t,x))\,u(t,x) - Ru(t,x)$$

with $u(1,x) = \min\{x_1,\ldots,x_{100}\}$, $\bar{\mu} = 0.02$, $\bar{\sigma} = 0.2$ for $t \in [0,1]$, $x \in \mathbb{R}^{100}$.



(a) Relative L^1-error for $u(0,100,\ldots) \approx 57.3$

Deep BSDE solver ($N = 40$, $\gamma = \frac{8}{1000}$): L^1**-error:** 0.46%, **Runtime:** 617 **seconds**.

Numerics for forward stochastic differential equations

Theorem (Gerencsér, J, & Salimova 2017 PRSL A (minor revision))

Let $T \in (0, \infty)$, $d \in \{2, 3, 4, \dots\}$, $\xi \in \mathbb{R}^d$, $(a_N)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy $\lim_{N \to \infty} a_N = 0$. Then there exist globally bounded $\mu, \sigma \in \mathcal{C}^\infty(\mathbb{R}^d, \mathbb{R}^d)$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every Brownian motion $W\colon [0, T] \times \Omega \to \mathbb{R}$, every solution $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ of

$$\tfrac{\partial}{\partial t} X_t = \mu(X_t) + \sigma(X_t)\, \tfrac{\partial}{\partial t} W_t, \qquad t \in [0, T], \qquad X_0 = \xi,$$

and every $N \in \mathbb{N}$ we have

$$\inf_{s_1, \dots, s_N \in [0,T]} \inf_{\substack{u\colon \mathbb{R}^N \to \mathbb{R}^d \\ \text{measurable}}} \mathbb{E}\Big[\big\| X_T - u\big(W_{s_1}, \dots, W_{s_N}\big)\big\|\Big] \geq a_N.$$

- **Dimension** $d \geq 4$ and **Euler scheme**: Hairer, Hutzenthaler, & J 2015 AOP
- **Dimension** $d \geq 4$**:** J, Müller-Gronbach & Yaroslavtseva 2016 CMS
- **Weak convergence and** $d \geq 4$**:** Müller-Gronbach & Yaroslavtseva 2016 SAA (to appear)
- **Adaptive approximations and** $d \geq 4$**:** Yaroslavtseva 2016

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Let $T \in (0, \infty)$, $d \in \{2, 3, 4, \ldots\}$, $\xi \in \mathbb{R}^d$, $(a_N)_{N \in \mathbb{N}} \subseteq \mathbb{R}$ *satisfy* $\lim_{N \to \infty} a_N = 0$*. Then there exist globally bounded* $\mu, \sigma \in C^\infty(\mathbb{R}^d, \mathbb{R}^d)$ *such that for every probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, every Brownian motion* $W \colon [0, T] \times \Omega \to \mathbb{R}$*, every solution* $X \colon [0, T] \times \Omega \to \mathbb{R}^d$ *of*

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and every $N \in \mathbb{N}$ *we have*

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Theorem (Hefter & J 2017)

Let $T, \delta, \beta \in (0, \infty)$, $\gamma, \xi \in [0, \infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $W\colon [0,T] \times \Omega \to \mathbb{R}$ be a Brownian motion, let $X\colon [0,T] \times \Omega \to \mathbb{R}$ be a solution of

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Deelstra & Delbaen 1998 Appl Stoch Models Data Anal: Let $Y^N\colon \{0,1,\ldots,N\} \times \Omega \to \mathbb{R}$, $N \in \mathbb{N}$, satisfy for all $N \in \mathbb{N}$, $n \in \{0,1,\ldots,N-1\}$ that $Y_0^N = \xi$ and

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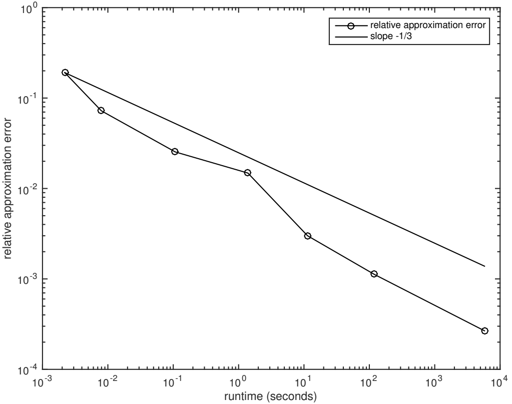

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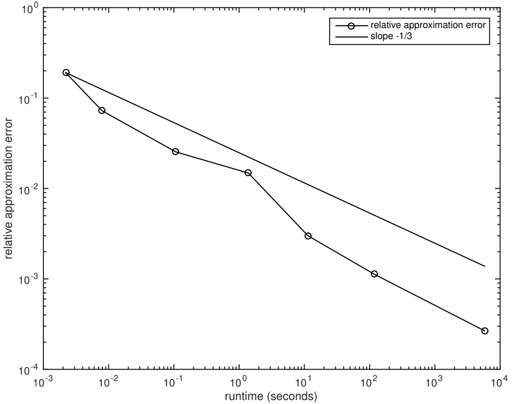

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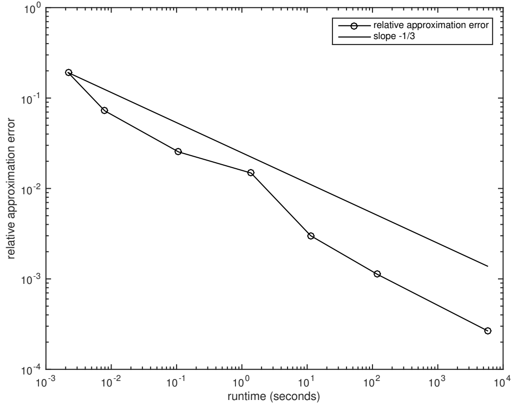

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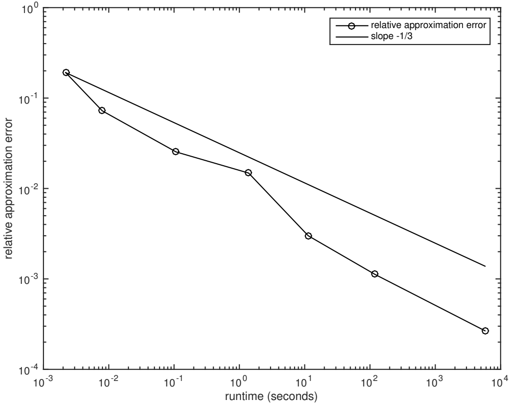

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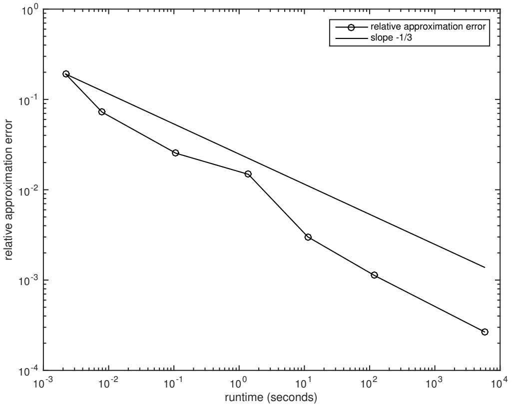

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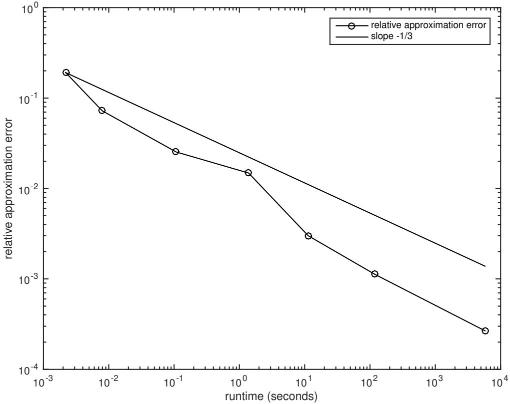

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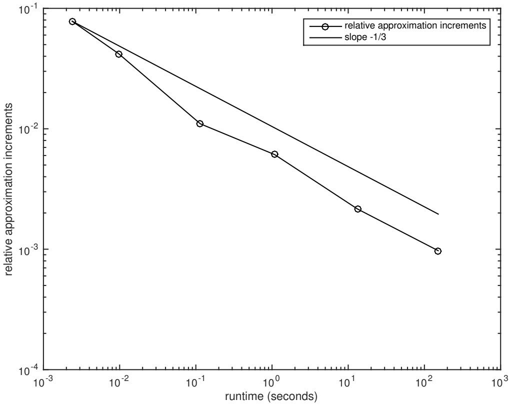
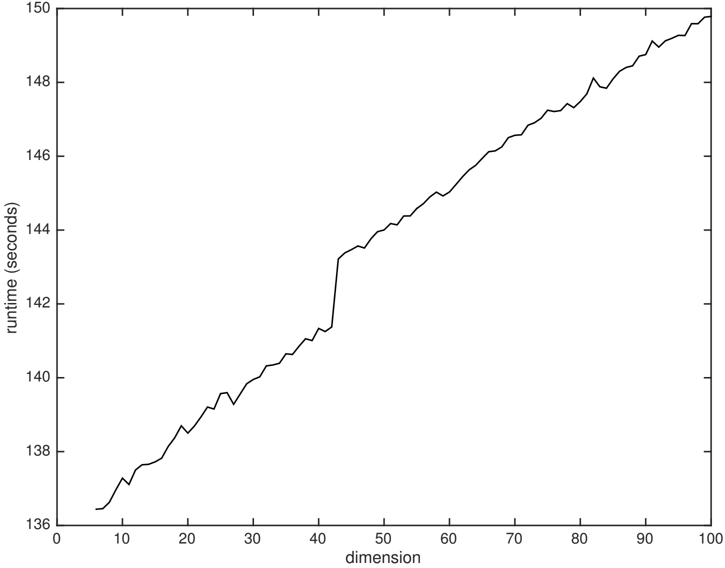


Pricing with default risk Runtime for one realization of $\mathbf{U}^{1}_{6,6}(0,\xi)$ against dimension $d \in \{5, 6, \ldots, 100\}$.



Pricing with default risk (Duffie, Schroder, & Skiadas 1996 AAP, Bender, Schweizer, & Zhuo 2015 MF)

Consider $\delta = \frac{2}{3}$, $R = \frac{2}{100}$, $\gamma^h = \frac{2}{10}$, $\gamma^l = \frac{2}{100}$, $\bar{\mu} = \frac{2}{100}$, $\bar{\sigma} = \frac{2}{10}$, $v^h, v^l \in (0, \infty)$ satisfy $v^h < v^l$, and assume for all $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ that

$$\mu(x) = \bar{\mu} x, \qquad \sigma(x) = \bar{\sigma}\,\mathrm{diag}(x),$$

and

$$f(x, y) = -(1-\delta)\, y \Big[\gamma^h\, \mathbb{1}_{(-\infty, v^h)}(y) + \gamma^l\, \mathbb{1}_{[v^l, \infty)}(y) + \Big[\tfrac{(\gamma^h - \gamma^l)}{(v^h - v^l)} \big(y - v^h\big) + \gamma^h \Big] \mathbb{1}_{[v^h, v^l)}(y) \Big] - Ry.$$

- We consider $v^h = 50$, $v^l = 120$ in the case $d = 1$.
- Bender et al. consider $v^h = 54$, $v^l = 90$ in the case $d = 5$.
- We consider $v^h = 47$, $v^l = 65$ in the case $d = 100$.