Kusuoka-Stroock gradient bounds for the filtering equation

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Kusuoka-Stroock estimates for diffusion semigroups

Kusuoka and Stroock analysed the smoothness properties of the (perturbed) semigroup associated to a diffusion process:

$$(P_t^c \varphi)(x) = \mathbb{E}\left[\varphi(X_t^x) \exp\left(\int_0^t c\left(X_s^x\right) ds\right)\right], \quad t \geq 0, \quad x \in \mathbb{R}^{d_1},$$

where

$$X_t^x = x + \int_0^t V_0(X_s^x) ds + \sum_{i=1}^N \int_0^t V_i(X_s^x) \circ dB_s^i, \quad t \geq 0, \qquad (1)$$

- $\{V_i \quad i = 0, ..., N\}$ are C_b^∞ satisfying Kusuoka's UFG condition.
- B be a N-dimensional standard Brownian motion
- $c \in C_b^\infty\left(\mathbb{R}^{d_1}\right)$

The UFG condition

Notation:

- $$V_{[i]} = V_i, \ V_{[\alpha \star i]} = \left[V_{[\alpha]}, V_i \right], \quad i \in \{0, \ldots, N\},$$
- "lengths" of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ are used:

  $$|\alpha| = |(\alpha_1, \ldots, \alpha_n)| = n, \quad \|\alpha\| = \|(\alpha_1, \ldots, \alpha_n)\| = n + \sharp\{i : \alpha_i = 0\}$$
- $\mathcal{A}_1(m) =$ the set of multi-indices α different from (0) for which $\|\alpha\| \leq m$.

Definition

The vector fields $\{V_i, 0 \leq i \leq N\}$ satisfy *the UFG condition of order m* if for any $\alpha \in \mathcal{A}_1$ there exist $\varphi_{\alpha,\beta} \in \mathcal{C}_b^\infty(\mathbb{R}^{d_1})$ such that

$$V_{[\alpha]} = \sum_{\beta \in \mathcal{A}_1(m)} \varphi_{\alpha,\beta} V_{[\beta]}.$$

The Kusuoka Stroock gradient estimate

Theorem (Kusuoka and Stroock, 1987; Kusuoka 2003)

Suppose the V_i, $i = 0, \ldots, N$ satisfy the UFG condition. For any $j, m > 0$ and $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m \in \mathcal{A}_1$ there there exists $c > 0$ such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} P_t^c \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_{L^p(dx)}$$

$$\leq ct^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_{L^p(dx)}$$

for all $\varphi \in C_0^\infty \left(\mathbb{R}^{d_1} \right)$, $p \in [1, \infty]$, $t \in (0, 1]$.

A randomly perturbed semigroup

Define the randomly perturbed semigroup

$$\rho_t^{Y(\omega)}(\varphi)(x) = \mathbb{E}\left[\varphi(X_t^x) Z_t(X^x, Y) \middle| \mathcal{Y}_t\right](\omega), \ t \geq 0, \ x \in \mathbb{R}^{d_1}, \quad (2)$$

where

$$Z_t(X^x, Y) = \exp\left(\sum_{i=1}^{d_2} \int_0^t h^i(X_s^x)\, dY_s^i - \frac{1}{2}\sum_{i=1}^{d_2} \int_0^t h^i(X_s^x)^2\, ds\right).$$

- $Y = \left\{ \left(Y_t^i\right)_{i=1}^{d_2}, t \geq 0 \right\}$ is a d_2-dim Bm independent of X, $\mathcal{Y}_t = \sigma\{Y_s, s \in [0, t]\}$.
- $h^i \in C_b^\infty\left(\mathbb{R}^{d_2}\right),\ i = 1, ..., d_2$

Application to the filtering equation

- Let $(\Omega, \mathcal{F}, P)$ be the probability space on which we have defined Y.
- Y drives the following linear parabolic stochastic PDE

$$d\rho_t^x(\varphi) = \rho_t^x(A\varphi)dt + \sum_{k=1}^{d_2} \rho_t^x(h_k\varphi)dY_t^k, \tag{3}$$

$$\rho_0^x = \delta_x.$$

- here ρ_t^x is a measure valued process, A is the following differential operator

$$A\varphi = V_0\varphi + \frac{1}{2}\sum_{i=1}^{N} V_i^2\varphi \tag{4}$$

  and φ is a suitably chosen test function.
- equation (3) is called the Duncan-Mortensen-Zakai equation.
- plays a central role in non-linear filtering: The normalized solution of gives the conditional distribution of a partially observed stochastic process.

The non-linear filtering problem

$(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$ probability space, $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions.

- the *signal* process:

$$dX_t = V_0(X_t)dt + \sum_{i=1}^{N} V_i(X_t) \circ dB_t^i, \quad X_0 = x, \quad t \geq 0, \qquad (5)$$

W an $\mathcal{F}_t$-adapted d_2-dimensional Brownian motion independent of X.

- the *observation* process:

$$Y_t = \int_0^t h(X_s)ds + W_t, \qquad (6)$$

$\mathcal{Y}_t = \sigma(Y_s,\ s \in [0,t]) \vee \mathcal{N}$, $\mathcal{N}$ comprises all $\tilde{\mathbb{P}}$-null sets.

The filtering problem. Determine π_t, the conditional distribution of the signal X at time t given Y in the interval $[0,t]$.

The filtering problem continued

Let $\mathbb{P}$ be absolutely continuous with respect to $\tilde{\mathbb{P}}$ such that

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = Z_t(X, Y).$$

$$Z_t(X, Y) = \exp\left(\sum_{i=1}^{d_2}\int_0^t h^i(X_s)\,dY_s^i - \frac{1}{2}\sum_{i=1}^{d_2}\int_0^t h^i(X_s)^2\,ds\right).$$

By Girsanov's theorem, under $\mathbb{P}$, Y is a Brownian motion independent of X; additionally the law of X under $\tilde{\mathbb{P}}$ is the same as its law under $\mathbb{P}$.

Kallianpur-Striebel formula

$$\pi_t = \frac{\rho_t^{Y(\omega)}}{\rho_t^{Y(\omega)}(\mathbf{1})} \quad \tilde{\mathbb{P}}(\mathbb{P}) - \text{a.s.}, \tag{8}$$

$$\rho_t^{Y(\omega)}(\varphi) = \mathbb{E}\left[\varphi(X_t)Z_t(X, Y)\middle|\,\mathcal{Y}_t\right](\omega), \; t \geq 0, \tag{9}$$

- $\mathbf{1}$ is the constant function $\mathbf{1}(x) = 1$ for any $x \in \mathbb{R}^{d_1}$.
- $\rho_t^{Y(\omega)}$ the *unnormalised* conditional distribution the signal.

Main Results

- Suppose the V_i, $i = 0, \ldots, N$ satisfy the UFG condition

Theorem

Let $h \in C_b^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$ and $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_m \in \mathcal{A}_1$. Then there exists $c(\omega)$ a.s. finite such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \rho_t^{Y(\omega)} \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_\infty$$

$$\leq c(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_\infty$$

for any $\varphi \in C_b^\infty\left(\mathbb{R}^{d_1}\right)$ and $t \in (0, 1]$.
If $h \in C_0^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. There exists $c(\omega)$ a.s. finite such that

$$\left\| \left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \rho_t^{Y(\omega)} \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi \right) \right) \right\|_p$$

$$\leq c(\omega)\, t^{-(\|\alpha_1\| + \cdots + \|\alpha_m\|)/2} \|\varphi\|_p$$

for all $\varphi \in C_0^\infty\left(\mathbb{R}^{d_1}\right)$, $p \in [1, \infty]$, $t \in (0, 1]$.

Results continued

Corollary

Let $h \in C_b^\infty\left(\mathbb{R}^{d_1}, \mathbb{R}^{d_2}\right)$. *There exists* $c(\omega)$ *a.s. finite such that*

$$\left\|\left(V_{[\alpha_1]} \cdots V_{[\alpha_j]} \pi_t \left(V_{[\alpha_{j+1}]} \cdots V_{[\alpha_m]} \varphi\right)\right)\right\|_\infty$$

$$\leq c(\omega)\, t^{-(\|\alpha_1\|+\cdots+\|\alpha_m\|)/2} \|\varphi\|_\infty$$

for any $\varphi \in C_b^\infty\left(\mathbb{R}^N\right)$ *and* $t \in (0,1]$.

- $V_i, i = 0, \ldots, N$ satisfy the Hörmander condition: an estimate for the product of the likelihood function and the density of the signal follows

Step 1. Expand $\rho_t^{Y(\omega)}$

- Let $S(k)$ the set of all multi-indices $\bar{q}$ with k entries in the set $\{1, ..., d_2\}$.
- Introduce operators $R_{q,\bar{q}}$ where $q = (t_1, t_2, \ldots, t_k)$ is a non-empty multi-index with entries $0 < t_1 < t_2 < \cdots < t_k < 1$ and $\bar{q} = (i_1, ..., i_{k-1})$ is a multi-index in $S(k-1)$
- $$R_{(s,t),\varnothing}(\varphi) = P_{t-s}(\varphi)$$

  and, inductively, for $k > 1$,

$$\begin{aligned} R_{(s,t_1,t_2,\ldots,t_k),(i_1,\ldots,i_{k-1})}(\varphi) &= R_{(s,t_1,t_2,\ldots,t_{k-1})}\left(h_{i_{k-1}} P_{t_k - t_{k-1}}(\varphi)\right) \\ &= P_{t_1-s}\left(h_{i_1} P_{t_2-t_1} \ldots \left(h_{i_{k-1}} P_{t_k-t_{k-1}}(\varphi)\right)\right. \end{aligned}$$

Lemma

We have almost surely that

$$\rho_t^x(\varphi) = P_t(\varphi)(x) + \sum_{m=1}^{\infty} \sum_{\bar{q} \in S(m)} R_{0,t}^{m,\bar{q}}(\varphi) \tag{10}$$

where, for $\bar{q} = (i_1, \ldots, i_m)$,

$$R_{0,t}^{m,\bar{q}}(\varphi) = \underbrace{\int_0^t \int_0^{t_m} \cdots \int_0^{t_2}}_{m \ \text{times}} R_{(0,t_1,\ldots,t_m,t),\bar{q}}(\varphi)(x) dY_{t_1}^{i_1} \ldots dY_{t_m}^{i_m}.$$

Step 2. Pathwise representation of the $R_t^m(\varphi)$

$$q_{s,t}^{k}(Y) = \underbrace{\int_s^t \int_s^{t_k} \cdots \int_s^{t_2}}_{k \text{ times}} dY_{t_1} \ldots dY^i_{t_k}$$

and $q_{\bar{t}}^{\bar{k}}(Y), \bar{k} = (k_1, ..., k_r)\ t = (t_1, ...t_r)$ be the products of iterated integrals

$$q_{s,\bar{t}}^{\bar{k}}(Y) = \prod_{i=1}^{r} q_{s,t_i}^{k_i}(Y).$$

We define a formal degree on these products of iterated integrals by letting

$$\deg\left(q_{s,\bar{t}}^{\bar{k}}(Y)\right) = \sum_{i=1}^{r} k_i.$$

Next define the sets Θ_k

$$\Theta_k = \mathrm{sp}\left\{ q_{\bar{t}}^{\bar{k}}(Y),\ \ \bar{k} = (k_1, ..., k_r), \sum_{i=1}^{r} k_i \leq k \right\}.$$

Also for $\bar{q} \in S(k)$ let $\bar{q} \in (i_1, ..., i_k)$ define $\Phi_{\bar{q}}$, $\Psi_{\bar{q}}$, be the following operators

$$\begin{aligned} \Phi_{\bar{q}}\varphi &= h^{i_1}...h^{i_k}\varphi \\ \Psi_{\bar{q}}\varphi &= A(h^{i_1}...h^{i_k})\varphi + \sum_{i=1}^{d} V_i(h^{i_1}...h^{i_k})V_i\varphi. \end{aligned}$$

and Γ be the set of operators

$$\Gamma = \{\Phi_{\bar{q}_1}, \Psi_{\bar{q}_2}, \Psi_{\bar{q}_1}\Phi_{\bar{q}_2}, \quad \bar{q}_1, \bar{q}_2 \in S(k),\ k \geq 1\}.$$

Theorem

$$R^m_{s,t}(\varphi) = P_{t-s}(h^m\varphi)(x)\underbrace{\int_s^t\int_s^{t_m}\dots\int_s^{t_2}}_{m\ \text{times}} dY_{t_1}\dots dY_{t_m}$$
$$+\sum_{k=1}^{m-1} q^{k,m}_{s,t}(Y)\underbrace{\int_s^t\int_s^{t_k}\dots\int_s^{t_2}}_{k\ \text{times}} q^{k,m}_{(s,t_1,\dots,t_k)}(Y)\,\bar{R}^k_{(s,t_1,\dots,t_k,t)}(\varphi)(x)dt_1\dots dt_k$$
$$+\sum_{k=1}^{m}\underbrace{\int_s^t\int_s^{t_k}\dots\int_s^{t_2}}_{k\ \text{times}} \bar{q}^{k,m}_{(s,t_1,\dots,t_k)}(Y)\,\hat{R}^k_{(s,t_1,\dots,t_k,t)}(\varphi)(x)dt_1\dots dt_k, \qquad (11)$$

and $q^{k,m}_{(s,t_1,\dots,t_k)}(Y)$, $\bar{q}^{k,m}_{(s,t_1,\dots,t_k)}(Y) \in \Theta_m$ *are linear combinations of (products of) iterated integrals of* Y *and* $\bar{R}^k_{(t_1,\dots,t_k,t)}(\varphi)$ *are given by*

$$\bar{R}^k_{(s,t_1,\dots,t_k,t)}(\varphi) = P_{t_1-s}\left(\bar{\Phi}_1 P_{t_2-t_1}\dots\left(\bar{\Phi}_k P_{t-t_k}(\varphi)\right)\right)$$

and $\bar{\Phi}_i \in \Gamma$, $i = 1,..,k$.

A first a priori estimate

Proposition

Under the assumptions of Theorem 3 let $\alpha, \beta \in A_1(\ell)$, $\gamma \in (1/3, 1/2)$ *then there exists a r.v.* $C(\omega, m, \gamma) > 0$ *a.s. finite such that*

$$\left\| V_{[\alpha]} R^m_{0,t} V_{[\beta]} \varphi \right\|_\infty \leq C(\omega, m, \gamma)\, t^{-(\|\alpha\| + \|\beta\|)/2 + m\gamma} \|\varphi\|_\infty$$

for all $\varphi \in C^\infty_b(\mathbb{R}^N)$ *and* $t \in (0, 1]$.

Proof ideas

- We have Hölder type controls on the iterated integrals

$$\left|q^k_{s,t}(Y)\right| \leq \frac{(c(\omega,\gamma)\,|s-t|)^{k\gamma}}{\theta(k\gamma)!}$$

for all $0 \leq s \leq t \leq 1$,

- estimate the regularity of the integral kernels $\bar{R}^k_{(s,t_1,\ldots,t_k,t)}$ using the Kusuoka-Stroock techniques
- the kernels roughly have the form

$$P_{t_1-t_0} W_1 P_{t_2-t_1} W_2 \cdots W_k P_{t-t_k},$$

where $W_j = u_i V_{i_j} + v_j$ for some $u_i, v_i \in C_b^\infty$

Proof ideas continued

In spirit we have two kinds of regularity estimates for the heat kernel

- first:
$$\|\nabla P_t \varphi\|_\infty \leq C\left(\|\varphi\|_\infty + \|\nabla \varphi\|_\infty\right)$$
- second:
$$\|\nabla P_t \varphi\|_\infty \leq C t^{-\ell/2} \|\varphi\|_\infty$$
- kernels are integrated over a simplex
- use the second estimate on the largest interval of the partition
- this interval is always at least T/k

Proof of the main theorem ct'ed

- the estimate for the R^m follows
- Back to the perturbation expansion: the asymptotics of the series are determined by the lower order terms
- Are we done?
- **no** : the estimates are not summable!

take a step back:

- define Sobolev and distribution type spaces H^1 and H^{-1}: encode the effect of $V_{[\alpha]}$ and $V_{[\beta]}$
- regard the $R^m_{s,t}$ as operators from H^{-1} to H^1
- The $R^m_{s,t}$ satisfy for $0 \leq s < u < t \leq T$

$$R^m_{s,t} = \sum_{j=0}^{m} R^j_{s,u} R^{m-j}_{u,t}$$

  an algebraic relation known as the multiplicative property in the rough path context
- such functionals also arise in the work of Deya, Gubinelli, Tindel et al when analysing the rough heat equation
- the a priori estimates provide us with bounds for the first few R^m
- Can we use the arguments of the extension theorem to get factorial decay?

Rough paths

- introduced by T. Lyons in the 90s to model and analyse the interaction of highly oscillatory and potentially non-differentiable systems
$$dY_t = f(Y_t)dX_t$$
- A rough path $\mathbf{X}$ of order $p \in [2,3)$ with values in a Banach space W is a pair of functions
$$\mathbf{X}_{s,t} := (x_{s,t}, \mathbb{X}_{s,t}) \in W \oplus W \otimes W,$$
where $0 \leq s \leq t \leq T$.
- think of $x_{s,t}$ as the increment of the path x itself and $\mathbb{X}_{s,t}$ as an area term.
- satisfy an analytic $p-$ variation type constraint on increments and area
- satisfy an algebraic constraint
$$\mathbb{X}_{s,t} - \mathbb{X}_{u,t} - \mathbb{X}_{s,u} = x_{s,u} \otimes x_{u,t}$$

Stochastic processes lifted to rough paths

Motivating this definition: The (truncated) signature

$$S_{s,t}(\varphi) = \sum_{j=0}^{\infty} \int_{s<t_1<\cdots<t_j<t} d\varphi_{t_1} \otimes \cdots \otimes d\varphi_{t_j}$$

- A great variety of stochastic processes lift to rough paths e.g. every $\mathbb{R}^d$ valued continuous semi-martingale, a great number of Gaussian processes, etc. If x_t is a semi-martingale we may define

$$\mathbb{X}_{s,t} = \int_{s\leq\tau\leq t} x_\tau \otimes dx_\tau$$

- the choice of the stochastic integration matters!
- obtain a pathwise approach to stochastic calculus

The rough path extension theorem

Theorem (Lyons)

Let $p \geq 1$ and $n \geq \lfloor p \rfloor$ and suppose $X : \Delta_T \to T^n(V)$ is a multiplicative function with finite p -variation controlled by ω. Then for every $m \geq \lfloor p \rfloor + 1$ there exists a unique continuous function $X^m : \Delta_T \to V^{\otimes m}$ such that

$$(s,t) \to X_{s,t} = \left(1, X^1_{s,t}, \ldots, X^{\lfloor p \rfloor}_{s,t}, \ldots, X^m_{s,t}, \ldots\right) \in T((V))$$

is a multiplicative functional with

$$\left\|X^i_{s,t}\right\| \leq \frac{\omega(s,t)^{i/p}}{\theta(i/p)!}$$

for all $i \geq 1$, $(s,t) \in \Delta_T$.

The neo-classical inequality

Theorem (Neo-classical inequality, Lyons 98)

For any $q \in [1,\infty)$, $n \in \mathbb{N}$ *and* $s,t \geq 0$

$$\frac{1}{q^2}\sum_{i=0}^{n}\frac{s^{\frac{i}{q}}t^{\frac{n-i}{q}}}{\left(\frac{i}{q}\right)!\left(\frac{n-i}{q}\right)!} \leq \frac{(s+t)^{n/q}}{(n/q)!}.$$

Consequences of the a priori estimates

Let ℓ be the smallest number for which we can satisfy UFG.

Lemma

For any $0 < \gamma < 1/2$, $m \geq 1$ *there exist random variables* $c(\gamma, m, \omega)$ *such that, almost surely*

$$\left\| R_{s,t}^m \right\|_{H^{-1} \to H^{-1}} \leq c\left(\gamma, m, \omega\right) \left|t - s\right|^{m\gamma}. \tag{12}$$

$$\left\| R_{s,t}^m \right\|_{H^{1} \to H^{1}} \leq c\left(\gamma, m, \omega\right) \left|t - s\right|^{m\gamma}. \tag{13}$$

and finally

$$\left\| R_{s,t}^m \right\|_{H^{-1} \to H^{1}} \leq c\left(\gamma, m, \omega\right) \left|t - s\right|^{m\gamma - 2\ell}. \tag{14}$$

for all $0 < s < t < 1$.

Preliminary factorial decay estimates

Lemma

For any $1/3 < \gamma < 1/2$ there exist a constant $\theta > 0$ and random variables $c(\gamma, \omega)$, almost surely finite, such that

$$\left\| R^n_{s,t} \right\|_{H^1 \to H^1} \leq \frac{\left(c\left(\gamma, \omega\right) |t-s| \right)^{n\gamma}}{\theta \left(n\gamma \right)!}. \tag{15}$$

and

$$\left\| R^n_{s,t} \right\|_{H^{-1} \to H^{-1}} \leq \frac{\left(c\left(\gamma, \omega\right) |t-s| \right)^{n\gamma}}{\theta \left(n\gamma \right)!} \tag{16}$$

for all $n \in \mathbb{N}$, $0 < s < t \leq 1$.

The main decay estimate

Proposition

Under the assumptions of Theorem 3 there exist constants $\theta > 0$, $\gamma' \in (1/3, 1/2)$, $m_0(\gamma', \ell) \in \mathbb{N}$ *and a random variable* $c(\gamma', \omega)$, *almost surely finite, such that*

$$\left\| R^m_{0,t} \right\|_{H^{-1} \to H^1} \leq \frac{\left(c\left(\gamma', \omega\right) t\right)^{m\gamma'}}{\theta\left(m\gamma'\right)!} \tag{17}$$

for all $m \geq m_0$ *and* $t \in (0, 1]$.

Proof ideas

- pick $\gamma' \in (1/3, \gamma)$
- for $m \geq m_0$ sufficiently large $m\gamma - 2\ell \geq m\gamma'$ and

$$\left\| R^m_{s,t} \right\|_{H^{-1} \to H^1} \leq c\left(\gamma, m, \omega\right) |t-s|^{m\gamma'} . \tag{18}$$

- m_0 depends on γ' and ℓ
- the estimate in (18) with $m \in [m_0, 2m_0]$ serves as inductive hypothesis
- construct the extension for $n > 2m_0$

Proof ideas continued

- Two ways to estimate the operator norm $H^{-1} \to H^1$ of the composition $R^m_{s,u} R^m_{u,t}$
- Suppose $m \geq 2m_0$

$$\|R^j_{s,u} R^{m-j}_{u,t}\|_{H^{-1}\to H^1}$$

$$\leq \min\left(\|R^j_{s,u}\|_{H^1\to H^1}\|R^{m-j}_{u,t}\|_{H^{-1}\to H^1}, \|R^j_{s,u}\|_{H^{-1}\to H^1}\|R^{m-j}_{u,t}\|_{H^{-1}\to H^{-1}}\right)$$

- consequence: for $m \geq 2m_0$ we can always find a Hölder type bound for $\|R^j_{s,u} R^{m-j}_{u,t}\|_{H^{-1}\to H^1}$
- use the preliminary factorial decay estimates and the inductive hypothesis
- the missing ingredient to apply the arguments of the extension theorem

Putting it all together

- recall

$$\rho_t^{Y(\omega)}(\varphi) = P_t(\varphi) + \sum_{m=1}^{\infty} R_{0,t}^m(\varphi) \tag{19}$$

- the second, factorially decaying, estimate holds for $m \geq m_0$
- m_0 depends on γ' and the geometry of the problem
- the small time asymptotics of these estimates are not sharp
- consider a mix of the a priori and the factorially decaying estimates
- cut off depends in an explicit way on the number of derivatives we are considering
- the resulting estimate has sharp small time asymptotics

Proof of the L^p estimate

- follow the arguments of Kusuoka and Stroock
- prove an L^1 estimate by duality arguments and deduce the claim using Riesz-Thorin interpolation
-
$$\|\varphi\|_1 = \sup_{\substack{\|g\|_\infty \leq 1 \\ g \in C_0^\infty(R^N)}} \left| \int \varphi g \right| . \tag{20}$$
- identify the (formal) adjoint of the semi-group P_t

Proof of the L^p estimate continued

Let

$$\widetilde{c} = \operatorname{div}(V_0) + \frac{1}{2}\sum_{j=1}^{d} V_j(\operatorname{div}(V_j)) + \frac{1}{2}\sum_{j=1}^{d}(\operatorname{div}(V_j))^2$$

and

$$\widetilde{V}_0 = -V_0 + \frac{1}{2}\sum_{j=1}^{d} V_j(\operatorname{div}(V_j)).$$

Let $\widetilde{X}_t$ be the diffusion associated to the vector fields $\left(\widetilde{V}_0, V_1, \ldots, V_d\right)$. Set

$$P_t^*\varphi(x) := E\left(\exp\left(\int_0^t \widetilde{c}\left(\widetilde{X}_s^x\right) ds\right)\varphi\left(\widetilde{X}_t^x\right)\right).$$

Theorem (Kusuoka-Stroock)

Let $\varphi \in C_0^\infty\left(R^N\right)$ *and* $g \in C_0^\infty\left(R^N\right)$ *then we have*

$$\int P_t\varphi\left(x\right)g\left(x\right)dx = \int \varphi\left(x\right)P_t^* g\left(x\right)dx,$$

i.e. the semi group P_t^* *is the (formal) adjoint to* P_t.

- iteratively use the adjoint relation on each term R^m in the perturbation expansion
- conclude the estimate by applying (appropriately modified versions of) the forward estimates