Quasiconformal Mappings and two-dimensional diffusion processes

joint work with N. Ikeda

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1 One-dim diffusion processes and aim of talk

Let $X = \{X_t\}$ be a diffusion process generated by

$$L = \frac{1}{2}a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} = \frac{d}{dm}\frac{d}{ds}.$$

$m(dx)$ is the speed measure and $s(x)$ is the scale function.

♣ $\{s(X_t)\}$ is a time-changed one-dimensional Brownian motion.

The aim of this talk is to show that,

for a class of 2-dim diffusion process $\{X_t\}$, $\exists\Phi$, a function (quasiconformal mapping),

such that $\{\Phi(X_t)\}$ is a time-changed Brownian motion.

2 Some of related works

(1) P.Lévy. $g : \mathbf{C} \supset U \to \mathbf{C}$ analytic, $\{B_t\}$: BM with $B_0 \in U$

Then, $\{g(B_t)\}$ is a time-changed BM in $\mathbf{C}$.

(2) Csink and Øksendale (1983) et al. For diffusion processes $\{X_t\}$, $\{Y_t\}$ on $\mathbf{R}^d$, $\mathbf{R}^p$,

characterization a mapping φ such that $\{\varphi(X_t)\}$ is a time change of $\{Y_t\}$ is given

by means of harmonic morphisms

and other characterizations of φ are given.

Given such φ, the corresponding diffusion is used to study φ (e.g., boundary value)

(3) Canonical forms of 2nd order ellptic operators (Courant-Hilbert, Petrovsky)

♣ We concentrate on the case where $d = p = 2$ and $\{Y_t\}$ is a BM.

We construct the mapping φ for a class of $\{X_t\}$ by using quasiconformal mappings.

3 Rough story

Let $ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2$ be a Riemannian metric on $U \subset \mathbf{R}^2$,

Assumption. (g_{ij}) is bounded, measurable, uniformly elliptic.

Then we always have a coordinate system (u, v) on U such that

$$ds^2 = \rho(u, v)(du^2 + dv^2) \qquad \text{for some function } \rho(u, v) > 0$$

(u, v) is an isothermal coordinate and

$\Phi(x, y) = (u(x, y), v(x, y))$ is a quasiconformal mapping.

The Laplace-Beltrami operator is $\dfrac{1}{\rho(u, v)}\Big(\dfrac{\partial^2}{\partial u^2} + \dfrac{\partial^2}{\partial v^2}\Big)$ (2-dim !)

and the corresponding diffusion is a time-changed Brownian motion.

4 Plan

1. Isothermal coordinate (Beltrami equation)

2. Dirichlet forms

3. 2-dim diffusions with drifts

4. Remarks on 3-dim spaces

5 Isothermal coordinate (Beltrami equation)

Write in complex form

$$ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2 = \left(\sqrt{g_{11}}dx + \frac{g_{12}}{\sqrt{g_{11}}}dy\right)^2 + \left(\frac{\sqrt{G}}{\sqrt{g_{11}}}dy\right)^2$$
$$= \left(\sqrt{g_{11}}dx + \frac{g_{12} + \mathrm{i}\sqrt{G}}{\sqrt{g_{11}}}dy\right)\left(\sqrt{g_{11}}dx + \frac{g_{12} - \mathrm{i}\sqrt{G}}{\sqrt{g_{11}}}dy\right),$$

($G = \det(g_{ij}) = g_{11}g_{22} - g_{12}^2$) and set

$$\lambda = g_{11} + g_{22} + 2\sqrt{G}, \quad \alpha = \frac{g_{11} - g_{22}}{\lambda} \qquad \text{and} \qquad \beta = \frac{2g_{12}}{\lambda}.$$

Put $\mu = \alpha + \mathrm{i}\beta$. Then

$$ds^2 = \frac{\lambda}{4}|dz + \mu d\overline{z}|^2,$$

where, as usual, $z = x + \mathrm{i}y$ and $\overline{z} = x - \mathrm{i}y$.

We have $ds^2 = \dfrac{\lambda}{4}|dz + \mu d\overline{z}|^2$ and

$$|\mu|^2 = \frac{g_{11} + g_{22} - 2\sqrt{G}}{g_{11} + g_{22} + 2\sqrt{G}} \leqq {}^{\exists}k < 1.$$

Then, we see that the equation, called the Beltrami equation,

$$\frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z} \qquad \text{or} \qquad \mu = \frac{w_{\overline{z}}}{w_z},$$

has a unique solution w $\big(w(O) = 0$ for a fixed $O \in U\big)$ with $w_z \in L^p(U)$ $(\exists p > 2)$

and, setting

$$\rho = \frac{1}{4}\lambda \left|\frac{\partial w}{\partial z}\right|^2 \qquad \text{and} \qquad w = u + \mathrm{i}v,$$

we obtain

$$ds^2 = \frac{\lambda}{4}|dz + \mu d\overline{z}|^2 = \rho|w_z dz + w_{\overline{z}} d\overline{z}|^2 = \rho|dw|^2 = \rho(du^2 + dv^2)$$

and w is the desired quasiconformal mapping or the isothermal coordinate.

6 Dirichlet forms

We have shown ($x_1 = x, x_2 = y, u_1 = u, u_2 = v$)

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} dx_i dx_j = \rho(du_1^2 + du_2^2), \quad \text{where } w = u_1 + \mathrm{i}u_2 \text{ satisfies } \frac{\partial w}{\partial \overline{z}} = \mu \frac{\partial w}{\partial z}.$$

The Jacobian is

$$\frac{\partial(u_1, u_2)}{\partial(x_1, x_2)} = \frac{1}{\rho}\sqrt{G}, \qquad G = g_{11}g_{22} - g_{12}^2.$$

Theorem. For any C^1 functions φ and ψ on U with compact support,

$$\sum_{i,j=1}^{2} \int_U g^{ij}(x_1, x_2) \frac{\partial \varphi}{\partial x_i}(x_1, x_2) \frac{\partial \psi}{\partial x_j}(x_1, x_2) \sqrt{G}(x_1, x_2) dx_1 dx_2$$

$$= \sum_{i=1}^{2} \int_{\Phi(U)} \frac{\partial \varphi}{\partial u_i}(u_1, u_2) \frac{\partial \psi}{\partial u_i}(u_1, u_2) du_1 du_2.$$

Extention to a subspace of $L^2(\sqrt{G} dx_1 dx_2) = L^2(\rho du_1 du_2)$!!

7 2-dim diffusions with drifts

Let U be convex and L is given by

$$L = \lambda(x,y)\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) + b_1(x,y)\frac{\partial}{\partial x} + b_2(x,y)\frac{\partial}{\partial y}.$$

Fix y and define the functions s_y and m_y on U by

$$s_y(x) = \int_c^x \exp\Big(-\int_c^v \frac{b_1(u,y)}{a(u,y)}du\Big)dv, \quad \text{and} \quad m'_y(x) = \frac{1}{a(x,y)}\exp\Big(\int_c^x \frac{b_1(u,y)}{a(u,y)}du\Big).$$

Then, setting $\xi = s_y(x)$, we have

$$L = a_1(\xi,y)\frac{\partial^2}{\partial \xi^2} + \widetilde{a}(\xi,y)\frac{\partial^2}{\partial y^2} + \widetilde{b}_2(\xi,y)\frac{\partial}{\partial y},$$

where

$$a_1(\xi,y) = \frac{s'_y(s_y^{-1}(\xi))}{m'_y(s_y^{-1}(\xi))}, \quad \widetilde{a}(\xi,y) = a(s_y^{-1}(\xi),y) \quad \text{and} \quad \widetilde{b}_2(\xi,y) = b_2(s_y^{-1}(\xi),y).$$

Next fix ξ. Then, in the same way, L is transformed into the form

$$L = a_1(\xi, \widehat{s_\xi}^{-1}(\eta))\frac{\partial^2}{\partial \xi^2} + \widehat{a}_2(\xi, \eta)\frac{\partial^2}{\partial \eta^2}.$$

Finally, using the result on the isothermal coordinate again, we see that there exists a coordinate $(\widetilde{x}, \widehat{y})$ and a function $a^*(\widehat{x}, \widehat{y})$ satisfying

$$L = a^*(\widehat{x}, \widehat{y})\Big(\frac{\partial^2}{\partial \widehat{x}^2} + \frac{\partial^2}{\partial \widehat{y}^2}\Big).$$

8 Remarks on 3-dim spaces

$\sum_{i,j=1}^{3} g_{ij} dx_i dx_j$ has an isothermal coordinate $\Longleftrightarrow$ The Cotton tensor R_{ijk} is zero, where

$$R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{4}(g_{ik}R_{,j} - g_{ij}R_{,k}), \qquad R : \text{scalar curvature}$$

A map $\Phi : M \to N$ is a harmonic morphism (preserves the harmonic structure)

if and only if Φ is harmonic and conformal ($^\exists$ isothermal coordinate).

References. [1] L.P.Eisenhart, Riemannian Geometry, 1925.

[2] L.V.Ahlfors, Lectures on Quasiconformal Mappings, 1966.

[3] B. Fuglede, Harmonic morphisms between Riemannian manifolds, Fourier (1978).

(1) Riemannian metric $^2 = dx^2 + dy^2 + f(x,y)^2 dz^2$ has an isothermal coordinate iff

$$-f_{xx}f_y + 2f_{xy}f_x + y_{yy}f_y - (f_{xx}+f_{yy})_y f = 0,$$

$$-f_{yy}f_x + 2f_{xy}f_y + f_{xx}f_x - (f_{xx}+f_{yy})_x f = 0.$$

(i) $f(x,y) = 1 + x^2 + y^2$ OK (ii) $f(x,y) = \sqrt{1+x^2+y^2}$ does not satisfy.

(2) Let $(B_1(t), B_2(t), B_3(t))$ be a 3-dim BM and

$S(t) = \dfrac{1}{2}\displaystyle\int_0^t (B_2(s)dB_1(s) - B_1(s)dB_2(s))$ be the stochastic area of $B_1(t)$ and $B_2(t)$.

♣ $(B_1(t), B_2(t), B_3(t) + S(t))$ cannnot be written as a time-change of 3-dim BM,

because this process corresponds to the metric

$$ds^2 = (1 + \frac{1}{4}y^2)dx^2 + (1 + \frac{1}{4}x^2)dy^2 + dz^2 - \frac{1}{2}xydxdy + xdydz - ydxdz$$

and the Cotton tensor of this metric is not zero.