Yang–Mills measure and the master field on the sphere

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James Norris – joint work with Antoine Dahlqvist

University of Cambridge

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Yang–Mills measure

Yang–Mills measure is a probability measure on connections, motivated by physical gauge theories, given formally by

$$\mu_T(d\omega) \propto e^{-S(\omega)/T} D\omega$$

where T is a positive parameter, S is the *Yang–Mills action*

$$S(\omega) = \frac{1}{2} \int_M \|\Omega\|^2 d\sigma$$

and Ω is the curvature of the connection ω

$$\Omega(X, Y) = d\omega(X, Y) + [\omega(X), \omega(Y)].$$

Here $D\omega$ is a formal 'translation-invariant measure' on connections.

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This has been formulated as a rigorous object when the underlying space M is two-dimensional.

For comparison ...

Wiener measure of speed T is a probability measure on paths, given formally by

$$\mu_T(dx) \propto e^{-E(x)/T} Dx$$

where E is the kinetic energy

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But note that

$$\{E = 0\} = \text{ constant paths}, \quad \{S = 0\} = \text{ flat connections}$$

and the second space is much bigger.

Outline

- Yang–Mills holonomy field $H : \{\text{paths}\} \to U(N)$
- Small-area limit $T \to 0$
- High-dimensional limit $N \to \infty$
- Master field $\Phi_T : \{\text{loops}\} \to [-1, 1]$
- Makeenko–Midgal equations
- Discrete Coulomb gas
- Characterization of the master field
- High-dimensional limit of the Brownian bridge in $U(N)$

Yang–Mills holonomy fields

- M a compact $d = 2$ smooth manifold (e.g. the sphere $\mathbb{S}$)
- $T \in (0, \infty)$, σ a smooth positive probability measure on M
- G a compact Lie group, with Lie algebra $\mathfrak{g}$ (e.g. $U(N)$, $\mathfrak{u}(N)$)
- $\|.\|$ an invariant metric on $\mathfrak{g}$ (e.g. $\|g\|^2 = N \sum_{i,j=1}^{N} |g_{ij}|^2$)

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- $P(M)$ set of rectifiable (continuous) paths in M
- $\mathcal{M}(P(M), G)$ set of *multiplicative functions*

$$h : P(M) \to G, \quad h_{\gamma_1\gamma_2} = h_{\gamma_2} h_{\gamma_1}$$

  where $\gamma_1\gamma_2$ is the extension of γ_1 by γ_2
- $(p_t(g) : t \in (0, \infty), g \in G)$ heat kernel on G associated to $\|.\|$

A random process $H = (H_\gamma : \gamma \in P(M))$ is a *Yang-Mills holonomy field* in G of parameter T if

(a) $H(\omega) \in \mathcal{M}(P(M), G)$ for all $\omega \in \Omega$

(b) for any discretization (V, E, F) of M

$$\mathbb{P}(H_e \in dh_e \text{ for all } e \in E) \propto \prod_{f \in F} p_{T\sigma(f)}(h_{\partial f}) \prod_{e \in E} dh_e$$

(c) $H(\gamma_n) \to H(\gamma)$ in probability whenever $\gamma_n \to \gamma$ in 1-variation with fixed endpoints.

Theorem (Lévy 2003, Driver 1989, Sengupta 1997)

There is a unique probability measure μ_T on $\mathcal{M}(P(M), G)$ under which the coordinate process $H_\gamma(h) = h_\gamma$ is a Yang–Mills holonomy field in G of parameter T.

For comparison ...

A random process $B = (B_t : t \in [0,1])$ is a Brownian bridge in G from 1 to 1 at speed T if

(a) $B(\omega) \in C([0,1], G)$ for all $\omega \in \Omega$

(b) for any partition $0 < t_1 < \cdots < t_{n-1} < 1$, setting $g_0 = g_n = 1$ and $s_k = t_k - t_{k-1}$, where $t_0 = 0$ and $t_n = 1$,

$$\mathbb{P}(B_{t_k} \in dg_k \text{ for all } k) \propto \prod_{k=1}^{n} p_{Ts_k}(g_k g_{k-1}^{-1}) \prod_{k=1}^{n-1} dg_k.$$

In fact there are many such Brownian bridges embedded in a Yang–Mills holonomy field $(H_\gamma : \gamma \in P(\mathbb{S}))$ in G of parameter T ...

Large deviations of the Yang–Mills measure in the small-area limit

We give $\mathcal{M}(P(M), G)$ the weakest topology making the coordinate maps continuous. Thus $h(n) \to h$ iff $h_\gamma(n) \to h_\gamma$ for all paths γ.

Theorem (Lévy & N. 2005)

In the limit $T \to 0$, the family of Yang–Mills measures $(\mu_T : T \in (0, \infty))$ satisfies a large deviations principle with speed T and rate function S.

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This makes a rigorous link between the Yang–Mills measure and the Yang–Mills action similar to that made by Schilder's theorem between Wiener measure and the kinetic energy.

The Yang-Mills measures disintegrate over bundle topologies: the LDP holds also conditional on the bundle topology.

High-dimensional limit of the Yang–Mills holonomy field

Let $(H_\gamma : \gamma \in P(\mathbb{S}))$ be a Yang-Mills holonomy field in $U(N)$ of parameter T. Write $L(\mathbb{S})$ for the set of loops in $P(\mathbb{S})$. Set

$$\mathrm{tr}(g) = \frac{1}{N}\sum_{i=1}^{N} g_{ii}.$$

Theorem

There is a function $\Phi_T : L(\mathbb{S}) \to \mathbb{C}$ such that

$$\mathrm{tr}(H_\ell) \to \Phi_T(\ell)$$

in probability as $N \to \infty$ for all $\ell \in L(\mathbb{S})$.

The function Φ_T is the *master field on the sphere*.

Brian Hall has independently obtained such a statement for regular loops, conditional on its validity for simple loops.

Easy properties of the master field

The master field inherits a number of properties from its finite N approximations

- $\Phi_T = 1$ on constant loops
- $\Phi_T(\gamma_1\gamma_2) = \Phi_T(\gamma_2\gamma_1)$ whenever $\gamma_1\gamma_2 \in L(\mathbb{S})$
- Φ_T is invariant under reduction: $\Phi_T(\ell_1) = \Phi_T(\ell_2)$ whenever $\ell_1 \sim \ell_2$
- $\Phi_T(\theta(\ell)) = \Phi_T(\ell)$ whenever θ is an area-preserving diffeomorphism of $\mathbb{S}$.

Here, we write $\ell_1 \sim \ell_2$ if ℓ_1 and ℓ_2 have a common reduction ℓ_0, where ℓ_0 is a reduction of ℓ if it may be obtained by cutting finitely many treelike paths from ℓ.

Makeenko–Migdal equations

Given a regular loop ℓ and a point v of self-intersection of ℓ, let

$$(\theta(\tau, .) : \tau \in (-\varepsilon, \varepsilon))$$

be a *Makeenko–Migdal flow* at (ℓ, v), that is, a smooth family of diffeomorphisms of $\mathbb{S}$ which preserve the areas of all faces of ℓ, except for the faces $f_1, \dots, f_4$ around v, for which we have

$$\frac{d}{d\tau}\sigma(\theta(\tau, f_i)) = (-1)^{i+1}.$$

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Let $(H_\gamma : \gamma \in P(\mathbb{S}))$ be a Yang-Mills holonomy field in $U(N)$ of parameter T.

Theorem (Driver, Gabriel, Hall & Kemp 2016)

Set $\ell(\tau) = \theta(\tau, \ell)$ and write $\ell_v, \hat{\ell}_v$ for the loops obtained by splitting ℓ at v. Then

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \mathbb{E}(\mathrm{tr}(H_{\ell(\tau)})) = T\mathbb{E}(\mathrm{tr}(H_{\ell_v})\,\mathrm{tr}(H_{\hat{\ell}_v})).$$

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On letting $N \to \infty$, we deduce that the master field Φ_T satisfies the *Makeenko-Migdal equations*

$$\left.\frac{d}{d\tau}\right|_{\tau=0} \Phi_T(\ell(\tau)) = T\Phi_T(\ell_v)\Phi_T(\hat{\ell}_v).$$

Representation by a discrete Coulomb gas

Let $\ell \in L(\mathbb{S})$ be a simple loop which divides $\mathbb{S}$ into components of areas a and b. Then, for all $m, n \in \mathbb{Z}$,

$$\mathbb{E}(\mathrm{tr}(H_\ell^{-m})\,\mathrm{tr}(H_\ell^n)) = \mathbb{E}(I_m^a(\Lambda) I_n^b(\Lambda)).$$

Here Λ is the discrete Coulomb gas in $\mathbb{Z}$ given by

$$\mathbb{P}(\Lambda = \lambda) \propto \prod_{1 \leqslant j < k \leqslant N} (\lambda_j - \lambda_k)^2 \prod_{j=1}^N e^{-N\lambda_j^2 T/2}$$

where λ runs over increasing sequences $(\lambda_1, \ldots, \lambda_N)$ in $\mathbb{Z}$.

Also, for $a \in [0,1]$, $I_0^a(\lambda) = 1$ and, for $n \in \mathbb{Z} \setminus \{0\}$,

$$I_n^a(\lambda) = \frac{e^{-aTn^2/(2N)}}{2\pi i n} \int_\gamma \exp\{-n(aTz - G_\lambda^{N/n}(z))\} dz$$

where γ is a contour around the set $[\lambda_1, \lambda_N] + \{|z| \leqslant |n|/N\}$ and

$$G_\lambda^\alpha(z) = \frac{\alpha}{N} \sum_{j=1}^N \mathrm{Log}\left(1 + \frac{1}{\alpha(z - \lambda_j)}\right).$$

Large deviations of the Coulomb gas

For $\mu \in \mathcal{M}_1(\mathbb{R})$, set

$$\mathcal{I}_T(\mu) = \int_{\mathbb{R}^2} \left\{ (x^2 + y^2) T + \log |x - y| \right\} \mu(dx)\mu(dy)$$

if $\mu([a, b]) \leqslant b - a$ for all intervals $[a, b]$, and set $\mathcal{I}_T(\mu) = \infty$ otherwise.

Theorem (Guionnet & Maïda 2005)

The laws of the empirical distributions

$$\mu_\Lambda = \frac{1}{N} \sum_{i=1}^{N} \delta_{\Lambda_i}$$

on $\mathcal{M}_1(\mathbb{R})$ satisfy a large deviations principle with speed N^2 and rate function $\mathcal{I}_T$.

Bulk scaling limit of the Coulomb gas

Theorem (Lévy & Maïda 2015)

The functional $\mathcal{I}_T$ has a unique minimizer μ_T on $\mathcal{M}_1(\mathbb{R})$, which has a continuous, symmetric, unimodal and compactly supported density ρ_T with respect to Lebesgue measure, with $\rho_T(x) \in [0, 1]$ for all x.

For $T \in (0, \pi^2]$,

$$\rho_T(x) = \frac{T}{2\pi}\sqrt{\frac{4}{T} - x^2}, \quad |x| \leqslant \frac{2}{\sqrt{T}}.$$

For $T \in (\pi^2, \infty)$, the density ρ_T may be expressed in terms of the complete elliptic integrals K and E of the first and second kind. In particular, there is a non-trivial interval around 0 where $\rho_T = 1$.

The master field on simple loops

Set

$$G_T(z) = \int_{\mathbb{R}} \frac{\rho_T(x)}{z - x} dx.$$

The following limit holds in probability as $N \to \infty$ for all $n \in \mathbb{N}$

$$\begin{aligned} I_n^a(\Lambda) &\to \frac{1}{2\pi i n} \int_\gamma \exp\{-n(aTz - G_T(z))\} dz \\ &= \frac{2}{n\pi} \int_0^\infty \cosh\{n(a-b)Tx/2\} \sin\{n\pi\rho_T(x)\}\, dx. \end{aligned}$$

So, by the representation formula, for any simple loop ℓ which divides $\mathbb{S}$ into components of areas a and b, $\mathrm{tr}(H_\ell^n)$ also converges in probability, with the same limit.

Characterization of the master field on the sphere

Theorem

The master field Φ_T has the following properties, which characterize it uniquely among functions $L(\mathbb{S}) \to \mathbb{C}$:

(a) *Φ_T is continuous for the 1-variation topology on $L(\mathbb{S})$*

(b) *Φ_T is invariant under reduction*

(c) *Φ_T satisfies the Makeenko-Migdal equations*

(d) *for all simple loops ℓ, dividing $\mathbb{S}$ into components of areas a and b, and all $n \in \mathbb{N}$,*

$$\Phi_T(\ell^n) = \frac{2}{n\pi} \int_0^\infty \cosh\{n(a-b)Tx/2\} \sin\{n\pi\rho_T(x)\}\, dx.$$

High-dimensional limit of the Brownian bridge in $U(N)$

There is a unique family of probability measures $(\nu_T(t) : t \in [0,1])$ on the unit circle $\mathbb{T} = \{|z| = 1\}$ such that, for all $n \in \mathbb{N}$,

$$\int_{\mathbb{T}} z^n \nu_T(t, dz) = \frac{1}{2\pi i n} \int_\gamma \exp\{-n(tTz - G_T(z))\} dz$$

For $T \in (0, \pi^2]$ and $t \in [0,1]$, consider the random variable

$$\beta_T(t) = e^{i\sqrt{Tt(1-t)}X}, \quad X \sim \frac{\sqrt{4-x^2}}{2\pi} \text{ on } [-2,2].$$

Then $\beta_T(t)$ has law $\nu_T(t)$ on $\mathbb{T}$.

Theorem

Let $(B_t : t \in [0,1])$ be a Brownian bridge in $U(N)$ from 1 to 1 at speed T. The empirical distribution of eigenvalues of B_t converges weakly in probability to $\nu_T(t)$ as $N \to \infty$ for all $t \in [0,1]$.