The stochastic Cauchy problem driven by cylindrical Lévy processes

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Some parts are based on joint work with Umesh Kumar

In this talk

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in Hilbert space V;
- $G : U \to V$ linear and bounded;
- $(L(t):\, t \geqslant 0)$ cylindrical Lévy process in Hilbert space U.

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Literatur with specific examples of L:

- Peszat and Zabczyk. 2007
- Brzeźniak, Goldys, Imkeller, Peszat, Priola and Zabczyk. 2010.
- Brzeźniak and Zabczyk. PA, 2010.
- Priola and Zabczyk. PTRF, 2011.
- Liu and Zhai, C.R.A.Sci., 2012.
- $\cdots\cdots$

In this talk

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

Problems:

- L does not attain values in U

  Consequences: • no semimartingale decomposition in U

  • no stopping times in U

- solution and L may not have finite moments

  Consequences: • no $L_P^p(\Omega)$ spaces

- solution may have unbounded paths

  Consequences: • no integration by parts formula

  • no usual Fubini argument

Cylindrical random variables

and

cylindrical measures

Cylindrical processes

Let U be a Banach space with dual space U^* and dual pairing $\langle \cdot, \cdot \rangle$ and let $(\Omega, \mathscr{A}, P)$ denote a probability space.

Definition: A cylindrical random variable X in U is a mapping

$$X : U^* \to L^0_P(\Omega; \mathbb{R}) \quad \text{linear and continuous.}$$

A cylindrical process in U is a family $(X(t) : \ t \geqslant 0)$ of cylindrical random variables.

- I. E. Segal, 1954
- I. M. Gel'fand 1956: Generalized Functions
- L. Schwartz 1969: seminaire rouge, radonifying operators

Cylindrical processes

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A cylindrical process in U is a family $(X(t) : \ t \geqslant 0)$ of cylindrical random variables.

A cylindrical random variable $X : U^* \to L_P^0(\Omega; \mathbb{R})$ is uniquely described by its characteristic function

$$\varphi_X : U^* \to \mathbb{C}, \qquad \varphi_X(u^*) := E\big[e^{iXu^*}\big].$$

Example: induced cylindrical random variable

Example: Let $X : \Omega \to U$ be a (classical) random variable. Then

$$Z : U^* \to L^0_P(\Omega; \mathbb{R}), \qquad Zu^* = \langle X, u^* \rangle$$

defines a cylindrical random variable with characteristic function

$$\varphi_Z(u^*) = E\big[e^{iZu^*}\big] = E\big[e^{i\langle X, u^*\rangle}\big] = \varphi_X(u^*).$$

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But: not for every cylindrical random variable $Z\colon U^* \to L^0_P(\Omega;\mathbb{R})$ there exists a classical random variable $X\colon \Omega \to U$ satisfying

$$Za = \langle X, u^*\rangle \qquad \text{for all } u^* \in U^*.$$

Example: cylindrical Brownian motion

Definition:

A cylindrical process $(W(t): \; t \geqslant 0)$ is called a *cylindrical Brownian motion*, if for all $u_1^*, \dots, u_n^* \in U^*$ and $n \in \mathbb{N}$ the stochastic process

$$\Big((W(t)u_1^*, \dots, W(t)u_n^*): \; t \geqslant 0 \Big)$$

is a centralised Brownian motion in $\mathbb{R}^n$.

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is a centralised Brownian motion in $\mathbb{R}^n$.

Example: in the standard case, the cylindrical random variable $W(1)\colon U^* \to L_P^0(\Omega;\mathbb{R})$ is called canonical cylindrical Gaussian and has characteristic function

$$\varphi_{W(1)}(u^*) = \exp\left(-\tfrac{1}{2} \|u^*\|^2 \right).$$

Cylindrical Lévy processes

Definition: cylindrical Lévy process

Definition: (Applebaum, Riedle (2010))
A cylindrical process $(L(t):\ t \geqslant 0)$ is called a *cylindrical Lévy process*, if for all $u_1^*, \dots, u_n^* \in U^*$ and $n \in \mathbb{N}$ the stochastic process :

$$\Big((L(t)u_1^*, \dots, L(t)u_n^*) :\ t \geqslant 0 \Big)$$

is a Lévy process in $\mathbb{R}^n$.

Lévy-Khintchine formula

Theorem: The characteristic function $\varphi_{L(t)}\colon U^* \to \mathbb{C}$ of a cylindrical Lévy process L is given by

$$\begin{aligned}
&\varphi_{L(t)}(u^*) \\
&= \exp\left(t\left(i\,p(u^*) - \tfrac{1}{2}q(u^*) + \int_U \left(e^{i\langle u,u^*\rangle} - 1 - i\langle u,u^*\rangle\,\mathbb{1}_{B_1}(\langle u,u^*\rangle)\right)\nu(du)\right)\right) \\
&=: \exp\left(t\,\Psi_{p,q,\nu}(u^*)\right)
\end{aligned}$$

where
- $p : U^* \to \mathbb{R}$ is (non-linear) continuous and $p(0) = 0$;
- $q : U^* \to \mathbb{R}$ is a quadratic form;
- ν cylindrical measure, $\displaystyle\int_U \left(\langle u,u^*\rangle^2 \wedge 1\right)\nu(du) < \infty$ for all $u^* \in U^*$;
- $B_1 := \{\beta \in \mathbb{R} :\ |\beta| \leqslant 1\}$

Example: series approach

Theorem Let U be a Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$ and $(\sigma_k)_{k\in\mathbb{N}}\subseteq\mathbb{R}$;

$(h_k)_{k\in\mathbb{N}}$ be a sequence of independent, real-valued Lévy processes.

If for all $u^*\in U^*$ and $t\geqslant 0$ the sum

$$L(t)u^* := \sum_{k=1}^{\infty}\langle e_k, u^*\rangle \sigma_k h_k(t)$$

converges P-a.s. then it defines a cylindrical Lévy process $(L(t):\, t\geqslant 0)$.

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Example 0: for h_k standard, real-valued Brownian motion:

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{\infty} \iff$ cylindrical (Wiener) Lévy process

$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{2} \iff$ honest (Wiener) Lévy process

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Example 1: for h_k symmetric, standardised, α-stable:

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{(2\alpha)/(2-\alpha)} \iff \text{cylindrical Lévy process}$$

$$(\sigma_k)_{k\in\mathbb{N}} \in \ell^{\alpha} \iff \text{honest Lévy process}$$

Example: subordination

Theorem

Let W be a cylindrical Brownian motion in a Banach space U,

ℓ be an independent, real-valued, $\alpha/2$-stable Lévy subordinator;

Then, for each $t \geqslant 0$,

$$L(t) : U^* \to L_P^0(\Omega; \mathbb{R}), \qquad L(t)u^* = W(\ell(t))u^*$$

defines a cylindrical Lévy process $(L(t) : t \geqslant 0)$ in U with

$$\varphi_{L(t)} : U^* \to \mathbb{C}, \qquad \varphi_{L(t)}(u^*) = \exp\left(-t \left\| u^* \right\|^{\alpha}\right).$$

Stochastic integration for deterministic integrands

Integration: motivation

Assume: Y classical Lévy process in a Banach space U

$$\Phi(s) := \sum_{k=0}^{n-1} \mathbb{1}_{(t_k, t_{k+1}]}(s)\varphi_k \quad \text{for} \quad \varphi_k \in \mathscr{L}(U, V).$$

Integration: motivation

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Then

$$\begin{aligned}
\langle \int_0^T \Phi(s)\,dY(s), v^*\rangle &= \sum \langle \varphi_k\big(Y(t_{k+1}) - Y(t_k)\big), v^*\rangle \\
&= \sum \varphi_k^* v^* \big(Y(t_{k+1}) - Y(t_k)\big) \\
&= \int_0^T \underbrace{\Phi^*(s)v^*}_{U^*-\text{valued}}\,dY(s)
\end{aligned}$$

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For a cylindrical Lévy process:

1st step: define the integral for U^*-valued integrands

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Then

$$\langle \int_0^T \Phi(s)\,dY(s), v^*\rangle = \sum \langle \varphi_k\big(Y(t_{k+1}) - Y(t_k)\big), v^*\rangle$$

$$= \sum \varphi_k^* v^*\big(Y(t_{k+1}) - Y(t_k)\big)$$

$$= \int_0^T \underbrace{\Phi^*(s)v^*}_{U^*-\text{valued}} dY(s)$$

For a cylindrical Lévy process:

1st step: define the integral for U^*-valued integrands

2nd step: interpret this integral as a cylindrical random variable

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$$= \int_0^T \underbrace{\Phi^*(s)v^*}_{U^*-\text{valued}} dY(s)$$

For a cylindrical Lévy process:

1st step: define the integral for U^*-valued integrands

2nd step: interpret this integral as a cylindrical random variable

3rd step: call a function stochastically integrable if this cylindrical variable is induced by a classical random variable (Pettis idea).

The cylindrical integral

Denote by $S(U^*)$ the space of all U^*-valued simple function

$$f(s) := \sum_{k=0}^{n-1} \mathbb{1}_{(t_k,t_{k+1})}(s)u_k^* \quad \text{for } u_k^* \in U^*,$$

equipped with $\|\cdot\|_\infty$ and define

$$J(f) := \sum_{k=0}^{n-1} \big(L(t_{k+1}) - L(t_k)\big)\big(u_k^*\big).$$

Then we obtain

$$J\colon S(U^*) \to L_P^0(\Omega;\mathbb{R})$$

is continuous.

The cylindrical integral

Theorem: If $\Phi : [0,T] \to \mathscr{L}(U,V)$ is a mapping such that

$$\Phi^*(\cdot)v^* : [0,T] \to U^* \quad \text{is regulated for all } v^* \in V^*,$$

then

$$Z(\Phi) : V^* \to L^0_P(\Omega;\mathbb{R}), \qquad Z(\Phi)v^* := J(\Phi^*(\cdot)v^*)$$

defines a cylindrical random variable with characteristic function

$$\varphi_{Z(\Phi)}(v^*) = \exp\left(\int_0^T \Psi_{p,q,\nu}\Big(\Phi^*(s)v^*\Big)\, ds\right) \qquad \text{for all } v^* \in V^*,$$

where $\Psi_{p,q,\nu}$ is the Lévy symbol of L.

The stochastic integral

Definition:

A function $\Phi : [0, T] \to \mathscr{L}(U, V)$ is called stochastically integrable if there exists a random variable $I(\Phi) : \Omega \to V$ such that P-a.s.

$$\langle I(\Phi), v^* \rangle = Z(\Phi) v^* \qquad \text{for all } v^* \in V^*.$$

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Conclusion: The following are equivalent:

(a) Φ is stochastically integrable;

(b) the characteristic function $\varphi_{Z(\Phi)}$ is the characteristic function of a genuine probability measure μ on $\mathfrak{B}(V)$.

In this case, μ is an infinitely divisible measure.

The stochastic integral

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Theorem: Assume V is a Hilbert space with ONB $(e_k)_{k\in\mathbb{N}}$. Then $\Phi : [0,T] \to \mathscr{L}(U,V)$ is stochastically integrable if and only if:

(a) $v^* \mapsto p(\Phi^*(\cdot)v^*)$ is weak-weakly sequentially continuous;

(b) $\displaystyle\int_0^T \text{tr}[\Phi(s)q\Phi^*(s)]\, ds < \infty$;

(c) $\displaystyle\limsup_{m\to\infty} \sup_{n\geqslant m} \int_0^T \int_U \left(\sum_{k=m}^{n} \langle u, \Phi^*(s)e_k \rangle^2 \wedge 1 \right) \nu(du)\, ds < \infty$;

A stochastic Fubini theorem

Theorem: (with Umesh Kumar)
Let $S = [a, b]$ and $f \colon S \times [0, T] \to U^*$ a function satisfying:
(i) f is jointly measurable;

(ii) for almost all $s \in S$, the map $t \mapsto f(s, t)$ is in $D_-([0, T], U^*)$;

(iii) the map $t \mapsto f(t, \cdot)$ is in $D_-([0, T]; \, L^2(S; U^*))$.

Then we have P-a.s. that

$$\int_S \int_0^T f(s, t)\, dL(t)\, ds = \int_0^T \int_S f(s, t)\, ds\, dL(t).$$

Here: $D_-([0,T];B)$ is the space of all càglàd, functions.

Proof: Fubini theorem

The function

$$F\colon [0,T] \to \mathscr{L}\big(U, L^2(S)\big), \qquad F(t)u = \langle u, f(\cdot,t)\rangle$$

is stochastically integrable w.r.t. L.

Proof: Fubini theorem

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$$\int_S \left(\int_0^T f(s,t)\, dL(t) \right) ds = \int_S \left(\int_0^T F(t)\, dL(t) \right)(s)\, ds$$

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$$\begin{aligned}\int_S \left(\int_0^T f(s,t)\,dL(t)\right)\,ds &= \int_S \left(\int_0^T F(t)\,dL(t)\right)\!(s)\,ds \\ &= \left\langle \int_0^T F(t)\,dL(t) \right\rangle \left\langle 1 \right\rangle_{L^2(S)}\end{aligned}$$

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&= \left\langle \int_0^T F(t)\,dL(t) \right\rangle \left\langle 1 \right\rangle_{L^2(S)} \\
&= \int_0^T F^*(t)1\,dL(t)
\end{aligned}$$

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&= \int_0^T F^*(t) 1\, dL(t) \\
&= \int_0^T \int_S f(s,t)\, ds\, dL(t).
\end{aligned}$$

Ornstein-Uhlenbeck process

Stochastic evolution equations

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in Hilbert space V;
- $G : U \to V$ linear and bounded;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in Hilbert space U.

Definition: A stochastic process $(X(t) : t \in [0,T])$ in V is called a **weak solution** if it satisfies for all $v^* \in \mathrm{D}(A^*)$ and $t \in [0,T]$ that

$$\langle X(t), v^*\rangle = \langle X(0), v^*\rangle + \int_0^t \langle X(s), A^*v^*\rangle\,ds + L(t)(G^*v^*).$$

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Theorem: The following are equivalent:

(a) $t \mapsto S(t)G$ is stochastically integrable;

(b) there exists a weak solution $(X(t) : t \in [0,T])$.

In this case, the weak solution is given by

$$X(t) = S(t)X(0) + \int_0^t S(t-s)G\,dL(s) \qquad \text{for all } t \in [0,T].$$

Stochastic evolution equations

Define for each $t \in [0,T]$

$$Y(t) := \int_0^t T(t-s)G\,dL(s).$$

Stochastic evolution equations

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Then we obtain for each $v^* \in \text{Dom}(A^*)$:

$$\int_0^t \langle Y(s), A^*v^* \rangle \, ds = \int_0^t \left(\int_0^s G^*T^*(s-r)A^*v^* \, dL(r) \right) ds$$

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&= \int_0^t \left(\int_r^t G^*T^*(s-r)A^*v^*\, ds \right) dL(r) \\
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\end{aligned}$$

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\end{aligned}$$

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Example: Let L be the canonical α-stable cylindrical Lévy process with characteristic function $\varphi_{L(t)}(u^*) = \exp\left(-t\,\|u^*\|^{\alpha}\right)$. Then the following are equivalent:

(1) there exists a weak solution;

(2) $\displaystyle\int_0^T \|S(s)G\|_{HS}^{\alpha}\,ds < \infty.$

Irregularity of trajectories

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In specific examples of cylindrical Lévy processes it was observed that the solution exits but with very irregular paths in V:

- Brzeźniak, Goldys, Imkeller, Peszat, Priola and Zabczyk. 2010:
  sum of independent real-valued Lévy processes
  no left or right limits in V
- Brzeźniak and Zabczyk. PA, 2010.
  canonical α-stable cylindrical process
  no càdlàg modification in V:
- ……

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Theorem: If there exists a constant $c > 0$ such that

$$\sup_{n\in\mathbb{N}} \nu\left(\left\{u \in U : \sum_{k=1}^{n} \langle u, e_k\rangle^2 > c\right\}\right) = \infty,$$

then there does not exists a modification $\tilde{X}$ of X such that

$$\big(\langle \tilde{X}(t), v^*\rangle : t \in [0,T]\big) \quad \text{has càdlàg paths for all } v^* \in V^*.$$

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Theorem: Assume that the weak solution X exists. Then

(a) X is continuous in probability.

Regularity of trajectories

$$dX(t) = AX(t)\,dt + G\,dL(t) \qquad \text{for all } t \in [0,T]$$

- A generator of C_0-semigroup $(S(t))_{t\geqslant 0}$ in Hilbert space V;
- $G : U \to V$ linear and bounded;
- $(L(t) : t \geqslant 0)$ cylindrical Lévy process in Hilbert space U.

Theorem: Assume that the weak solution X exists. Then

(a) X is continuous in probability.

(b) X is cylindrically square integrable, i.e. for each $v^* \in V^*$ there exists a modification $\widetilde{X}_{v^*}$ of $(\langle X(t), v^*\rangle : t \in [0,T])$ with

$$\int_0^T |\widetilde{X}_{v^*}(s)|^2\,ds < \infty$$