Discrete rough paths and limit theorems

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Durham Symposium – 2017

Joint work with Yanghui Liu

Outline

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Definition of fBm

Definition 1.

A 1-d fBm is a continuous process $B = \{B_t;\ t \geq 0\}$ such that $B_0 = 0$ and for $\nu \in (0,1)$:

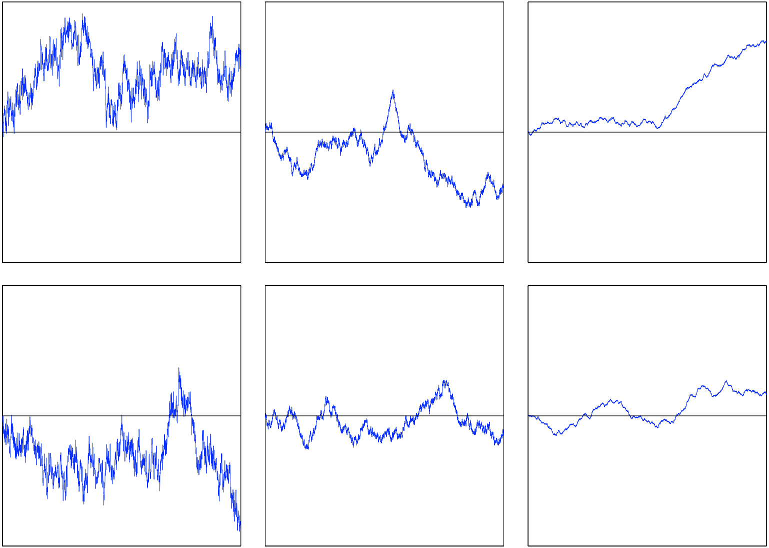
- B is a centered Gaussian process
- $\mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2\nu} + |t|^{2\nu} - |t-s|^{2\nu})$

m-dimensional fBm: $B = (B^1, \ldots, B^m)$, with B^i independent 1-d fBm

Variance of increments:

$$\mathbf{E}[|B_t^j - B_s^j|^2] = |t-s|^{2\nu}$$

Examples of fBm paths



$\nu = 0.35$ $\nu = 0.5$ $\nu = 0.7$

Some notation

Uniform partition of $[0,1]$: For $n \geq 1$ we set

$$t_k = \frac{k}{n}$$

Increment of a function: For $f : [0,1] \to \mathbb{R}^d$, we write

$$\delta f_{st} = f_t - f_s$$

Hermite polynomial of order q: defined as

$$H_q(t) = (-1)^q e^{\frac{t^2}{2}} \frac{d^q}{dt^q} e^{-\frac{t^2}{2}}$$

Hermite rank

Definition 2.

Consider

- $\gamma = \mathcal{N}(0, 1)$.
- $f \in L^2(\gamma)$ such that f is centered.

Then there exist:

- $d \geq 1$
- A sequence $\{c_q;\ q \geq d\}$

such that f admits the expansion:

$$f = \sum_{q=d}^{\infty} c_q H_q.$$

The parameter d is called Hermite rank of f.

Breuer-Major's theorem for fBm increments

Theorem 3.

Let

- $f \in L_2(\gamma)$ with rank $d \geq 1$
- B a 1-d fBm with Hurst parameter $\nu < \frac{1}{2}$

For $0 \leq s \leq t \leq 1$ and $n \geq 1$, we set:

$$h_{st}^n = n^{-\frac{1}{2}} \sum_{s \leq t_k < t} f\left(n^\nu \delta B_{t_k t_{k+1}}\right)$$

Then the following convergence holds true:

$$h^n \xrightarrow{f.d.d.} \sigma_{d,f} W \quad \text{as} \quad n \to \infty$$

Breuer-Major with weights (1)

Motivation for the introduction of weights:

- Analysis of numerical schemes
- Parameter estimation based on quadratic variations
- Convergence of Riemann sums in rough contexts

Weighted sums (or discrete integrals): For a function g, we set

$$
\begin{aligned}
\mathcal{J}_s^t(g(B); h^n) &= \sum_{s \le t_k < t} g(B_{t_k})\, h^n_{t_k t_{k+1}} \\
&= n^{-\frac{1}{2}} \sum_{s \le t_k < t} g(B_{t_k})\, f(n^{\nu} \delta B_{t_k t_{k+1}})
\end{aligned}
$$

Breuer-Major with weights (2)

Recall:

$$\mathcal{J}_s^t(g(B); h^n) = n^{-\frac{1}{2}} \sum_{s \le t_k < t} g(B_{t_k}) \, f(n^{\nu} \delta B_{t_k t_{k+1}})$$

Expected limit result: For W as in Breuer-Major,

$$\lim_{n \to \infty} \mathcal{J}_s^t(g(B); h^n) = \sigma_{d,f} \int_s^t g(B_u) \, dW_u \qquad (1)$$

Unexpected phenomenon:

The limits of $\mathcal{J}_s^t(g(B); h^n)$ can be quite different from (1)

Breuer-Major with weights (3)

Theorem 4.

For $d \geq 1$ and g smooth enough we set

$$V_{st}^{n,d}(g) = \mathcal{J}_s^t(g(B); h^{n,d}) = n^{-\frac{1}{2}} \sum_{s \leq t_k < t} g(B_{t_k}) \, H_d(n^{\nu} \delta B_{t_k t_{k+1}})$$

Then the following limits hold true:

1. If $d > \frac{1}{2\nu}$ then
   $V_{st}^{n,d}(g) \xrightarrow{(d)} c_{d,\nu} \int_s^t g(B_u) \, dW_u$
2. If $d = \frac{1}{2\nu}$ then
   $V_{st}^{n,d}(g) \xrightarrow{(d)} c_{1,d,\nu} \int_s^t g(B_u) \, dW_u + c_{2,d,\nu} \int_s^t f^{(d)}(B_u) \, du$
3. If $1 \leq d < \frac{1}{2\nu}$ then
   $n^{-(\frac{1}{2} - \nu d)} V_{st}^{n,d}(g) \xrightarrow{\mathbf{P}} c_d \int_s^t f^{(d)}(B_u) \, du$

Breuer-Major with weights (3)

Remarks on Theorem 4:

- Obtained in a series of papers by
  Corcuera, Nualart, Nourdin, Podolskij, Réveillac, Swanson, Tudor
- Extensions to p-variations, Itô formulas in law

Limitations of Theorem 4:

- One integrates w.r.t $h^{n,d}$, in a fixed chaos
- Results available only for 1-d fBm
- Weights of the form $y = g(B)$ only

Aim of our contribution:

- Generalize in all those directions

Outline

Rough path

Notation: We consider

- $\nu \in (0,1)$, Hölder continuity exponent
- $\ell = \lfloor \frac{1}{\nu} \rfloor$, order of the rough path
- $p > 1$, integrability order
- $\mathbb{R}^m$, state space for a process x
- $\mathcal{S}_2 \equiv$ simplex in $[0,1]^2 = \{(s,t);\, 0 \le s \le t \le 1\}$

Rough path: Collection $\mathbf{x} = \{x^i; i \le \ell\}$ such that

- $x^i = \{x^i_{st} \in (\mathbb{R}^m)^{\otimes i};\, (s,t) \in \mathcal{S}_2\}$
- $x^i_{st} = \int_{s \le s_1 < \cdots < s_i \le t} dx_{s_1} \otimes \cdots \otimes dx_{s_i}$ (to be defined rigorously)
- We have

$$|x^i|_{p,\nu} \equiv \sup_{(u,v)\in\mathcal{S}_2} \frac{|x^i_{uv}|_{L_p}}{|v-u|^{\nu i}} < \infty$$

Controlled processes (incomplete definition)

Definition 5.

Let:

- $\ell = \lfloor \frac{1}{\nu} \rfloor$
- x a (L^p, ν, ℓ)-rough path
- A family $\mathbf{y} = (y, y^{(1)}, \ldots, y^{(\ell-1)})$ of processes

We say that $\mathbf{y}$ is a process controlled by $\mathbf{x}$ if

$$\delta y_{st} = \sum_{i=1}^{\ell-1} y_s^{(i)} x_{st}^i + r_{st}, \quad \text{and} \quad |r_{st}|_{L_p} \lesssim |t-s|^{\nu\ell}.$$

Remark: Typical examples of controlled process
$\hookrightarrow$ solutions of differential equations driven by x, or $g(x)$

Abstract transfer theorem: setting

Objects under consideration: Let

- α limiting regularity exponent. Typically $\alpha = \frac{1}{2}$ or $\alpha = 1$
- $\mathbf{x}$ rough path of order ℓ
- h^n such that uniformly in n:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2} \leq K(t-s)^{\alpha+\nu i} \tag{2}$$

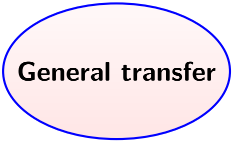
- $\mathbf{y}$ controlled process of order ℓ
- $(\omega^i, \, i \in \mathcal{I})$ family of processes independent of x
  $\hookrightarrow$ Typically $\omega_t^i =$ Brownian motion, or $\omega_t^i = t$

Abstract transfer theorem (1)

Recall: h^n satisfies:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2} \leq K(t-s)^{\alpha+\nu i}$$

Illustration:



Abstract transfer theorem (1)

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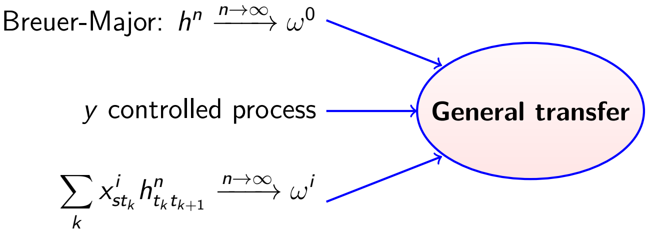
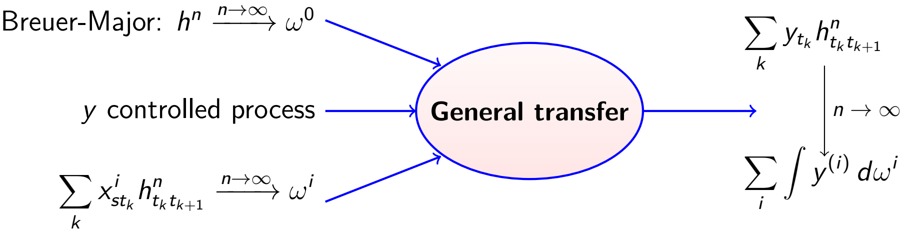


Abstract transfer theorem (1)

Recall: h^n satisfies:

$$|\mathcal{J}_s^t(x^i; h^n)|_{L_2} \leq K(t-s)^{\alpha+\nu i}$$

Illustration:



Abstract transfer theorem

Theorem 6.

We assume that (2) holds and:

1. As $n \to \infty$:
$$\left\{x, \mathcal{J}(x^i; h^n) \,;\, 0 \le i \le \ell-1\right\} \xrightarrow{f.d.d.} \{x, \omega^i \,;\, 0 \le i \le \ell-1\}.$$
2. One additional technical condition on $\int y \, d\omega^i$.

Then the following convergence holds true as $n \to \infty$:

$$\mathcal{J}(y; h^n) \xrightarrow{\text{f.d.d., stable}} \sum_{i=0}^{\ell-1} \int y^{(i)} \, d\omega^i.$$

Outline

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Notation

Setting: We consider

- A 1-d fractional Brownian motion B
- Hurst parameter: $\nu < \frac{1}{2}$
- 1-d rough path: $\mathbf{B}^i_{st} = \frac{(\delta B_{st})^i}{i!}$
- y controlled process
- f smooth enough with Hermite rank d
- W Wiener process independent of B

Quantity under consideration:

$$\mathcal{J}_s^t(y; h^{n,d}) = n^{-\frac{1}{2}} \sum_{s \le t_k < t} y_{t_k} \, f(n^{\nu} \delta B_{t_k t_{k+1}})$$

Breuer-Major with controlled weights

Theorem 7.

For f smooth with Hermite rank d and y controlled we set

$$\mathcal{J}_s^t(y; h^{n,d}) = n^{-\frac{1}{2}} \sum_{s \le t_k < t} y_{t_k} f(n^\nu \delta B_{t_k t_{k+1}})$$

Then the following limits hold true:

1. If $d > \frac{1}{2\nu}$ then
   $\mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{(d)} c_{d,\nu} \int_s^t y_u \, dW_u$
2. If $d = \frac{1}{2\nu}$ then
   $\mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{(d)} c_{1,d,\nu} \int_s^t y_u \, dW_u + c_{2,d,\nu} \int_s^t y_u^{(d)} \, du$
3. If $1 \le d < \frac{1}{2\nu}$ then
   $n^{-(\frac{1}{2} - \nu d)} \mathcal{J}_s^t(y; h^{n,d}) \xrightarrow{\mathbf{P}} c_d \int_s^t y_u^{(d)} \, du$

Breuer-Major with controlled weights (2)

Improvements of Theorem 7:

- One integrates w.r.t a general $f(n^{\nu}\delta B_{t_k t_{k+1}})$
  $\hookrightarrow$ with f smooth enough
- Results can be generalized to d-dim situations
- General controlled weights y

Other applications:

- Itô formulas in law, convergence of Riemann sums
- Asymptotic behavior of p-variations

Outline

Setting

Equation under consideration:

$$dy_t = \sum_{i=1}^{m} V_i(y_t) dB_t^i, \qquad y_0 \in \mathbb{R}^d, \tag{3}$$

where:

- V_i smooth vector fields
- B is a m-dimensional fBm with $\frac{1}{3} < \nu \leq \frac{1}{2}$
- Note: a drift could be included

Modified Euler scheme: for the uniform partition $\{t_k;\, k \leq n\}$,

$$y_{t_{k+1}}^n = y_{t_k}^n + \sum_{i=1}^{m} V_i(y_{t_k}^n) \delta B_{t_k t_{k+1}}^i + \frac{1}{2} \sum_{j=1}^{m} \partial V_j V_j(y_{t_k}^n) \frac{1}{n^{2H}} \tag{4}$$

CLT for the modified Euler scheme

Theorem 8.

Under the previous assumptions let

- y be the solution to (3)
- y^n be the modified Euler scheme defined by (4)

Let U be the solution to

$$U_t = +\sum_{j=1}^{m} \int_0^t \partial V_j(y_s) U_s dB_s^j + \sum_{i,j=1}^{m} \int_0^t \partial V_i V_j(y_s) dW_s^{ij}$$

Then the following weak convergence in $D([0,1])$ holds true:

$$n^{2H-\frac{1}{2}}(y - y^n) \xrightarrow{n\to\infty} U$$

Remarks on proofs

Convergence of Euler scheme: Reduced to a CLT for weighted sum

$$\sum_{i,j=1}^{m} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor - 1} \partial V_j V_i(y_{t_k}^n) \left[\mathbf{B}_{t_k t_{k+1}}^{\mathbf{2},ij} - \frac{1}{2}(t_{k+1} - t_k)^{2H} \right]$$

We are thus back to our general framework

Method of proof:

1. Get rid of negligible terms with rough paths expansions
2. Main contributions treated with
   - 4th moment method
   - Integration by parts