# Fluctuation results for planar random growth processes

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Work in progress with
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# DLA aggregate formed on electrode in copper sulphate solution

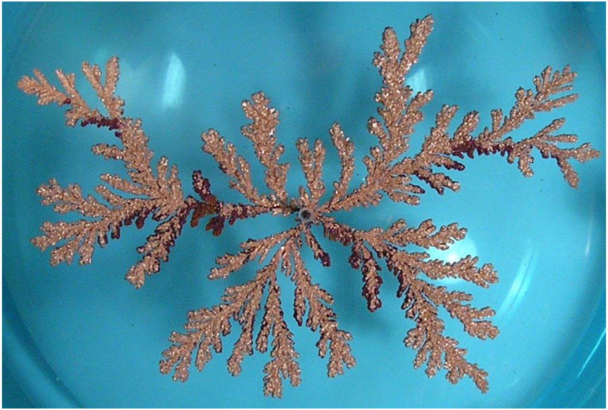


Photo by Kevin R Johnson

# Eden cluster formed by lichen growth

Photo by James Wearn

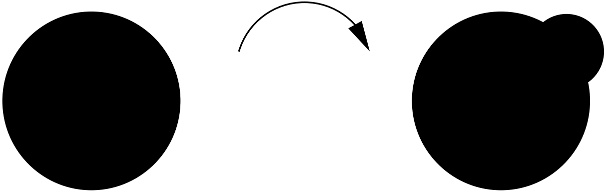
# Electrical "tattoo" on survivor of lightning strike

From "Lichtenberg Figures Due to a Lightning Strike" by Yves Domart, MD, and Emmanuel Garet, MD

# Conformal mapping representation of single particle

Let $D_0$ denote the exterior unit disk in the complex plane $\mathbb{C}$ and $P$ denote a particle of logarithmic capacity $c$ and attachment angle $\theta$.

Use the unique conformal mapping $f_P : D_0 \to D_0 \setminus P$ that fixes $\infty$ as a mathematical description of the particle.



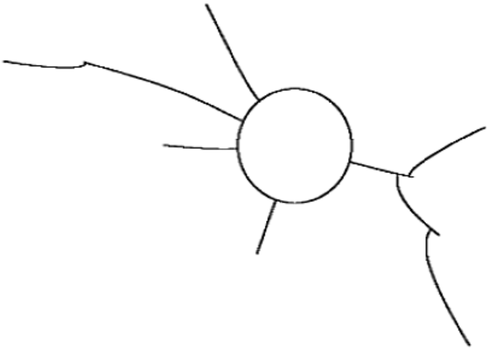
Results apply to any particle shape $P$ for which

$$f_P(z) = e^c \left( z + \frac{2cz}{e^{-i\theta} z - 1} \right) + O\left( \frac{c}{|z - e^{i\theta}|} \right)^2 .$$

# Conformal mapping representation of a cluster

- Suppose $P_1, P_2, \ldots$ is a sequence of particles, where $P_n$ has capacity $c_n$ and attachment angle $\theta_n$, $n = 1, 2, \ldots$.
  - Set $\Phi_0(z) = z$.
  - Recursively define $\Phi_n(z) = \Phi_{n-1} \circ f_{P_n}(z)$, for $n = 1, 2, \ldots$.
- This generates a sequence of conformal maps $\Phi_n : D_0 \to K_n^c$, where $K_{n-1} \subset K_n$ are growing compact sets, which we call clusters.
- By varying the sequences $\{\theta_n\}$ and $\{c_n\}$, it is possible to describe a wide class of growth models.

# Cluster formed by iteratively composing slit mappings

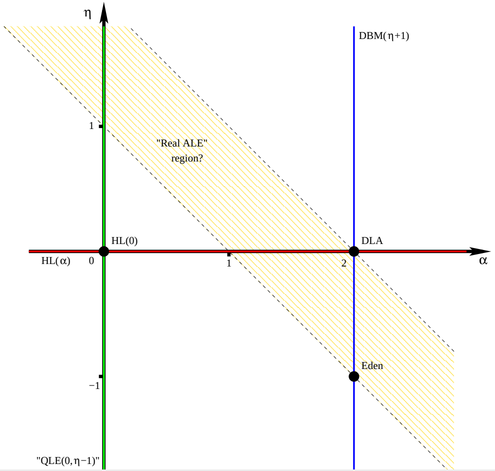


# Examples of models within this framework

- Hastings-Levitov family, HL($\alpha$) [1998]:
  - $\theta_n$ are i.i.d. $U(-\pi,\pi)$ random variables;
  - $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$.
- Dielectric-breakdown models, DBM($\eta$) [due to Mathiesen-Jensen, 2002]:
  - $\theta_n$ distributed $\propto |\Phi'_{n-1}(e^{i\theta})|^{1-\eta} d\theta$;
  - $c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-2}$.
- Quantum Loewner Evolution, QLE($\gamma,\eta$) [due to Miller-Sheffield, 2013]:
  - $\theta_n$ "distributed" $\propto e^{a(\gamma) h\circ\Phi_{n-1}(e^{i\theta})}|\Phi'_{n-1}(e^{i\theta})|^{b(\gamma)-1-\eta} d\theta$;
  - $c_n = c$ for all $n$, $P_n$ a $SLE_\kappa$ conditionally independent of the GFF $h$, given $\theta_n$ ($a$, $b$, functions depending on $\kappa$).

# Aggregate Loewner Evolution, ALE$(\alpha, \eta)$

- $\theta_n$ distributed $\propto |\Phi'_{n-1}(e^{i\theta})|^{-\eta} d\theta$; $\quad c_n = c|\Phi'_{n-1}(e^{i\theta_n})|^{-\alpha}$.



# Previous results

- Primary interest has been in asymptotic behaviour of large clusters.
- Almost all previous work relates to HL(0) as particle maps are i.i.d. so the model is mathematically the most tractable.
  - Norris and T. (2012) showed scaling limit of HL(0) is a growing disk with a branching structure related to the Brownian web.
  - Silvestri (2015) showed fluctuations converge to a log-correlated Fractional Gaussian Field.
- Results for HL($\alpha$) with $\alpha \neq 0$ have only been shown for regularized versions of the model.
  - Rohde and Zinsmeister (2005) analysed the dimension of scaling limits for a regularized version of HL($\alpha$) when $\alpha > 0$.
  - Sola, T., Viklund (2015) showed scaling limit of regularized HL($\alpha$) is a growing disk for all $\alpha$ provided regularization is strong enough.

# Phase transition

**Open Problem:**

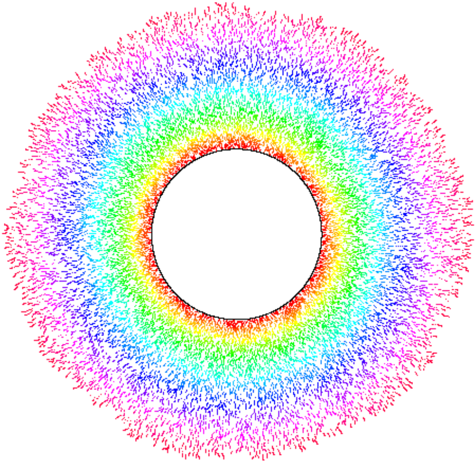
Does ALE$(\alpha, \eta)$ have a phase transition from disks to non-disks along the line $\alpha + \eta = 1$ (within some compact region)?

- Longstanding conjectures:
  - HL$(\alpha)$ has a phase transition at $\alpha = 1$.
  - DBM$(\eta)$ has a phase transition at $\eta = 0$.

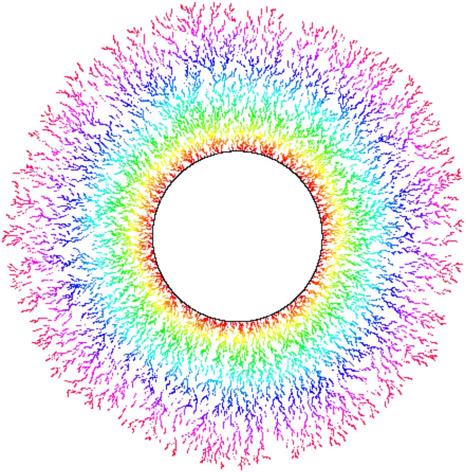
# Scaling limits for ALE(0,$\eta$)

- Natural to consider particle sizes that are very small compared to the overall size of the cluster and scaling limits where $n \to \infty$ while $c \to 0$.
- Models are difficult to analyse mathematically as all models (except HL(0)) exhibit long-range dependencies.
- Additional difficulty, when $\alpha \neq 0$, is total capacity of cluster is random and cannot, a priori, be bounded above or below, so unclear at what rate to let $n \to \infty$.
- When $\alpha = 0$, $K_n$ has capacity $cn$, so natural to look for scaling limits when $n = \lfloor T/c \rfloor$.

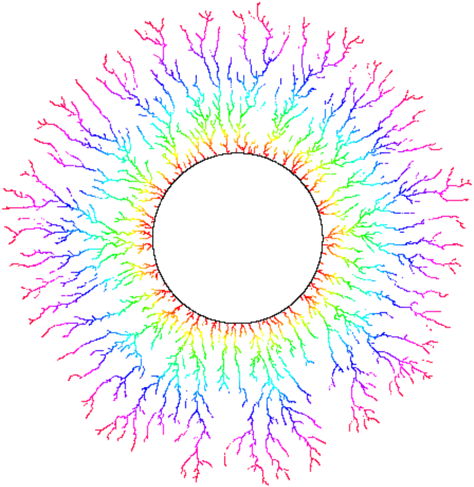
# ALE(0,-1) cluster with 10,000 particles for $d = 0.02$



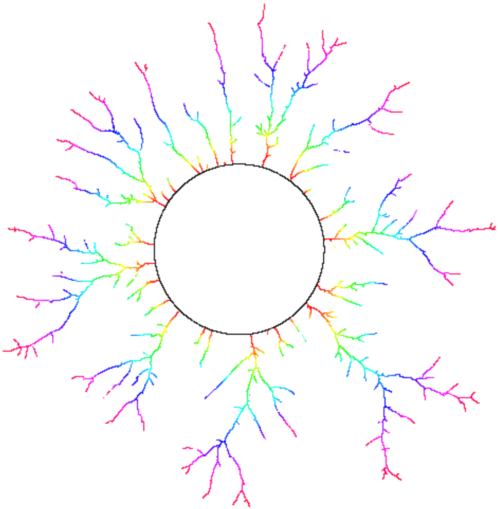
# ALE(0,0) cluster with 10,000 particles for $d = 0.02$



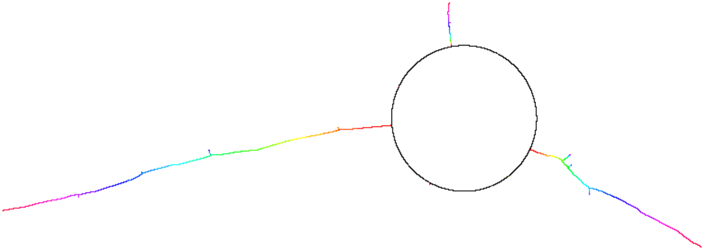
# ALE(0,1) cluster with 10,000 particles for $d = 0.02$



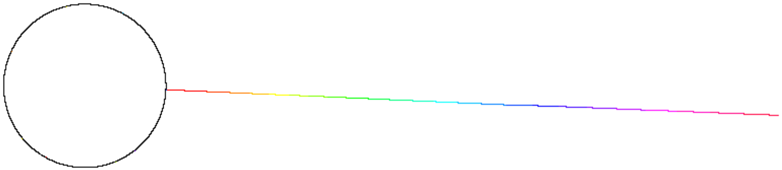
# ALE(0,1.5) cluster with 10,000 particles for $d = 0.02$



# ALE(0,2) cluster with 10,000 particles for $d = 0.02$



# ALE(0,4) cluster with 10,000 particles for $d = 0.02$



# Regularization for ALE($0,\eta$)

- Even after the arrival of a single slit particle, the map $\theta \mapsto |\Phi_n'(e^{i\theta})|$ is badly behaved and takes the values 0 and $\infty$.
- For some values of $\eta$,

$$\int_{\pi}^{\pi} |\Phi_{n-1}'(e^{i\theta})|^{-\eta} d\theta = \infty,$$

  so regularization is necessary to even define the measure.
- A solution is to let $\theta_n$ have distribution

$$\propto |\Phi_{n-1}'(e^{\sigma+i\theta})|^{-\eta} d\theta$$

  for $\sigma > 0$ and take the limit $\sigma \to 0$.
- Scaling limits are sensitive to rate at which $\sigma \to 0$.
  - If $\sigma \to 0$ very slowly, clusters converge to disks for all $\eta \in \mathbb{R}$;
  - If $\sigma \to 0$ very fast, scaling limits depend on the precise particles used.

# Disk Theorem

**Theorem:**

Suppose $N = \lfloor T/c \rfloor$ for some $T > 0$, and ALE$(0,\eta)$ is regularized by $\sigma$.

For each $\eta \in \mathbb{R}$, there exists a $\gamma = \gamma(\eta)$ such that, provided $\sigma \gg c^\gamma$,

$$e^{-cn}\Phi_n(z) - z \to 0$$

in probability as $c \to 0$, uniformly on $|z| \geq e^\sigma$ and $n \leq N$.

(Refining the value of $\gamma$ is work in progress, but at the moment $\gamma = 1/4(1 + |\eta|)$.)

# 1 minute proof ($\eta = 0$)

$$\Phi_n(z) - e^{cn}z = \sum_{k=1}^{n} \Phi_k(e^{c(n-k)}z) - \Phi_{k-1}(e^{c(n-k-1)}z).$$

But

$$\begin{aligned}\mathbb{E}\left[\Phi_k(z)\,|\mathcal{F}_k\right] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{k-1}(e^{i\theta} f_c(e^{-i\theta} z))d\theta \\ &= \frac{1}{2\pi i}\int_{|w|=1} \frac{\Phi_{k-1}(wf_c(zw^{-1}))}{w}dw \\ &= \lim_{w\to 0} \Phi_{k-1}\left(wf_c(zw^{-1})\right) \\ &= \Phi_{k-1}(e^c z).\end{aligned}$$

So $\Phi_n(z) - e^{cn}z$ is a martingale sum and the result follows by Bernstein's inequality.

# Pointwise Fluctuations ($\eta \leq 1$)

Set

$$\mathcal{F}_n(z) = c^{-1/2}(e^{-cn}\Phi_n(z) - z).$$

Then for fixed $|z| > 1$ and $t > 0$, in the limit as $c \to 0$ while $nc \to t$,

$$\mathcal{F}_n(z) \to \mathcal{N}\left(0, \sum_{m=0}^{\infty} \frac{1 - e^{-2(m(1-\eta)+1)t}}{m(1-\eta)+1} |z|^{-2m}\right).$$

(Note that if $\eta > 1$ would need $|z| > e^{(\eta-1)t}$ for this sum to converge – beginnings of a phase transition?)

# Global Fluctuations ($\eta \leq 1$)

Under the assumptions above, $\mathcal{F}_n(z) \to \mathcal{W}_t(z)$ where

$$\dot{\mathcal{W}_t}(z) = (1-\eta)z\mathcal{W}_t'(z) - \mathcal{W}_t(z) + \sqrt{2}\dot{\xi}_t(z)$$

where $\xi_t(z)$ is complex space-time white noise on the circle, analytically continued to the exterior unit disk.

Specifically

$$\mathcal{W}_t(z) = \sum_{m=0}^{\infty} (A_t^m + iB_t^m) z^{-m}$$

where

$$dA_t^m = -\left(m(1-\eta)+1\right) A_t^m dt + \sqrt{2} d\beta_t^m$$
$$dB_t^m = -\left(m(1-\eta)+1\right) B_t^m dt + \sqrt{2} d\beta'^m_t$$

where $\beta_t^m, \beta_t'^m$ are i.i.d. Brownian motions for $m = 0, 2, \ldots$.

# Remarks

- The map $z \mapsto \mathcal{W}_t(z)$ is determined (by analytic extension) by the boundary process $\theta \mapsto \mathcal{W}_t(e^{i\theta})$.
- When $\eta = 0$, these boundary fluctations are the same as for internal diffusion limited aggregation (IDLA).
- As $t \to \infty$, $\mathcal{W}_t(e^{i\theta})$ converges to a Gaussian field.
    - When $\eta = 0$, $\mathcal{W}_\infty(e^{i\theta})$ is known as the augmented Gaussian Free Field.
    - When $\eta < 1$, $\mathrm{Cov}\left(\mathcal{W}_\infty(\mathrm{e}^{\mathrm{ix}})\mathcal{W}_\infty(\mathrm{e}^{\mathrm{iy}})\right) = \Theta(\log|\mathrm{x} - \mathrm{y}|)$.
    - When $\eta = 1$, $\mathcal{W}_\infty(e^{i\theta})$ is complex white noise.

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