BMO and Morrey-Campanato estimates for stochastic singular integral operators and their applications to parabolic SPDEs

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Outline

Based on a recent joint work with
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Several kinds of interesting estimates for (mild) solutions of well posed SPDEs have been established. By using parabolic Littlewood-Paley inequality, Krylov [*An analytic approach to SPDEs*, pp 185-242 in: Stochastic Partial Differential Equations: Six Perspectives. 1999] proved that for

$$du = \Delta u dt + g dw_t$$

the following holds

$$\mathbb{E}\|\nabla u\|^p_{L^p((0,T)\times\mathbb{R}^d)} \leq C(d,p)\mathbb{E}\|g\|^p_{L^p((0,T)\times\mathbb{R}^d)}$$

for $p \in [2, \infty)$, where $g : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is measurable and w_t is a Brownian motion. Moreover, J. van Neerven, M. Veraar and L. Weis [*Stochastic maximal L^p-regularity*, Ann. Probab. **40** (2012) 788-812] made a significant extension of the above inequality to a class of operators A which admit a bounded H^∞-calculus of angle less than $\pi/2$.

I. Kim [JFA 2015] established a BMO (*Bounded Mean Oscillation*) estimate for stochastic singular integral operators and further proved that the q-th order BMO quasi-norm of the derivative of u is controlled by $\|g\|_{L^\infty}$.
I. Kim, K. Kim and S. Lim, [JMAA 2016] studied the parabolic Littlewood-Paley inequality for a class of time-dependent pseudo-differential operators of arbitrary order.
M. Yang [Proc. AMS 2015] considered the following

$$du = \Delta^{\frac{\alpha}{2}} u dt + f dL_t, \quad u_0 = 0, \ 0 < t < T$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$, $0 < \alpha < 2$, $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ is measurable and L_t is a Lévy process. A parabolic Triebel-Lizorkin space estimate for the convolution operator is obtained.

Our first task is to study the following stochastic integral operator

$$\begin{aligned}\mathcal{G}g(t,x) &:= \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(dz,ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t,s,x-y)g(s,y,z)dy\tilde{N}(dz,ds)\end{aligned}$$

for $\tilde{N}$ being the compensated martingale measure associated with a Poisson random measure over a σ-finite measure space $(Z,\mathcal{B}(Z),\nu)$ (as a marked point space).

We aim to seek appropriate conditions on the kernel K for the following BMO estimate

$$
\begin{aligned}
[\mathcal{G}g]_{\mathbb{BMO}(T,q)} \leq & N\Bigg(\left\|\left(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)}\nu(dz)\right)^{q/2}\right\|_{L^{\frac{\kappa}{\kappa-q}}} \\
& + \left\|\int_Z |g(\cdot,\cdot,z)|^{q_0}_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|^{q/q_0}_{L^{\tilde{\kappa}}(\Omega)} \\
& + \left\|\int_Z \|g(\cdot,\cdot,z)\|^q_{L^\infty(\mathcal{O}_T)}\nu(dz)\right\|_{L^{\frac{\kappa}{\kappa-q}}}\Bigg),
\end{aligned}
$$

where $q \in [2, p_0 \wedge \kappa]$, $\tilde{\kappa}$ is the conjugate of a positive constant κ, the constant N depends on q and d.

As an application, we show that the solution of the following equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x)dt + \sum_{k=1}^{\infty} \int_{\mathbb{R}^m} g^k(t,x) z\tilde{N}_k(dz,dt), \ t \in (0,T]$$

satisfies that for $q \in [2, q_0]$

$$[\nabla^{\beta} u]_{\mathbb{BMO}(T,q)} \leq N\hat{c} \left(\mathbb{E}[\| |g|_{\ell_2} \|^{q_0}_{L^{\infty}(\mathcal{O}_T)}] \right)^{q/q_0}$$

for $u_0 = 0$, where $\int_{\mathbb{R}^m} z\tilde{N}_k(t,dz) =: Y_t^k$ are independent m-dimensional pure jump Lévy processes with Lévy measure of ν^k, $\beta = \alpha/q_0$ and $\hat{c}$ is certain constant.

Moreover, for the following stochastic parabolic equation

$$du_t(x) = \Delta^{\frac{\alpha}{2}} u_t(x)dt + \sum_{k=1}^{\infty} h^k(t,x)dW_t^k,\ t \in (0,T]$$

for $u_0 = 0$, where W_t^k are independent one-dimensional Wiener processes. We have the following estimate, for any $q \in (0,p]$

$$[\nabla^{\frac{\alpha}{2}} u]_{\mathbb{BMO}(T,q)} \leq N \left(\mathbb{E}[\| |h|_{\ell_2} \|^p_{L^\infty(\mathcal{O}_T)}] \right)^{1/p}$$

under the condition that $h \in L^p(T,\ell_2)$. When $\alpha = 2$, we recover the result of Kim in JFA 2015.

Similar to the regularity of PDE, the regularity of SPDEs can be divided into two parts: the L^p-theory (initiated by Krylov [*On L_p-theory of stochastic partial differential equations in the whole space*, SIAM J. Math. Anal. **27** (1996)]) and the Schauder estimates. There are some extensions of Krylov's L_p-theory to SPDEs on bounded domains, by using the Moser's iteration scheme, see [L. Denis, A. Matoussi and L. Stoica, *L^p estimates for the uniform norm of solutions of quasilinear SPDE's*, PTRF **133** (2005)] and I. Kim and K. Kim [SPA2016]. As for the the Schauder estimates, a usual strategy is to get the Morrey-Campanato estimates and then by utilising embedding theorem to derive the Schauder estimates. A. Debussche, M. Hofmanová and J. Vovelle, [*Degenerate parabolic stochastic partial differential equations: quasilinear case*. AP **44** (2016)] proved that the solution of SPDEs is Hölder continuous in both time and space variables.

Du and Liu [*Schauder estimate for stochastic PDEs*, C. R. Math. Acad. Sci. Paris **354** (2016) 371-375] established the $C^{2+\alpha}$-theory for SPDEs on the whole space. Hsu and Wang [arXiv:1312.3311v3] used stochastic De Giorgi iteration technique to prove that the solutions of SPDEs are almost surely Hölder continuous in both space and time variables. The above mentioned results about the regularity of the solutions of SPDEs belongs to the space $L^p(\Omega; C^{\alpha,\beta}([0,T]\times G))$, where G is a bounded domain in $\mathbb{R}^d$. Now, there is an natural question, that is, can one get the Hölder estimate for the p-moment? Namely, can we derive the estimate in $C^{\alpha,\beta}([0,T]\times G; L^p(\Omega))$? We note that Du and Liu obtained the $C^{2+\alpha}$-theory for SPDEs in $C^{\alpha,\beta}([0,T]\times G); L^p(\Omega))$, where the Dini continuous is needed for the stochastic term. The method used is the Sobolev

embedding theorem and the iteration technique under the condition that the noise term satisfies Dini continuity.
Here we would like to consider a simple case, that is, the equation with additive noise. We aim to derive the Morrey-Campanato estimates for the stochastic convolution operators and then, by utilising the embedding theorem between Campanato space and Hölder space, we succeed to establish the norm of $C^{\theta,\theta/2}$. As an application, we show that the solutions of partial differential equations driven by Brownian motion or by Lévy noise are Hölder continuous in the both time and space variables on the whole space.

Fix $\gamma > 0$ and $T \in (0, \infty]$. Denote $\mathcal{O}_T := (0, T) \times \mathbb{R}^d$. For a measurable function h on $\Omega \times \mathcal{O}_T$, we define the q-th order stochastic bounded mean oscillation (BMO in short) quasi-norm of h on $\Omega \times \mathcal{O}_T$ as follows

$$[h]^q_{\mathbb{BMO}(T,q)} = \sup_Q \frac{1}{|Q|^2} \mathbb{E} \int_Q \int_Q |h(t,x) - h(s,y)|^q dtdxdsdy$$

where the sup is taken over all Q of the type

$$Q = Q_c(t_0, x_0) := (t_0 - c^\gamma, t_0 + c^\gamma) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0.$$

It is remarked that when $q = 1$, this is equivalent to the classical BMO semi-norm which is introduced by John-Nirenberg [*On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14** (1961) 415-426].

Let $K(\omega, t, s, x)$ be a measurable function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d$ such that for each $t \in \mathbb{R}_+$, $(\omega, s) \mapsto K(\omega, t, s, \cdot)$ is a predictable L^1_{loc}-valued process.

Kim proposed the following sufficient conditions for deriving the BMO estimate of $\mathcal{G}g$.

Assumption 2.1 There exist a $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_\lambda^t \left| \int_{|x| \geq c} |K(t, r, x)| dx \right|^2 dr \right\|_{L^{\kappa/2}(\Omega)} \leq \varphi((t-\lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\begin{aligned} & \left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(t,r,x) - K(s,r,x)| dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \\ \leq \quad & \varphi((t-s)(t \wedge s - \lambda)^{-1}); \end{aligned}$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\begin{aligned} & \left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(s,r,x+h) - K(s,r,x)| dx \right)^2 dr \right\|_{L^{\kappa/2}(\Omega)} \\ \leq \quad & N\varphi(|h|(s-\lambda)^{-1/\gamma}). \end{aligned}$$

Assumption 2.2 Suppose that $\mathcal{G}g$ is well-defined (a.e.) and the following holds

$$\mathbb{E}\int_0^T \|\mathcal{G}g(t,\cdot)\|_{L^{p_0}}^{p_0} dt \le N_0 \left\| \int_0^T \| |g(t,\cdot)|_{l_2} \|_{L^{p_0}}^{p_0} dt \right\|_{L^{\tilde{\kappa}}(\Omega)}$$

where $\tilde{\kappa}$ is the conjugate of κ, and

$$\mathcal{G}g(t,x) = \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} K(t,s,x-y) g^k(s,y) dy dw_s^k$$

Based on the Kunita's inequality (see e.g., Applebaum's monograph 2009) and the BDG inequality of Lévy noise (see Marinnelli,Prévôt and Röckner [JFA 2010];Marinnelli and Röckner [*On the maximal inequalities of Burkholder, Davis and Gundy*, Expo. Math. **34** (2016) 1-26]), we formulate the following assumptions

Assumption 2.3 There exist constants $q_0 \geq 2$, $\kappa \in [1, \infty]$ and a nondecreasing function $\varphi(t) : (0, \infty) \mapsto [0, \infty)$ such that

(i) for any $t > \lambda > 0$ and $c > 0$,

$$\left\| \int_\lambda^t \left| \int_{|x|\geq c} |K(t,r,x)| dx \right|^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t-\lambda)c^{-\gamma});$$

(ii) for any $t > s > \lambda > 0$,

$$\left\| \int_0^\lambda \left(\int_{\mathbb{R}^d} |K(t,r,x) - K(s,r,x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq \varphi((t-s)(t\wedge s-\lambda)^{-1});$$

(iii) for any $s > \lambda \geq 0$ and $h \in \mathbb{R}^d$,

$$\left\| \int_0^{\lambda} \left(\int_{\mathbb{R}^d} |K(s,r,x+h) - K(s,r,x)| dx \right)^{q_0} dr \right\|_{L^{\kappa/q_0}(\Omega)} \leq N\varphi(|h|(s-\lambda)^{-1/\gamma}).$$

Assumption 2.4 Similar Assumption 2.2, suppose that $\mathcal{G}g$ is well-defined (a.e.) and the following holds

$$\mathbb{E} \int_0^T \|\mathcal{G}g(t,\cdot)\|_{L^{q_0}}^{q_0} dt \leq N_0 \left\| \int_0^T \int_Z \|g(t,\cdot,z)\|_{L^{q_0}}^{q_0} \nu(dz) dt \right\|_{L^{\tilde{\kappa}}(\Omega)}.$$

Our main result is the following

Theorem 1

Let Assumptions 2.3 and 2.4 hold. Assume that the function g satisfies

$$\left\| \int_Z \|g(\cdot,\cdot,z)\|^{\varpi}_{L^\infty(\mathcal{O}_T)} \nu(dz) \right\|_{L^\varsigma(\Omega)} < \infty, \quad \varpi = 2 \ \text{ or } \ q_0$$

where $\varsigma = q_0 \tilde{\kappa} \vee \frac{q_0 \kappa}{2(\kappa - q_0)^+}$ ($\varsigma = \infty$ if $\kappa \leq q_0$). Then for any $q \in [2, q_0 \wedge \kappa]$, one has

Theorem 1 (cont'd)

$$
\begin{aligned}
[\mathcal{G}g]_{\mathbb{BMO}(T,q)} \leq \; & N\Bigg(\Bigg\| \Big(\int_Z \|g(\cdot,\cdot,z)\|^2_{L^\infty(\mathcal{O}_T)} \nu(dz) \Big)^{q/2} \Bigg\|_{L^{\frac{\kappa}{\kappa-q}}} \\
& + \Bigg\| \int_Z \|g(\cdot,\cdot,z)\|^{q_0}_{L^\infty(\mathcal{O}_T)} \nu(dz) \Bigg\|^{q/q_0}_{L^{\tilde{\kappa}}(\Omega)} \\
& + \Bigg\| \int_Z \|g(\cdot,\cdot,z)\|^{q}_{L^\infty(\mathcal{O}_T)} \nu(dz) \Bigg\|_{L^{\frac{\kappa}{\kappa-q}}} \Bigg)
\end{aligned}
$$

where $N = N(N_0, d, q, q_0, \gamma, \kappa, \varphi)$.

For $X = (t, x) \in \mathbb{R} \times \mathbb{R}^d$ and $Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d$, set

$$\delta(X, Y) := \max \left\{ |x - y|, |t - s|^{\frac{1}{2}} \right\}.$$

Let $Q_c(X)$ be the ball centered in $X = (t, x)$ and of radius c, i.e.,

$$Q_c(X) := \{Y = (s, y) \in \mathbb{R} \times \mathbb{R}^d : \ \delta(X, Y) < R\} = (t - c^2, t + c^2) \times B_c(x).$$

Fix $T \in (0, \infty]$ arbitrarily. Denote $\mathcal{O}_T = (0, T) \times \mathbb{R}^d$. Let D be a bounded domain in $\mathbb{R}^{d+1}$ and for $X \in D$, $D(X, r) := D \cap Q_r(X)$ and $d(D) := diam D$. We first introduce the definition of Campanato space.

Definition

(Campanato Space) Let $p \geq 1$ and $\theta \geq 0$. u belongs to Campanato space $\mathcal{L}^{p,\theta}(D;\delta)$ if $u \in L^p(D)$ and

$$[u]_{\mathcal{L}^{p,\theta}(D;\delta)} := \left(\sup_{X\in D, d(D)\geq\rho>0} \frac{1}{|D(X,\rho)|^\theta} \int_{D(X,\rho)} |u(Y) - u_{X,\rho}|^p dY \right)^{1/p} < \infty$$

and

$$\|u\|_{\mathcal{L}^{p,\theta}(D;\delta)} := \left(\|u\|^p_{L^p(D)} + [u]^p_{\mathcal{L}^{p,\theta}(D;\delta)} \right)^{1/p}$$

where $|D(X,\rho)|$ stands for the Lebesgue measure of $D(X,\rho)$ and

$$u_{X,\rho} = \frac{1}{|D(X,\rho)|} \int_{D(X,\rho)} u(Y)dY.$$

It is clear that Campanato space is a Banach space and has the following property if $1 \leq p \leq q < \infty$, $(\theta - p)/p \leq (\sigma - p)/q$, it holds

$$\mathcal{L}^{q,\sigma}(D;\delta) \subset \mathcal{L}^{p,\theta}(D;\delta).$$

Next we recall the definition of Hölder space.

Definition

(Hölder Space) Let $0 < \alpha \leq 1$. u belongs to Hölder space $C^{\alpha}(\bar{D};\delta)$ if u satisfies

$$[u]_{C^{\alpha}(\bar{D};\delta)} := \sup_{X \in D, d(D) \geq \rho > 0} \frac{|u(X) - u(Y)|}{\delta(X,Y)^{\alpha}} < \infty$$

and

$$\|u\|_{C^{\alpha}(\bar{D};\delta)} := \sup_{D} |u| + [u]_{C^{\alpha}(\bar{D};\delta)}.$$

Definition

Let $D \subset \mathbb{R}^{d+1}$. Domain D is called A-type if there exists a constant $A > 0$ such that $\forall X \in D$, $0 < \rho \leq diamD$, it holds that

$$|D(X,\rho)| \geq A|Q_\rho(X)|.$$

Comparing with the two space, we have the following relations.

Proposition

Assume that D is an A-type bounded domain. Then we have the following relation: when $1 < \theta \leq 1 + \frac{p}{d+2}$ and $p \geq 1$,

$$\mathcal{L}^{p,\theta}(D;\delta) \cong C^{\alpha}(\bar{D};\delta)$$

with $\alpha = \frac{(d+2)(\theta-1)}{p}$, where d is the dimension of the space and $A \cong B$ means that both $A \subseteq B$ and $B \subseteq A$ hold.

We aim to obtain the Campanato estimates under some assumptions on the kernel K. Noting that

$$\begin{aligned} &\left(\sup_{X\in D, d(D)\geq\rho>0}\frac{1}{|D(X,\rho)|^{\theta}}\int_{D(X,\rho)}|u(Y)-u_{X,\rho}|^{p}dY\right)^{1/p} \\ =\ &\left(\sup_{X\in D, d(D)\geq\rho>0}\frac{1}{|D(X,\rho)|^{\theta}}\int_{D(X,\rho)}\left|u(Y)-\frac{1}{|D(X,\rho)|}\int_{D(X,\rho)}u(Z)dZ\right|^{p}dY \\ \leq\ &\left(\sup_{X\in D, d(D)\geq\rho>0}\frac{1}{|D(X,\rho)|^{1+\theta}}\int_{D(X,\rho)}\int_{D(X,\rho)}|u(Y)-u(Z)|^{p}dZdY\right)^{1/p}, \end{aligned}$$

so the definition of semi-norm of the Campanato space can be replaced by the above inequality. We also remark that in order to get the Hölder estimate, the range of θ must be lager than 1.

For a measurable function h on $\Omega \times \mathcal{O}_T$, we define the Campanato quasi-norm of h on $\Omega \times \mathcal{O}_T$

$$[h]^p_{\mathcal{L}^{p,\theta}((Q;\delta);L^p(\Omega))} := \sup_Q \frac{1}{|Q|^{1+\theta}} \mathbb{E} \int_Q \int_Q |h(t,x) - h(s,y)|^p dtdxdsdy$$

where the sup is taken over all $Q = D \cap Q_c$ of the type

$$Q_c(t_0, x_0) := (t_0 - c^2, t_0 + c^2) \times B_c(x_0) \subset \mathcal{O}_T, \quad c > 0, t_0 > 0.$$

It is remarked that when $\theta = 1$, this is equivalent to the classical BMO semi-norm which is introduced in by John and Nirenberg. If the Campanato quasi-norm of h is finite, we then say that h belongs to the space $\mathcal{L}^{p,\theta}((Q;\delta);L^p(\Omega))$.

Given a kernel $K : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, set for $g : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$

$$\mathcal{K}g(t,x) := \int_0^t \int_{\mathbb{R}^d} K(t-r,y)g(r,x-y)dydW(r).$$

Assume $\gamma_i > 0$ $(i = 1,2)$ such that for any $t \in (0,T]$

$$\begin{aligned} & \int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z) - K(s-r,z)|(1+|z|^\beta)dz \right)^2 dr \\ \leq \; & N(T,\beta)(t-s)^{\gamma_1} \\ & \int_0^s \left(\int_{\mathbb{R}^d} |K(s-r,z)|dz \right)^2 dr \leq N_0 \\ & \int_s^t \left(\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta)dz \right)^2 dr \leq N(T,\beta)(t-s)^{\gamma_2} \end{aligned}$$

where N_0 is a positive constant.

Theorem 2

Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Suppose that $g \in C^\beta(\mathbb{R}_+ \times \mathbb{R}^d)$, $0 < \beta < 1$, is a non-random function and $g(0,0) = 0$. Then we have, for $p \geq 1$ and $\beta < \gamma$

$$[\mathcal{K}g]_{\mathcal{L}^{p,\theta}((D;\delta);L^p(\Omega))} \leq N(N_0, \beta, T, d, p)$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

This result shows that $\mathcal{K}g(t,x) \in \mathcal{L}^{p,\theta}((Q;\delta); L^p(\Omega))$. That is, $\|\mathcal{K}g\|_{L^p(\Omega)} \in \mathcal{L}^{p,\theta}(Q;\delta)$. Furthermore, by above Proposition, one gets

$$\mathcal{K}g(t,x) \in C^{\gamma}((\bar{D};\delta); L^p(\Omega)).$$

In above Theorem, the noise term g depends on the times and spatial variables. A natural question is: if g does not depend on the time t, what can we get?

Theorem 3

With the preamble as Thorem 2. Suppose that $g \in C^{\beta}(\mathbb{R}^{d+1})$, $0 < \beta < 1$ and $g(0) = 0$. Let D be a A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Then we have, for $p \geq 1$,

$$[\mathcal{K}g]_{\mathcal{L}^{p,\theta}(D;\delta)} \leq N(N_0, \beta, T, d, p)$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1, \gamma_2, \beta\}$.

Remark By above Proposition, one can get $\mathcal{K}g(t,x) \in C^{\gamma}((\bar{D};\delta); L^p(\Omega))$. In particular, taking $g = constant$, we have the regularity of time variable is $C^{\frac{1}{2}-}$ and the regularity of spatial variable is C^{∞}.

Consider the following stochastic singular integral operator

$$\begin{aligned}\mathcal{G}g(t,x) &= \int_0^t \int_Z K(t,s,\cdot) * g(s,\cdot,z)(x)\tilde{N}(dz,ds) \\ &= \int_0^t \int_Z \int_{\mathbb{R}^d} K(t-s,x-y)g(s,y,z)dy\tilde{N}(dz,ds)\end{aligned}$$

for $\mathbb{F}$-predictable processes $g : [0,T] \times \mathbb{R}^d \times Z \times \Omega \to \mathbb{R}$. For simplicity, we assume that the kernel function is deterministic. We first recall the Kunita's first inequality.

Definition

(Kunita's first inequality see e.g., Applebaum 2009) For any $p \geq 2$, there exists $N(p) > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|I(t)|^p\right) \leq N(p)\left\{\mathbb{E}\left[\left(\int_0^T\int_Z |H(t,z)|^2\nu(dz)dt\right)^{p/2}\right] + \mathbb{E}\left[\int_0^T\int_Z |H(t,z)|^p\nu(dz)dt\right]\right\}$$

where $H \in \mathcal{P}_2(t,E)$ and $I(t) = \int_0^t\int_Z H(s,z)\tilde{N}(dz,ds)$. Here, $\mathcal{P}_2(T,E)$ denotes the set of all equivalence classes of mappings $F : [0,T]\times E\times\Omega \to \mathbb{R}$ which coincide almost everywhere with respect to $\rho\times P$ and which satisfy the following conditions: (i) F is $\mathbb{F}$-predictable; (ii) $P\left(\int_0^T\int_E |F(t,x)|^2\rho(dt,dx) < \infty\right) = 1$.

Theorem 4

Let $g_1 : Z \times \Omega \to \mathbb{R}$ be measurable and fulfil the following

$$\mathbb{E}\left[\left(\int_Z |g_1(z)|^2 \nu(dz)\right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz)\right] < \infty$$

for some constant $p_0 > 2$. Suppose that the function g satisfies that

$$|g(t,x,z) - g(s,y,z)| \leq C_g \max\left\{(t-s)^{\frac{1}{2}}, |x-y|\right\}^{\beta} g_1(z)\, a.s.$$

for all $z \in Z$, and $g(0,0,z) = 0$ uniformly for $z \in Z$ almost surely. Assume further that there exist positive constants γ_i $(i = 1, 2)$ such that the non-random kernel function satisfies that for any $t \in (0, T]$,

Theorem 4 (cont'd)

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(t-r,z)-K(s-r,z)|(1+|z|^\beta)dz\right)^p dr \leq N(T,\beta)(t-s)^{\frac{\gamma_1 p}{2}}$$

$$\int_0^s \left(\int_{\mathbb{R}^d} |K(s-r,z)|dz\right)^p dr \leq N_0$$

$$\int_s^t \left(\int_{\mathbb{R}^d} |K(t-r,z)|(1+|z|^\beta)dz\right)^p dr \leq N(T,\beta)(t-s)^{\frac{\gamma_2 p}{2}}$$

where N_0 is a positive constant. Let D be an A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Then we have, for $2 \leq p \leq p_0$ and $\beta < \alpha$

$$[\mathcal{K}g(t,x)]_{\mathcal{L}^{p,\theta}(D;\delta)} \leq N(N_0,\beta,T,d,p)$$

where $\theta = 1 + \frac{\gamma p}{d+2}$ and $\gamma = \min\{\gamma_1,\gamma_2,\beta\}$. Furthermore, $\mathcal{G}g(t,x) \in C^\gamma((\bar{D};\delta);L^p(\Omega))$.

It is clear to see that

$$\sum_{k=1}^{\infty}\int_0^t\int_{\mathbb{R}^m}\int_{\mathbb{R}^d}K(t,s,x-y)g^k(s,y)dyz\tilde{N}_k(dz,ds)$$

is the fundamental solution to the following equation

$$du_t(x)=\Delta^{\frac{\alpha}{2}}u_t(x)dt+\sum_{k=1}^{\infty}\int_{\mathbb{R}^m}g^k(t,x)z\tilde{N}_k(dz,dt),\quad u_0=0,\ 0\leq t\leq T$$

where $\int_{\mathbb{R}^m}z\tilde{N}_k(t,dz)=:\ Y_t^k$ are independent m-dimensional pure jump Lévy processes with Lévy measure of ν^k.

Theorem 5

Let $q_0 \geq 2$. Then for any $g \in \mathbb{H}_p^{\eta+\alpha-\alpha/p}(T, \ell_2)$, the above equation has a unique solution u in $\mathcal{H}_p^{\eta+\alpha}$ ($\eta \in \mathbb{R}$), and for this solution

$$\|u\|_{\mathcal{H}_p^{\eta+\alpha}(t)} \leq N(p, T)\|g\|_{\mathbb{H}_p^{\eta+\alpha-\alpha/p}(t,\ell_2)}$$

for every $t \leq T$. Moreover, we have for $q \in [2, q_0]$

$$[\nabla^\beta u]_{\mathbb{BMO}(T,q)} \leq N\hat{c}\left(\mathbb{E}[\||g|_{\ell_2}\|^{q_0}_{L^\infty(\mathcal{O}_T)}]\right)^{q/q_0}$$

where $\beta = \alpha/q_0$ and $\hat{c} := \sup_{q\geq 2, k\geq 1}\left(\int_{\mathbb{R}^m} |z|^q \nu^k(dz)\right)^{\frac{1}{q}}$.

Let us consider the following SPDEs

$$\begin{cases} du(t,x) = \Delta^{\frac{\alpha}{2}} u(t,x)dt + \int_Z g(t,x,z)\tilde{N}(dt,dz), \quad t>0,\ x\in\mathbb{R}^d, \\ u(0,x) = u_0(x), \quad x\in\mathbb{R}^d, \end{cases}$$

where $\Delta^{\frac{\alpha}{2}} = -(-\Delta)^{\frac{\alpha}{2}}$. The well-posedness is known and the solution can be written as

$$\begin{aligned} & u(t,x) \\ = \; & (\mathcal{G} * u_0)(t,x) + (\mathcal{G} * g)(t,x) \\ = \; & \int_{\mathbb{R}^d} p(t;x,y)u_0(y)dy \\ & \quad + \int_0^t \int_{\mathbb{R}^d} \int_Z p(t,r;x,y)g(r,y,z)dy\tilde{N}(dt,dz). \end{aligned}$$

Theorem 6

Suppose that $u_0 \in C^\beta(\mathbb{R}^d)$ with $\beta < \alpha$ and the function g satisfies

$$|g(t,x,z) - g(s,y,z)| \leq C_g \max\left\{(t-s)^{\frac{1}{2}}, |x-y|\right\}^\beta g_1(z) \ a.s.$$

for all $z \in Z$, and $g(0,0,z) = 0$ uniformly for $z \in Z$ almost surely, where there exists a constant $p_0 > 1$ such that $g_1(z)$ satisfies that

$$\mathbb{E}\left[\left(\int_Z |g_1(z)|^2 \nu(dz)\right)^{p_0/2} + \int_Z |g_1(z)|^{p_0} \nu(dz)\right] < \infty.$$

Let D be a A-type bounded domain in $\mathbb{R}^{d+1}$ such that $\bar{D} \subset \mathcal{O}_T$. Then the solution u is Hölder continuous in domain D_T.

Thank You!