Particle representations for SPDEs with applications

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(Based on joint works with Kurtz; Chang & Liu; etc)

Outline

1. Weighted particle representation
2. System by atomic measure
3. Poisson particle representation
4. Application to stochastic filtering
5. Application to FBSDEs
6. Application to superprocesses

1. Weighted particle representation

SPDE we shall study

$$
\frac{\partial V(t,x)}{\partial t} = L_2(V(t))V(t,x)
$$
$$
+ \int_U L_1(V(t),u)V(t,x)\frac{W(dudt)}{dt} \tag{1.1}
$$

$W(duds)$ noise in space-time
$L_1(V(t),u)$ first order diff. operator
$L_2(V(t))$ second order diff. operator

$$\langle \phi, V(t) \rangle = \langle \phi, V(0) \rangle + \int_0^t \langle L_2(V(s))\phi, V(s) \rangle \, ds \qquad (1.2)$$
$$+ \int_{U \times [0,t]} \langle L_1(V(s), u)\phi, V(s) \rangle \, W(duds)$$

where

$$L_1(\nu, u)\phi(x) = \beta(x, \nu, u)\phi(x) + \alpha^T(x, \nu, u)\nabla\phi(x),$$

$$L_2(\nu)\phi(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x, \nu)\partial_{x_i}\partial_{x_j}\phi(x)$$
$$+ b(x, \nu)^* \nabla\phi(x) + d(x, \nu)\phi(x).$$

Choose σ and γ s.t.

$$\begin{aligned} a(x,\nu) &= \sigma(x,\nu)\sigma^T(x,\nu) \\ &+ \int_U \alpha(x,\nu,u)\alpha^T(x,\nu,u)\mu(du) \end{aligned}$$

and

$$\begin{aligned} b(x,\nu) &= c(x,\nu) + \sigma(x,\nu)\gamma(x,\nu) \\ &+ \int_U \beta(x,\nu,u)\alpha(x,\nu,u)\mu(du) \end{aligned}$$

Consider system with locations

$$
\begin{aligned}
X_i(t) &= X_i(0) + \int_0^t \sigma(X_i(s), V(s)) dB_i(s) \\
&\quad + \int_0^t c(X_i(s), V(s)) ds \\
&\quad + \int_{U \times [0,t]} \alpha(X_i(s), V(s), u) W(duds)
\end{aligned}
\tag{1.3}
$$

with

$$
V(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n A_i(t) \delta_{X_i(t)} \tag{1.4}
$$

and weights

$$A_i(t) \tag{1.5}$$
$$= A_i(0) + \int_0^t A_i(s)\gamma^T(X_i(s), V(s))dB_i(s)$$
$$+ \int_0^t A_i(s)d(X_i(s), V(s))ds$$
$$+ \int_{U \times [0,t]} A_i(s)\beta(X_i(s), V(s), u)W(duds).$$

Theorem (Kurtz-X)

$V(t)$ is sol. to (1.4) iff it is a sol. to SPDE (1.2).

Most with constant weight 1:

McKean ('67)

$$\begin{cases} \text{Hitsuda \& Mitoma ('86) conti.} \\ \text{Graham ('96) jump} \end{cases}$$

$$\infty\text{-dim.} \begin{cases} \text{Chiang, Kallianpur \& Sundar ('91), Cont.} \\ \text{Kallianpur \& Xiong ('94) jump} \\ \text{Xiong ('95)} \end{cases}$$

$$\text{no-constant} \begin{cases} \text{Non-random: Dawson \& Vaillancourt ('95)} \\ \text{Random: Kotelenez ('95) } A_i(t) \equiv A_i(0) \end{cases}$$

Conditions

$$|\sigma(x,\nu)|^2 \le K^2$$

$$|\sigma(x_1,\nu_1) - \sigma(x_2,\nu_2)|^2 \le K^2(|x_1 - x_2|^2 + \rho(\nu_1,\nu_2)^2)$$

where

$$\rho(\nu_1,\nu_2) = \sup\{|\langle \phi, \nu_1 - \nu_2 \rangle| : \|\phi\|_\infty \le 1, Lip(1)\}.$$

Similar conditions for other coeff.

Theorem (Kurtz-X)

The system (1.3-1.5) has a solution.

Idea of proof

Sequence of approximation

$$B_i^n(t) = B_i\left(\frac{[nt]}{n}\right), \qquad D_n(t) = \frac{[nt]}{n}$$

$$W^n(A \times [0,t]) = W\left(A \times [0, \frac{[nt]}{n}]\right)$$

Solution (X^n, A^n, V^n) weak convergence to (X, A, V)

Theorem (Kurtz-X)

The system (1.3-1.5) has at most one solution.

Idea of proof
Lipschitz condition is not satisfied
Local Lipschitz, i.e. on

$$\{(a_i) : |a_i| \leq M, \ i = 1, 2, \cdots\}$$

How to truncate?

Theorem (Kurtz-X)

The system (1.3-1.5) has at most one solution.

Idea of proof
Lipschitz condition is not satisfied
Local Lipschitz, i.e. on

$$\{(a_i) : |a_i| \leq M, \ i = 1, 2, \cdots\}$$

How to truncate?
Truncate the average:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} A_i(t)^2$$

How to prove uniqueness for SPDE (1.2)?
Consider Linear SPDE

$$
\begin{aligned}
\langle \phi, U(t) \rangle &= \langle \phi, U(0) \rangle + \int_0^t \langle L_{2,s}\phi, U(s) \rangle \, ds \\
&\quad + \int_{U \times [0,t]} \langle L_{1,s,u}\phi, U(s) \rangle \, W(du\,ds) \quad (1.6)
\end{aligned}
$$

where

$$
L_{2,s}\phi = L_2(V(s))\phi
$$
$$
L_{1,s,u}\phi = L_1(V(s), u)\phi
$$

Theorem (Kurtz-X)

If $V_0 \in L^2(\mathbb{R}^d)$, then the linear SPDE (1.6) has at most one solution.

Key: Smooth out by $Z_\delta(t) = T_\delta U(t)$.

Theorem (Kurtz-X)

If $V_0 \in L^2(\mathbb{R}^d)$, then SPDE (1.2) has at most one solution.

Proof

Let $V_1(t)$ be another solution.

Consider

$$X_i(t) = X_i(0) + \int_0^t \sigma(X_i(s), V_1(s)) dB_i(s) + \cdots$$

$$A_i(t) = \cdots$$

Let

$$V_2(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{\infty} A_i(t) \delta_{X_i(t)}$$

Then V_2 is a solution to

$$
\begin{aligned}
\langle \phi, U(t) \rangle &= \langle \phi, U(0) \rangle + \int_0^t \langle L_2(V_1(s))\phi, U(s) \rangle \, ds \qquad (1.7) \\
&\quad + \int_{U \times [0,t]} \langle L_1(V_1(s), u)\phi, U(s) \rangle \, W(du\,ds).
\end{aligned}
$$

V_1 is also a solution to (1.7). By uniqueness, $V_1 = V_2$. V_1 is a solution to (1.3-1.5). $V_1 = V$

For $i = 1, 2, \cdots$, the position of the ith particle is

$$X_t^i = X_0^i + \int_0^t \sigma(X_s^i)dW_s^i + \int_0^t b(X_s^i, N_s)ds + \int_0^t \alpha(X_s^i)dW_s,$$
$$(2.1)$$

where

$$N_s = \sum_i \delta_{X_s^i}.$$

Condition (B): $\exists\ c,\ r > 0$ s.t., $\forall\ x,\ z^i \in \mathbb{R},\ i = 1, 2, \cdots$, we have

$$|\sigma(x)| + |\alpha(x)| + \left| b\left(x, \sum \delta_{z^i}\right) \right| \leq K.$$

Condition (Lip): σ, α Lip., and $\exists$ c, $r > 0$ s.t.,
$\forall x$, $\tilde{x}$, z^i, $\tilde{z}^i \in \mathbb{R}$, $i = 1, 2, \cdots$, we have

$$
\begin{aligned}
&\left| b\left(x, N\right) - b\left(\tilde{x}, \tilde{N}\right) \right| \\
\leq\ & c|x - \tilde{x}| \left(N(S_r(x)) + \tilde{N}(S_r(\tilde{x})) \right) \\
&+ c \left(\sum_{|z^i - x| \leq r} |z^i - \tilde{z}^i| + \sum_{|\tilde{z}^i - \tilde{x}| \leq r} |z^i - \tilde{z}^i| \right)
\end{aligned} \tag{2.2}
$$

where

$$
N = \sum \delta_{z^i}.
$$

Typical example

$$b(x, \nu) = \phi \left(x, \int h(x - y)\nu(dy) \right)$$

where ϕ bounded Lipschitz in both variables, h bounded Lipschitz w/ compact support.

Condition (I): $X_0^i = i$, $i \in \mathbb{Z}$.

Lemma

For any $c_1 > 2$ and $t > 0$, there exists a positive constant c_2 such that

$$\mathbb{P}\left(N_t([-n, n]) > c_1 n\right) \leq c_2 e^{-(c_1 - 2)n}, \qquad \forall n \in \mathbb{N}. \qquad (2.3)$$

Corollary

There exists a constant c_3 such that for all n,

$$\mathbb{E}\left(N_t([-n, n])^2 1_{N_t([-n,n]) > 3n}\right) \leq c_2 n e^{-n}.$$

Let c be s.t.

$$\rho(a) = c 1_{|a|<1} \exp\left(-1/(1-a^2)\right), \qquad a \in \mathbb{R}$$

becomes a p.d.f. Let

$$\phi(x) = \int_{\mathbb{R}} e^{-|a|} \rho(x-a) da, \qquad x \in \mathbb{R}.$$

Then, ϕ is smooth and

$$c_n e^{-|x|} \le \phi^{(n)}(x) \le C_n e^{-|x|}, \qquad \forall \ x \in \mathbb{R}.$$

Also

$$\phi(x^i) \le \phi(\tilde{x}^j) \exp\left(|x^i - \tilde{x}^i| + |\tilde{x}^j - \tilde{x}^i|\right). \tag{2.4}$$

Let $\{\tilde{X}_t^i, \ i \in \mathbb{Z}\}$ be another solution.

Lemma

There exists a constant $c_1 > 0$ such that $\forall \, n$,

$$\mathbb{E}\left\{ \sum_{j: \ |X_s^j| > n} |X_s^j - \tilde{X}_s^j|^2 e^{-|X_s^j|} \#\{i: \ |X_s^j - X_s^i| \le 1\} \right\} \le c_1 e^{-n/2}.$$

(2.5)

Theorem (Kurtz-X)

The system (2.1) has at most one solution.

Idea of proof

$$f(t) \equiv \mathbb{E} \sum_i |X_t^i - \tilde{X}_t^i|^2 \phi(X_t^i) \le I_0 + I_1 + I_2 \qquad (2.6)$$

where

$$I_0 = c \int_0^t \mathbb{E} \sum_i |X_s^i - \tilde{X}_s^i|^2 \phi(X_s^i) ds,$$

$$I_1 = c \int_0^t \mathbb{E} \sum_i \sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) 1_{|X_s^j - X_s^i| \le 1} ds,$$

$$I_2 = c \int_0^t \mathbb{E} \sum_i \sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(\tilde{X}_s^j) 1_{|\tilde{X}_s^j - \tilde{X}_s^i| \le 1} ds.$$

$$I_1$$

$$\leq c \int_0^t \mathbb{E} \sum_j |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) \#\{i: \ |X_s^j - X_s^i| \leq 1\} ds$$

$$\leq c \int_0^t \mathbb{E} \left\{ \left(\sum_{j: \ |X_s^j| < n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) \right) N_s([-n, n]) \right\} ds$$

$$+ c \int_0^t \mathbb{E} \left\{ \sum_{j: \ |X_s^j| \geq n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) \#\{i: \ |X_s^j - X_s^i| \leq 1\} \right\} ds$$

$$\equiv I_{11} + I_{12}.$$

$$I_{12} \leq ce^{-n/2},$$

$$
\begin{aligned}
I_{11} &\leq 3cn \int_0^t \mathbb{E} \sum_{j:\ |X_s^j| < n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) ds \\
&\quad + c \int_0^t \mathbb{E}\bigg\{ \bigg(\sum_{j:\ |X_s^j| < n} |X_s^j - \tilde{X}_s^j|^2 \phi(X_s^j) \bigg) \\
&\qquad \times N_s([-n,n]) 1_{N_s([-n,n]) > 3n} \bigg\} ds \\
&\leq cn \int_0^t f(s) ds + ce^{-\delta n}.
\end{aligned}
$$

Then,

$$I_1 \le cn \int_0^t f(s)ds + 2ce^{-\delta n}.$$

Similarly

$$I_2 \le cn \int_0^t \tilde{f}(s)ds + ce^{-\delta n}. \tag{2.7}$$

Hence,

$$f(t) \le cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

Similarly, we can prove that

$$\tilde{f}(t) \le cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

Adding two equations, we get

$$f(t) + \tilde{f}(t) \le cn \int_0^t \left(f(s) + \tilde{f}(s) \right) ds + ce^{-\delta n}.$$

By Gronwall's inequality, for $t < \delta$, we have

$$f(t) + \tilde{f}(t) \le ce^{-\delta n} e^{nt} \to 0$$

This gives us uniqueness for $t < \delta$. We get uniqueness by choosing replacing $e^{-|x|}$ by $e^{-\lambda|x|}$.

Consider system: For $i = 1, 2, \cdots,$

$$
\begin{aligned}
X_t^i &= X_0^i + \int_0^t \sigma(X^i) dW_i(s) + \int_0^t b(X_s^i, V_s) ds \\
&\quad + \int_{U \times [0,t]} \alpha(X_s^i, u) W(duds)
\end{aligned} \tag{3.1}
$$

and

$$
\begin{aligned}
U_t^i &= U_0^i + \int_0^t U_s^i \gamma(X_s^i) dW_i(s) + \int_0^t U_s^i a(X_s^i, V_s) ds \\
&\quad + \int_{U \times [0,t]} U_s^i \beta(X_s^i, u) W(duds),
\end{aligned} \tag{3.2}
$$

where V is the measure-valued process given by

$$V(t) = \lim_{u \to \infty} \frac{1}{u} \sum_{U_t^i \leq u} \delta_{X_t^i}. \qquad (3.3)$$

Remark

i) Given V,

$$\sum_i \delta_{(X_t^i, U_t^i)}$$

is a Poisson random measure on $\mathbb{R} \times \mathbb{R}_+$ with intensity measure $V(t) \times m$, where m is the Lebesgue measure on $\mathbb{R}_+$.

ii) Advantages of Poisson representation: $V(t)$ can be infinite measure.

iii)

$$
\begin{aligned}
V(t) &= \lim_{\epsilon \to 0+} \epsilon \sum_{i=1}^{\infty} e^{-\epsilon U_t^i} \delta_{X_t^i} \\
&= \lim_{\epsilon \to 0+} \epsilon^2 \sum_{i=1}^{\infty} U_t^i e^{-\epsilon U_t^i} \delta_{X_t^i}.
\end{aligned}
$$

Condition (B) remains true.
Wasserstein distance by

$$\rho_x(\nu_1, \nu_2) = \sup\left\{\left|\int_{|y-x|\leq 1} f(y)(\nu_1 - \nu_2)(dy)\right| : f \in \mathbb{B}_1\right\},$$

where $\mathbb{B}_1$ is the unit ball under Lipschitz norm, i.e.,

$$\mathbb{B}_1 = \{f : \mathbb{R} \to \mathbb{R}, \ |f(x)| \leq 1, \ |f(x) - f(y)| \leq |x - y|\}.$$

Condition (Lip): $\exists\, K$ s.t. $\forall\, x,\, \tilde{x} \in \mathbb{R}$ and $\nu,\, \tilde{\nu} \in M(\mathbb{R})$, we have

$$|b(x, \nu) - b(\tilde{x}, \tilde{\nu})|$$
$$\leq \quad K\left(|x - \tilde{x}|\,(1 + \nu(S_1(x)) + \tilde{\nu}(S_1(\tilde{x}))) + \rho_x(\nu, \tilde{\nu}) + \rho_{\tilde{x}}(\nu, \tilde{\nu})\right),$$

where $S_1(x)$ denotes the interval $(x - 1, x + 1)$.

Suppose that (3.1-3.3) has two solutions (X_i, U_i, V) and $(\tilde{X}_i, \tilde{U}_i, \tilde{V})$, $i = 1, 2, \cdots$.

We will need the following large ball type estimate.

Lemma

Suppose that

$$\sup_{y \in \mathbb{R}} \left(V_0(S_2(y)) + \int_{|x-y|>2} \frac{V_0(dx)}{|x-y-1|} \right) < \infty,$$

where $S_2(y)$ is the interval with center y and radius 2. Then, there are constants c, $\delta > 0$ such that for any $s \in [0, T]$ and $n \in \mathbb{N}$, we have

$$\mathbb{P} \left(\sup_{|y| \leq n} V_s(S_1(y)) > \sqrt{n} \right) \leq ce^{-\delta n}. \qquad (3.4)$$

Key in the proof i) V solution to (1.2).

ii) By the exponential formula of Boué-Dupuis, we have

$$- \log \mathbb{E} e^{f(W)} = \inf_u \mathbb{E} \left(\frac{1}{2} \|u\|^2 - f \left(W + \int_0^{\cdot} u_s ds \right) \right), \quad (3.5)$$

where the infimum is taken over all processes u that are $\mathcal{F}_t^W$-predictable such that

$$\|u\|^2 = \int_0^T u_t^2 dt < \infty, \qquad a.s.$$

Theorem (Kurtz-X)

The infinite system (3.1-3.3) has a unique solution.

Idea of proof Let

$$f(t) \equiv \lim_{\epsilon \to 0} \sum_i \epsilon e^{-\epsilon U_t^i} |X_t^i - \tilde{X}_t^i|^2 \phi(X_t^i),$$

and

$$g(t) = \lim_{\epsilon \to 0} \mathbb{E} \sum_i \epsilon^3 e^{-\epsilon U_t^i} (U_t^i - \tilde{U}_t^i)^2 \phi(X_t^i).$$

$\tilde{f}$ and $\tilde{g}$ are defined similarly.

Let $h(t)$ be the sum of $f(t)$, $g(t)$, $\tilde{f}(t)$ and $\tilde{g}(t)$. Then,

$$h(t) \leq cn \int_0^t h(s)ds + ce^{-\delta n}.$$

The uniqueness then follows from Gronwall's inequality.

Signal process

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t.$$

Observation process

$$Y_t = \int_0^t h(X_s)ds + W_t.$$

Information

$$\mathcal{G}_t = \sigma\{Y_s : \ s \leq t\}.$$

Optimal filter

$$\langle f, \pi_t \rangle = \mathbb{E}\left(f(X_t)|\mathcal{G}_t\right).$$

Kallianpur-Striebel formula,

$$\langle f, \pi_t \rangle = \frac{\langle f, \mu_t \rangle}{\langle 1, \mu_t \rangle},$$

where

$$\langle f, \mu_t \rangle = \hat{\mathbb{E}} \left(f(X_t) M_t | \mathcal{G}_t \right)$$

and $\hat{\mathbb{E}}$ is expectation wrt $\hat{P}$ under which (B, Y) is B.M.

Zakai equation: Special case of (1.1) with $W = Y$ and

$$L_2\phi(x) = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j}\phi(x) + b(x)^*\nabla\phi(x),$$

$$L_1\phi(x) = h(x)\phi(x).$$

Kushner-FKK equation: Special case of (1.1) with W being the innovation process and

$$L_1(\nu, u)\phi(x) = h(x) - \langle h, \nu \rangle .$$

Consider

$$\begin{cases} dX(t) &= b(X(t), Y(t), Z(t))dt + \sigma(X(t), Y(t), Z(t))dW(t), \\ dY(t) &= g(X(t), Y(t), Z(t))dt + Z(t)dW(t), \\ X(0) &= x, \ \ Y(T) = h(X(T)). \end{cases}$$

How to solve FBSDE numerically?

Try $Y(t) = u(t, X(t))$. By Itô's formula, we get

$$
\begin{aligned}
dY(t) &= \partial_t u\, dt + \partial_x u\, dX(t) + \frac{1}{2}\partial_x^2 u (dX(t))^2 \\
&= \partial_t u\, dt + \partial_x u\, b\, dt + \partial_x u \sigma\, dW_t + \frac{1}{2}\partial_x^2 u \sigma^2\, dt \\
&= \left(\partial_t u + b\partial_x u + \frac{1}{2}\sigma^2 \partial_x^2 u \right) dt + \partial_x u \sigma\, dW_t.
\end{aligned}
$$

Compare with FBSDE, we get

$$
\begin{aligned}
&h(t, X(t), u(t, X(t)), Z(t)) \\
=\ &\partial_t u(t, X(t)) + b(t, X(t), u(t, X(t)), Z(t))\partial_x u(t, X(t)) \\
&+\frac{1}{2}\sigma^2(t, X(t), u(t, X(t)), Z(t))\partial_x^2 u(t, X(t)) \qquad (5.6)
\end{aligned}
$$

and

$$
Z(t) = \sigma(t, X(t), u(t, X(t)), Z(t))\partial_x u(t, X(t)).
$$

Four step scheme (Ma-Protter-Yong):
Step 1: Find function $z(t, x, y, p)$ satisfying

$$z(t, x, y, p) = \sigma(t, x, y, z(t, x, y, p))p. \tag{5.7}$$

Step 2: Solve PDE

$$\begin{cases} \partial_t u + b(t, x, u, z(t, x, u, \partial_x u))\partial_x u \\ +\frac{1}{2}\sigma^2(t, x, u, z(t, x, u, \partial_x u))\partial_x^2 u \\ = h(t, x, u, z(t, x, u, \partial_x u)) \\ u(T, x) = g(x), \qquad x \in \mathbb{R}. \end{cases} \tag{5.8}$$

Step 3: Solve SDE

$$dX(t) = \tilde{b}(t, X(t))dt + \tilde{\sigma}(t, X(t))dW_t, \quad X_0 = x, \qquad (5.9)$$

where

$$\tilde{b}(t, x) = b(t, x, u(t, x), z(t, x, u, \partial_x u(t, x))).$$

Step 4:

$$\left\{ \begin{array}{rcl} Y(t) & = & u(t, X(t)) \\ Z(t) & = & z(t, X(t), Y(t), \partial_x u(t, X(t))). \end{array} \right. \qquad (5.10)$$

(A1) The generator g has the following form: for $z = (z_1, \cdots, z_l)$,

$$g(x, y, z) = C(x, y)y + \sum_{j=1}^{l} D_j(x, y)z_j,$$

and $b(x, y)$, $\sigma(x)$, $g(x, y, z)$, $f(x)$, $C(x, y)$ and $D(x, y)$ are all bounded and Lipschitz continuous maps with bounded partial derivatives up to order 2. Furthermore, the matrix $\sigma\sigma^*$ is uniformly positive definite, and the function f is integrable. Here σ^* denote the transpose of the matrix σ.

We consider the following FBSDE in the fixed duration $[0, T]$:

$$\begin{cases} dX(t) = b\left(X\left(t\right), Y\left(t\right)\right) dt + \sigma\left(X\left(t\right)\right) dW(t), \\ -dY(t) = g(X(t), Y(t), Z(t))dt - Z(t)dW(t), \\ X(0) = x, Y(T) = f(X(T)), \end{cases} \quad (5.11)$$

where $b : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times l}$,
$g : \mathbb{R}^d \times \mathbb{R}^k \times \mathbb{R}^{k \times l} \to \mathbb{R}^k$ and $f : \mathbb{R}^d \to \mathbb{R}^k$.

The PDE becomes

$$\begin{cases} -\frac{\partial u(t,x)}{\partial t} &= Lu(t,x) + C\left(x, u\left(t,x\right)\right) u\left(t,x\right) \\ & \quad + \sum_{j=1}^{l} \sigma_j\left(x\right) D_j(x, u\left(t,x\right)) \partial_x u\left(t,x\right) \\ u(T,x) &= f(x), \end{cases}$$

and

$$L = \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} \partial_{x_i, x_j} + \sum_{i=1}^{d} b_i \partial_{x_i},$$

with $a_{ij} = \left(\sigma\sigma^T\right)_{ij}$, $\sigma = (\sigma_1, \cdots, \sigma_l)$ and b_i being the ith coordinate of b.

For $0 \leq t \leq T$, assume $v\left(t,x\right) = u\left(T-t,x\right)$. Then

$$
\begin{cases}
\frac{\partial v(t,x)}{\partial t} &= Lv(t,x) + C\left(x, v\left(t,x\right)\right) v\left(t,x\right) \\
&\quad + \sum_{j=1}^{l} \sigma_j\left(x\right) D_j(x, v\left(t,x\right)) \partial_x v\left(t,x\right) \\
v\left(0,x\right) &= f(x).
\end{cases}
\tag{5.12}
$$

PDE (5.12) can be rewritten as

$$
\begin{aligned}
\frac{\partial v(t,x)}{\partial t} &= \frac{1}{2} \sum_{i,j=1}^{d} \partial_{x_i,x_j} \left(a_{ij}(x) v(t,x) \right) \\
&\quad - \sum_{i=1}^{d} \partial_{x_i} \left(\tilde{b}_i \left(x, v \right) v \left(t, x \right) \right) \\
&\quad + \tilde{c} \left(x, v \left(t, x \right) \right) v \left(t, x \right)
\end{aligned}
$$

where

$$\tilde{c}(x,v) = C(x,v) - \sum_{i=1}^{d} \partial_{x_i} b_i(x,v)$$
$$- \sum_{i=1}^{d} \partial_{x_i} \tilde{D}_i(x,v) + \frac{1}{2} \sum_{i,j=1}^{d} \partial_{x_i,x_j} a_{ij}(x),$$

$$\tilde{b}_i(x,v) = \sum_{j=1}^{d} \partial_{x_j} a_{ij}(x) - b_i(x,v) - \tilde{D}_i(x,v),$$

and

$$\tilde{D}_i(x,v) = \sum_{j=1}^{l} D_j(x,v)\, \sigma_{ij}(x).$$

Theorem (Chang-Liu-X)

For $0 < t \leq T$, $i = 1, 2, \cdots$, let

$$
\begin{cases}
dX_i(t) = \tilde{b}(X_i(t), v(t, X_i(t)))dt + \sigma(X_i(t))dB_i(t), \\
dA_i(t) = A_i(t)\tilde{c}\left(X_i(t), v\left(t, X_i(t)\right)\right) dt \\
V(t) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} A_j(t)\delta_{X_j(t)}
\end{cases}
\tag{5.13}
$$

with i.i.d initial random sequence $\{(X_i(0), A_i(0)), i \in \mathbb{N}\}$ taking values in $\mathbb{R}^d \times \mathbb{R}$, where $\{B_i(t),\ i \in \mathbb{N}\}$ are independent standard Brownian motions, $V(t)$ has density $v(t, x)$ and $v(0, x) = f(x)$. Then, $v(t, x)$ solves SPDE (5.12).

At time $t = 0$, k_n particles at locations

$$x_i^n \in \mathbb{R}, \qquad i = 1, 2, \cdots, k_n.$$

- Each has $expo.(n)$ clock (indep. from each other).

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- Each has $expo.(n)$ clock (indep. from each other).
- When time-up, the particle will split/die w/ equal prob.
- Denote particle by a multi-index α.
- Between branching times, particle α moves according to

$$dx_t^\alpha = b(x_t^\alpha)dt + c(x_t^\alpha)dW_t + e(x_t^\alpha)dB_t^\alpha$$

W, B^α indep. B.M.

Define measure-valued process

$$\mu_t^n = \frac{1}{n} \sum_{\alpha \sim t} \delta_{x_\alpha^n(t)}$$

where $\alpha \sim t$ means particle α is alive at time t.

Convergence Theorem

$(\mu_t^n) \Rightarrow (\mu_t)$ unique sol. to MP: μ_t is $\mathcal{M}(R)$-valued

$$M_t(f) \equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + af'' \rangle \, ds$$

continuous martingale with

$$\langle M(f) \rangle_t = \int_0^t \left(\langle \mu_s, f^2 \rangle + \langle \mu_s, cf' \rangle^2 \right) ds$$

where $a(x) = \frac{1}{2}(e(x)^2 + c(x)^2)$.

Studied by Adler and Skoulakis (2001) and Wang (1998) among others.

(CMP)

$$
\begin{aligned}
N_t(f) &\equiv \langle \mu_t, f \rangle - \langle \mu, f \rangle - \int_0^t \langle \mu_s, bf' + df'' \rangle \, ds \\
&\quad - \int_0^t \langle \mu_s, cf' \rangle \, dW_s
\end{aligned}
$$

P^W-martingale w/

$$
\langle N(f) \rangle_t = \int_0^t \langle \mu_s, f^2 \rangle \, ds
$$

Formally,

$$
\begin{aligned}
N_t(f) &= \langle \mu_t, f \rangle - \langle \mu, f \rangle \\
&\quad - \int_0^t \left\langle \mu_s, (b + c\dot{W}_s)f' + df'' \right\rangle ds,
\end{aligned}
$$

Therefore,

$$
E^W \exp\left(-\langle \mu_t, f \rangle\right) = \exp\left(-\langle \mu, y_{0,t} \rangle\right),
$$

where

$$
\begin{aligned}
y_{s,t} &= f + \int_s^t c\partial_x y_{r,t} \dot{W}_r \, dr \\
&\quad + \int_s^t \left(b\partial_x y_{r,t} + a\partial_x^2 y_{r,t} - y_{r,t}(x)^2 \right) dr.
\end{aligned}
$$

Stochastic integral is backward Itô integral.

Theorem

$$\mathbb{E}^W \exp\left(-\langle\mu_t, f\rangle\right) = \exp\left(-\langle\mu, y_{0,t}\rangle\right),$$

where $Lf = af'' + bf'$,

$$
\begin{aligned}
y_{s,t} &= f + \int_s^t c\partial_x y_{r,t}\hat{d}W_r \\
&\quad + \int_s^t \left(Ly_{r,t} - y_{r,t}(x)^2\right)dr.
\end{aligned}
$$

Forward version:

$$
\begin{aligned}
v_t(x) &= f(x) + \int_0^t \left(Lv_r(x) - \frac{\gamma}{2} v_r(x)^2 \right) dr \\
&\quad + \int_0^t \int_U h(y,x) \nabla v_r(x) W(drdy).
\end{aligned} \tag{6.14}
$$

> **Theorem**
>
> *i) The SLLE (6.14) has a unique solution $v_t(x)$.*
> *ii) v_t is the unique solution of the following infinite particle system: $i = 1, 2, \cdots$,*
>
> $$
> \begin{aligned}
> d\xi_t^i &= dB_i(t) - \int_U h(y, \xi_t^i) W(dt\,dy) \\
> &\quad + \left(2a'(\xi_t^i) - \int_U h(y, \xi_t^i) \nabla h(y, \xi_t^i) \mu(dy) \right) dt, \quad (6.15)
> \end{aligned}
> $$
>
> $$
> dm_t^i = m_t^i \left(\left(a''(\xi_t^i) - v_t(\xi_t^i) \right) dt - \int_U \nabla h(y, \xi_t^i) W(dt\,dy) \right), \quad (6.16)
> $$
>
> $$
> \nu_t = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n m_t^i \delta_{\xi_t^i}, \qquad a.s. \quad (6.17)
> $$
>
> *where for any $t \geq 0$, ν_t is absolutely continuous with respect to Lebesgue measure with v_t as the Radon-Nikodym derivative.*

Thanks!