Control of evolution equations with Lévy noise

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Joint work with A. Święch and E. Priola

July 8, 2017

1. Durham 1974
2. Controlled system
3. HJB equation. Viscosity solutions
4. Value function as viscosity solution
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2nd Durham Symposium July 22 - August1, 1974
Functional Analysis and Stochastic Processes
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2. C. Dellacherie

Stochastic integral representations of Poissonian martingales.

Controlled system

(1)
$$dX(s) = (AX(s) + g(X(s), a(s)))\,ds + G(X(s-), a(s))\,dZ(s),$$

$$X(t) = x \in H,\ \ s \in [t, T],\ \ Z \text{ Lévy process on } U$$

$$a(s), \qquad s \in [t, T],\ \text{control},\ a(s) \in \Lambda\text{-set of control parameters}$$

A is a linear operator, with domain $D(A) \subset H$,

$$\frac{dX(t)}{dt} = AX(t), \ \ X(0) = x \in H,$$

has unique, weak solution:

$$X(t) = e^{tA}x, \ \ t \geq 0.$$

$$H = L^2(\mathcal{O}), \ \ A = \Delta, \ \ D(A) = H_0^2(\mathcal{O}).$$

$$dx(s,\xi) = y(s,\xi)ds$$
$$dy(s,\xi) = (\Delta x(s,\xi) + k(x(s,\xi), a(s)))\ ds + h(\xi)\, dZ(s).$$

$$A = \begin{pmatrix} 0, & I \\ \Delta, & 0 \end{pmatrix}, \qquad X(s) = \begin{pmatrix} x(s,\cdot) \\ y(s,\cdot) \end{pmatrix}$$

Z, $\alpha-$ stable, real process.

Lévy processes

$$\mathbb{E}\left[e^{i\langle u, Z(t_2)-Z(t_1)\rangle_U}\right] = e^{-(t_2-t_1)\psi(u)},$$

where

$$\psi(u) = -i\langle a, u\rangle_U + \frac{1}{2}\langle Qu, u\rangle_U$$

$$+\int_{U\setminus\{0\}} \left(1 - e^{i\langle u,z\rangle_U} + 1_{\{\|z\|_U<1\}} i\langle u, z\rangle_U\right)\nu(dz). \tag{2}$$

$$\pi([t,s], B) = \sum_{t<\tau\leq s} 1_B(L(\tau)-L(\tau-)), \quad B \in \mathcal{B}(U),\ L(\tau-) = \lim_{r\uparrow\tau} L(r),$$

$$\widehat{\pi}(d\tau, dz) = \pi(d\tau, dz) - d\tau\,\nu(dz).$$

(3) $$Z(s) = a\,s + W(s) + Z_0(s) + Z_1(s),$$

where $a \in U$ W is a Wiener process, Z_0, Z_1 are independent Lévy processes,

$$Z_0(s) = \int_t^s \int_{0<\|z\|<1} z\widehat{\pi}(d\tau, dz), \quad Z_1(s) = \int_t^s \int_{\|z\|\geq 1} z\pi(d\tau, dz),$$

(4)
$$dX(s) = (AX(s)+g(X(s), a(s)))\,ds+\int_U G(X(s-), z, a(s))\,\widehat{\pi}(ds, dz)$$

$$X(t) = x, \quad s \in [t, T]$$

$$V(t,x) = \inf_{a(\cdot)} \mathbb{E}\left(\int_t^T f(X(s),a(s))\,ds + h(X(T))\right), \tag{5}$$

$t \in [0,T], \quad x \in H,$ value function.

HJB equation:

$$\frac{\partial u}{\partial t}(t,x) + \inf_{a\in\Lambda}(L^a u(t,x) + f(x,a)) = 0,$$
$$u(T,x) = h(x), \;\; x \in H,$$

(6)

$$L^a v(x) = \langle Ax, Dv(x)\rangle + \langle g(x,a), Dv(x)\rangle$$
$$+ \int_U [v(x + G(x,a)z) - v(x) - \langle Dv(x), G(x,a)z\rangle]\,\nu(dz)$$

(7)

$$L^a v(x) = \langle Ax, Dv(x)\rangle + L_0^a v(x),$$

$$\frac{\partial u}{\partial t}(t,x) + \langle Ax, Du(t,x)\rangle + \inf_{a\in\Lambda}\big(L_0^a u(t,x) + f(x,a)\big) = 0, \tag{8}$$
$$u(T,x) = h(x).$$

Meta theorem:

V is the unique solution to (6) and the minimizer: $\hat{a}(t,x)$, $t \in [0,T]$, $x \in H$,

$$\inf_{a \in \Lambda} \left(L^a V(t,x) + f(x,a) \right) = L^{\hat{a}(t,x)} V(t,x) + f(x, \hat{a}(t,x))$$

is the optimal feedback strategy.

Process $\widehat{X}$:

$$
\begin{aligned}
d\widehat{X}(s) &= \left[A\widehat{X}(s) + g(\widehat{X}(s), \hat{a}(s, \widehat{X}(s)))\right] ds \\
&\quad + G(\widehat{X}(s-), \hat{a}(s, \widehat{X}(s-)))\, dZ(s), \\
\widehat{X}(t) &= x
\end{aligned}
$$

is an optimal one.

(9)
$$dX(s) = (AX(s) + g(X(s), a(s)))\,ds + G(X(s-), a(s))\,dW(s),$$

Stochastic Optimal Control in Infinite Dimensions: Dynamic Programming and HJB Equations, Springer, 2017
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HJB equation:

1. viscosity solutions
2. mild solutions
3. solutions through BSDEs

1. **Viscosity solutions**: based on *A. Święch, J. Zabczyk*,

Uniqueness for integro-PDE in Hilbert spaces, Potential Anal., 38 (2013), no. 1, 233–255.

Integro-PDE in Hilbert spaces: Existence of viscosity solutions,Potential Anal., 45 (2016), 703–736

2.**Mild solutions**: based on *E. Priola and J. Zabczyk*,

Structural properties of semilinear SPDEs driven by cylindrical stable processes, PTRF, 149 (2011), 97–137.

3. **Solutions through BSDEs**: only comments

Test functions

$$L^a v(x) = \langle Ax, Dv(x)\rangle + \langle g(x,a), Dv(x)\rangle$$

$$+ \int_U [v(x + G(x,a)z) - v(x) - \langle Dv(x), G(x,a)z\rangle]\,\nu(dz)$$

$$(10) \qquad \begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \inf_{a\in\Lambda}[L^a u(t,x) + f(x,a)] = 0, \\ &u(T,x) = h(x), \;\; t \in [0,T], \;\; x \in H. \end{aligned}$$

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$\varphi(t,x)$, $t \in [0,T]$, $x \in H$ is a **test function** if φ is continuous on $(0,T) \times H$ and
φ_t, $D\varphi$, $A^* D\varphi$ $D^2\varphi$ are uniformly continuous on $(\varepsilon, T-\varepsilon) \times H$ for every $\varepsilon > 0$ and locally bounded.

Subsolution

A continuous function $u : (0, T] \times H \to R$ is a viscosity subsolution to (10) if $u(T, x) \leq h(x)$ and whenever $u - \varphi$ has a global maximum at a point (t, x) for a test function φ then

$$\frac{\partial \varphi}{\partial t}(t, x) + \inf_{a \in \Lambda} \left[L^a \varphi(t, x) + f(x, a) \right] \geq 0. \tag{11}$$

Supersolution and viscosity solution

A continuous function $u : (0, T] \times H \to R$ is a viscosity supersolution to (10) if $u(T, x) \geq h(x)$ and whenever $u - \varphi$ has a global minimum at a point (t, x) for a test function φ then

$$(12) \qquad \frac{\partial \varphi}{\partial t}(t, x) + \inf_{a \in \Lambda} [L^a \varphi(t, x) + f(x, a)] \leq 0.$$

A viscosity solution to (10) is a function which is both subsolution and supersolution.

Assumptions

(A3) $\|g(x,a) - g(y,a)\| + \|G(x,a) - G(y,a)\| \leq C\|x - y\|,$

(A4) $\|g(0,a)\| + \|G(0,a)\| \leq C,$

(A5) $|f(x,a) - f(y,a)\| + \|h(x) - h(y)\| \leq \sigma(\|x - y\|),$

f, h continuous, σ modulus of confinity

(A6) $\int_U (\|z\|^2 \wedge \|z\|)\nu(dz) < +\infty.$

Existence theorem

Theorem 1

Under (A3)–(A6) the value function V is a viscosity solution to (10).

B-continuity

Let B-positive definite, bounded operator on H such that for some c_0:

$$\langle(-A^*B + c_0 B)x, x\rangle \geq 0, \qquad \text{for all } x \in B.$$

Define

$$\|x\|_B = \|B^{1/2}x\| = \langle Bx, x\rangle^{1/2},$$

u is B-continuous on $(0, T] \times H$ if whenever $t_n \to t$, $Bx_n \to Bx$ and x_n bounded, $u(t_n, x_n) \to u(t, x)$.

Uniqueness

$$(\mathrm{A3})' \qquad \|g(x,a)-g(y,a)\| + \|G(x,a)-G(y,a)\| \le C\|x-y\|_B$$

$$(\mathrm{A5})' \qquad |f(x,a)-f(y,a)| + \|h(x)-h(y)| \le \sigma(\|x-y\|_B)$$

B-test functions, B-viscosity solution is B-continuous.

B-test functions

$$\psi = \varphi + \delta(t, x) h(\|x\|),$$

(i) $\varphi_t, D\varphi, D^2\varphi, A^* D\varphi, \delta_t, D\delta, D^2\delta, A^* D\delta$ are uniformly continuous on $(\epsilon, T - \epsilon) \times H$ for every $\epsilon > 0$, $\delta \geq 0$ and is bounded, φ is B-lower semicontinuous, δ is B-continuous.

(ii) h is even, h', h'' are uniformly continuous on $\mathbb{R}$, $h'(r) \geq 0$ for $r \in (0, +\infty)$.

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(ii) h is even, h', h'' are uniformly continuous on $\mathbb{R}$, $h'(r) \geq 0$ for $r \in (0, +\infty)$.

Viscosity subsolutions are requiered to be B- upper semicontinous.
Viscosity supersolutions are required to be B- lower semicontinous

Theorem 2

Under $(\mathrm{A3})'$, $(\mathrm{A4})$, $(\mathrm{A5})'$, $(\mathrm{A6})$ equation (10) has unique B-viscosity solution. It is identical with the value function.

Wave equation

$$\frac{\partial^2 x}{\partial s^2} = \left(\Delta x(s,\xi) + \bar{g}(x(s,\xi), a(s))\right) ds + \overline{G}(x(s-,\xi), a(s))\, dZ(s).$$

Initial and boundary conditions, $s \in [t, T]$:

$$x(t,\xi) = \bar{x}(\xi), \qquad \frac{\partial x}{\partial t}(s,\xi) = y(s,\xi) = 0, \qquad \xi \in \partial\mathcal{O},$$

$x(s,\xi)$ – position, $\frac{\partial x}{\partial s}(s,\xi) = y(s,\xi)$ – velocity

$$X(s) = \begin{pmatrix} x(s,\cdot) \\ y(s,\cdot) \end{pmatrix} \in H = \begin{pmatrix} H_0^1(\mathcal{O}) \\ \times \\ L^2(\mathcal{O}) \end{pmatrix}$$

$$A = \begin{pmatrix} 0, & I \\ \Delta, & 0 \end{pmatrix}, \qquad D(A) = \begin{pmatrix} H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}) \\ \times \\ H_0^1(\mathcal{O}) \end{pmatrix}$$

$$g\left(\begin{pmatrix} x \\ y \end{pmatrix}, a\right) = \begin{pmatrix} 0 \\ \bar{g}(x, a) \end{pmatrix}, \qquad G\left(\begin{pmatrix} x \\ y \end{pmatrix}, a\right) = \begin{pmatrix} 0 \\ \overline{G}(y, a) \end{pmatrix}$$

$$B = \begin{pmatrix} (-\Delta)^{-\frac{1}{2}}, & 0 \\ 0, & (-\Delta)^{-\frac{1}{2}} \end{pmatrix}$$

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|_B = \left(\|(-\Delta)^{-\frac{1}{4}} x\|^2_{L^2(\mathcal{O})} + \|(-\Delta)^{-\frac{1}{4}} y\|^2_{L^2(\mathcal{O})} \right)^{1/2}$$

Z – one dimensional Lévy process.

Additive noise

$$
(13) \qquad dX(s) = (AX(s) + g(X(s), a(s)))ds + dZ(s),
$$

$$
\begin{aligned}
\frac{\partial v}{\partial t}(t,x) = \langle Ax, Dv(t,x)\rangle &+ \int_H (v(t, x+z) \\
&- v(t,x) - \langle Dv(t,x), z\rangle)\nu(dz) \\
&+ \inf_{a\in\Lambda}\langle g(x,a), Dv(t,x)\rangle.
\end{aligned}
$$

$$
v(0,x) = h(x)
$$

Additive noise

$$(14) \qquad \frac{\partial v}{\partial t}(t,x) = \mathcal{A}v(t,x) + \mathcal{H}(x, Dv(t,x))$$
$$v(0,x) = h(x)$$

$\mathcal{H}(x,p) = \inf_{a \in \Lambda} \langle g(x,a), p \rangle$, $\mathcal{A}$ generator of

$$dY(s) = AY(s)\,ds + dZ(s),$$

P_t transition semigroup of Y,

$$(15) \qquad v(t,x) = P_t h(x) + \int_0^t P_{t-s}[\mathcal{H}(\cdot), Dv(s,\cdot)]\,ds.$$

α-Stable systems

$$dX(s) = (AX(s) + g(X(s), a(s)))ds + dZ(s)$$

Assumption (A1):

$$Ae_n = -\lambda_n e_n, \;\; n = 1, 2, \dots \qquad Z(s) = \sum_{n=1}^{+\infty} \beta_n Z_n(s) e_n,$$

Z_n independent, α -stable processes .

$$\sum_{n=1}^{+\infty} \frac{\beta_n^\alpha}{\lambda_n} < +\infty, \qquad \frac{\beta_n^\alpha}{\lambda_n} \geq c \frac{1}{\lambda_n^{\alpha\gamma}}, \;\; n = 1, \dots, \gamma \in (0, 1),$$

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Z_n independent, α -stable processes .

$$\sum_{n=1}^{+\infty} \frac{\beta_n^\alpha}{\lambda_n} < +\infty, \qquad \frac{\beta_n^\alpha}{\lambda_n} \geq c \frac{1}{\lambda_n^{\alpha\gamma}}, \;\; n = 1, \dots, \gamma \in (0, 1),$$

Assumption (A2): For some constants M, C

$$|\mathcal{H}(x,p) - \mathcal{H}(x,q)| \leq M\|p - q\|, \;\; p, q, x \in H,$$

$$|\mathcal{H}(x,p) - \mathcal{H}(y,p)| \leq C\|x - y\| \|p\|, \;\; q, x, y \in H.$$

Space $C^{1,\gamma}$, of continuous functions $u(t,x)$, $\in [0,T]$, $x \in H$ s.t.

$$\|u\|_{C^{1,\gamma}} = \sup_{0<t\leq T}\left[\sup_x |u(t,x)| + t^\gamma \sup_x \|Du(t,x)\|\right]$$

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$$\|u\|_{C^{1,\gamma}} = \sup_{0<t\leq T}\left[\sup_x |u(t,x)| + t^{\gamma}\sup_x \|Du(t,x)\|\right]$$

Theorem 3

Assume $\alpha \in (1,2)$. Under $(\mathrm{A1})$, $(\mathrm{A2})$ equation (15) has unique solution in $C^{1,\gamma}([0,T],H)$

$$X(s) = x + \int_t^s g(X(r))dr + \int_t^s \int_U G(X(r-), z)\widehat{\pi}(ds, dz)$$

$$Y(s) + \int_s^T \int_U Z(r, z)\widehat{\pi}(dr, dz)$$

$$= \int_s^T f(X(r), Y(r), Z(r, \cdot)dr + h(X(T)).$$

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Then

$$u(t, x) = Y(t; t.x)$$

satisfies a nonlinear parabolic type equation. Under proper choice of f it is a HJB equation.

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Then

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satisfies a nonlinear parabolic type equation. Under proper choice of f it is a HJB equation.
The controlled equation is of the form

$$dX(s) = g(X(s)))\, ds + \int_U G(X(s-), z)(1 + R(X(s-), z, a))\widehat{\pi}(ds, dz)$$

Proposition 1

Let $A_n = nA(nI + A)^{-1}$, be the Yosida approximations of A. If a predictable process $\phi(r)$, $r \in [t, T]$ is such that

$$\mathbb{E} \int_t^T \|\phi(r)\|_{\mathcal{H}}^2 \, dr < +\infty,$$

then the stochastic convolution

$$\psi(s) = \int_t^s \int_U e^{(s-r)A} \phi(r, u) \widehat{\pi}(dr, du), \qquad t \le s \le T$$

has a cádlág modification and
(16)

$$\lim_{n\to\infty} \mathbb{E} \sup_{t\le s\le T} \left\| \int_t^s \int_U (e^{(s-r)A_n} - e^{(s-r)A}) \phi(r, u) \widehat{\pi}(dr, du) \right\|^2 = 0.$$

Proposition 2

Let $\left(\Omega_i, \mathcal{F}_i, \mathcal{F}_s^{i,t}, \mathbb{P}_i, L_i\right), i = 1, 2,$ be two reference probability spaces and $\pi_i, i = 1, 2,$ be the Poisson random measures for $L_i, i = 1, 2.$ and a_i strategies. Let $\mathcal{L}_{\mathbb{P}_1}(a_1(\cdot), L_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(a_2(\cdot), L_2(\cdot))$ on some subset $D \subset [0, T]$ of full measure. Denote by $X_i(\cdot)$ the unique mild solutions of the corresponding controlled equations with $a(\cdot) = a_i(\cdot) i = 1, 2.$ Then $\mathcal{L}_{\mathbb{P}_1}(X_1(\cdot), a_1(\cdot)) = \mathcal{L}_{\mathbb{P}_2}(X_2(\cdot), a_2(\cdot))$ on D.

Dynamic Programming Principle

Yosida approximations are needed to use Ito's formula.
Unbounded A replaced by bounded A_n.

$$V(t,x) = \inf_{(a(\cdot),\tau_{a(\cdot)})} \mathbb{E}\left[\int_t^{\tau_{a(\cdot)}} f(X(s),a(s))\,ds + V(\tau_{a(\cdot)}, X(\tau_{a(\cdot)}))\right]$$

$$t \le \tau_{a(\cdot)} \le T, \quad \text{stopping time}$$

Dubling technique of Crandall and Lions

Show that if u, v are subsolution and supersolution then

$$u(t,x) - v(t,x) \leq 0, \qquad t \in [0,T], \; x \in H,$$

Dubling technique of Crandall and Lions

Show that if u, v are subsolution and supersolution then

$$u(t,x) - v(t,x) \leq 0, \qquad t \in [0,T], \; x \in H,$$

$$\phi_\gamma((t,x),(s,y)) = u(t,x) - v(s,y) - \psi_\gamma(t,z,s,y), \quad \gamma\text{-parameters}$$

$$\gamma = (\varepsilon, \beta), \qquad \psi_\gamma(t,x,s,y) = \frac{1}{2\varepsilon}\langle B(x-y), x-y\rangle - \frac{(t-s)^2}{2\beta},$$

$(\bar{t},\bar{x},\bar{s},\bar{y}) \in (0,T] \times H \times (0,T] \times H$ maximizer of ϕ_γ. Then

$$u(t,x) - v(t,x) = \phi_\gamma((t,x),(t,x)) \leq \phi_\gamma((\bar{t},\bar{x}),(\bar{s},\bar{y})).$$

Show that $\phi_\gamma((\bar{t},\bar{x}),(\bar{s},\bar{y}))$ is small for proper choice of γ.

Pushing maximas

Consider map

$$(t, x) \to \phi_\gamma((t, x), (\bar{s}, \bar{y}))$$

which attains maximum at $(\bar{t}, \bar{x})$; subsolution property of u gives relations between $(\bar{t}, \bar{x}), (\bar{s}, \bar{y})$.

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Gradient estimates:

Enrico Priola, J. Z. (PTRF, 149 (2011), 97–137).

$$dY(s) = AY(s)\,ds + dZ(s), \qquad Y(0) = x, \;\; Y(s,x)$$

$$P_t\varphi(x) = \mathbb{E}(\varphi(Y(s,x))),$$

$$\sup_{x\in H} \|DP_t\varphi\| \le c_\alpha C_t \sup_{x\in H} |\varphi(x)|, \qquad t > 0,$$

$$c_\alpha = \int \frac{(p'_\alpha(z))^2}{p_\alpha(z)}\,dz, \qquad C_t = \sup_n \frac{e^{-\lambda_n t}(\lambda_n)^{1/\alpha}}{\beta_n}$$

p_α , the canonical α-stable density

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p_α , the canonical α-stable density

Comparison property and maximum principle

$$(17)\qquad \frac{d}{dt}u(t,x) = \mathcal{A}(u(t,x)),\ \ u(0,x) = x,\ \ x \in C(E).$$

Equation (17) has comparison property iff $x \le y$ implies $u(t,x) \le u(t,y),\ \ t \ge 0$.

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Operator $\mathcal{A}$ has maximum property iff $x \le y$ and $x(\xi) = y(\xi)$ imply $\mathcal{A}x(\xi) \le \mathcal{A}y(\xi), \quad \xi \in E.$

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Fact: Comparison property and maximum principle are equivalent.

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