Weierstrass Institute for
Applied Analysis and Stochastics

Leibniz Association

Type II singular perturbation approximation for linear systems with Lévy noise

Martin Redmann

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PDE $(X(t,\zeta))$

- motion of viscous fluids,
- water or sound waves,
- distribution of heat...

$\longrightarrow$ Discretisation in space (e.g. Galerkin methods) $\downarrow$

ODE $(x(t))$

- might be sparse,
- might be of large order,
- high computational cost.

$\downarrow$ Model order reduction (MOR) (e.g. balancing related) $\longleftarrow$

Reduced Model $(\tilde{x}(t))$

- small order,
- low computational time,
- high accuracy desired.

Overview

SPDE $(X(t, \zeta, \omega))$

- motion of viscous fluids,
- water or sound waves,
- distribution of heat...

$\longrightarrow$ Discretisation in space (e.g. Galerkin methods)

$\downarrow$

SODE $(x(t, \omega))$

- might be sparse,
- might be of large order,
- high computational cost.

$\downarrow$

$\longleftarrow$ Model order reduction (MOR) (e.g. balancing related)

Reduced Model $(\tilde{x}(t, \omega))$

- small order,
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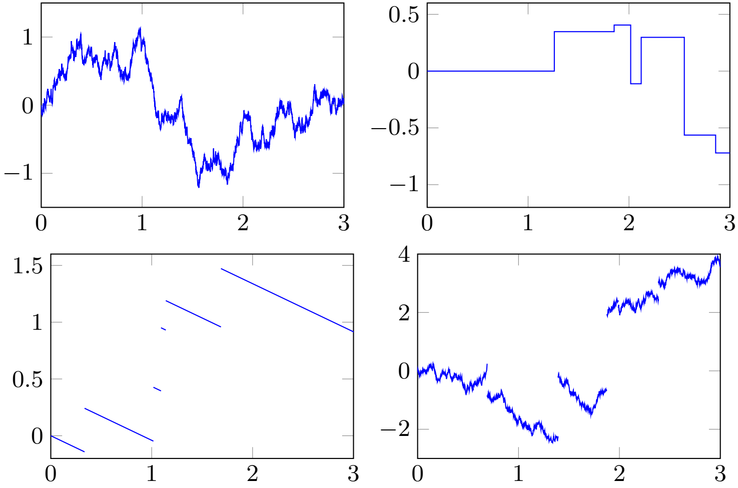

More information: [PESZAT, ZABCZYK '07] & [APPLEBAUM '09]

Let M be a q-dimensional Lévy process.

$$dx(t) = [Ax(t) + Bu(t)]\,dt + N_i x(t-) dM^i(t),$$
$$y(t) = Cx(t)$$

with $A, N_i \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $\mathbb{E}\left[M(t)\right] = 0$, $\mathbb{E}\left\|M(t)\right\|^2_{\mathbb{R}^q} < \infty$,

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where n is large. Replace this system by

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + E_i u(t-)\right] dM^i(t),$$
$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + Du(t)$$

with $\tilde{A}, \tilde{N}_i \in \mathbb{R}^{r\times r}$, $\tilde{B} \in \mathbb{R}^{r\times m}$, $\tilde{C} \in \mathbb{R}^{p\times r}$, $D \in \mathbb{R}^{p\times m}$ and $E_i \in \mathbb{R}^{r\times m}$,

where $r \ll n$ such that

$$y(t) \approx \tilde{y}(t).$$

(for $N_i = 0$ [ANTOULAS '05])

Asymptotic Mean Square Stability

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\begin{aligned}
dx(t) &= [Ax(t) + Bu(t)]\, dt + N_i x(t-) dM^i(t), \\
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Asymptotic Mean Square Stability

$$dx(t) = [Ax(t) + Bu(t)]\,dt + N_i x(t-)dM^i(t),$$
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Definition

- A càdlàg adapted process $x(t)$, $t \geq 0$, is a solution if
$$x(t) = x_0 + \int_0^t [Ax(s) + Bu(s)]\,ds + \int_0^t N_i x(s-)dM^i(s), \quad t \geq 0.$$
- *Notation*: $x(t, x_0, u)$ for control $u \in L_t^2$, time $t \geq 0$ and initial condition $x_0 \in \mathbb{R}^n$.

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Let $K = (k_{ij})_{i,j=1,\dots,q}$ be the covariance matrix of M, i.e., $\mathbb{E}\left[M(t)M^T(t)\right] = Kt$.

Asymp. mean square stability **[R. '17]**

$$\mathbb{E}\,\|x(t,x_0,0)\|_{\mathbb{R}^n}^2 \to 0 \text{ when } t \to \infty, \quad \forall x_0 \in \mathbb{R}^n$$

$$\Leftrightarrow \sigma(A \otimes I + I \otimes A + \sum_{i,j=1}^{q} N_i \otimes N_j\, k_{ij}) \subset \mathbb{C}_-.$$

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Proposition [R. ’17]

Q exists due to the mean square asymptotic stability and is the unique solutions to

$$A^T Q + QA + \sum_{i,j=1}^{q} {N_i}^T Q N_j \, k_{ij} = -C^T C.$$

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Proposition [BENNER, DAMM '11] for $M = w$

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We define the type I reachability Gramian by $P_1 := \mathbb{E} \int_0^\infty \Phi(t) B B^T \Phi^T(t) dt$.

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Proposition **[R. '17]**

Let $(p_{1,k})$ be an ONB of EV of P_1, then

$$\sup_{t\in[0,T]} \mathbb{E} \left|\langle x(t,0,u), p_{1,k}\rangle_{\mathbb{R}^n}\right| \leq \lambda_{1,k}^{\frac{1}{2}} \|u\|_{L_T^2}.$$

Type II reachability Gramian

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Definition **[Damm, Benner '14] for $k_{ij} = \delta_{ij}$ & [R. '17]**

We define the type II reachability Gramian P_2 as a positive definite solution to

$$A^T P_2^{-1} + P_2^{-1} A + \sum_{i,j=1}^{q} N_i^T P_2^{-1} N_j k_{ij} \leq -P_2^{-1} B B^T P_2^{-1}. \quad (1)$$

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Proposition [DAMM, BENNER '14] for $k_{ij} = \delta_{ij}$ & [R. '17]

There exists a positive definite solution to inequality (1) due to the assumption of mean square asymptotic stability for the system.

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$$\sup_{t\in[0,T]} \sqrt{\mathbb{E}\langle x(t,0,u), p_{2,k}\rangle^2_{\mathbb{R}^n}} \leq \lambda_{2,k}^{\frac{1}{2}} \left\| u \right\|_{L^2_T}.$$

Using the type II approach means that a balancing transformation based on the Gramians Q and P_2 is applied to the following system:

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Theorem

Suppose that $Q > 0$. Then, there is an invertible matrix $T = T(P_2, Q)$ such that

$$\begin{aligned} d\hat{x}(t) &= \left[TAT^{-1}\hat{x}(t) + TBu(t)\right] dt + TN_iT^{-1}\hat{x}(t-)dM^i(t), \\ y(t) &= CT^{-1}\hat{x}(t) \end{aligned}$$

with $\hat{P}_2 = TP_2T^T = T^{-T}QT^{-1} = \hat{Q} = \Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_n)$, where $\sigma_i = \sqrt{eig_i(P_2Q)}$.

Using the type II approach means that a balancing transformation based on the Gramians Q and P_2 is applied to the following system:

$$dx(t) = [Ax(t) + Bu(t)]\, dt + N_i x(t-) dM^i(t),$$
$$y(t) = Cx(t).$$

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From now on we assume to already have a balanced system.

Let (A, B, C, N_i) be balanced. Using a partition of the matrices we obtain

Balanced Partitioned Full Model

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \right) dt + \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1(t-) \\ x_2(t-) \end{bmatrix} dM^i(t),$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \geq 0.$$

Reduced Order Model

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + \tilde{E}_i u(t-)\right] dM^i(t),$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t).$$

Let (A, B, C, N_i) be balanced. Using a partition of the matrices we obtain

Truncate Second Line & Set $x_2(t) = 0$ — $N = 0 \rightarrow$ [Antoulas '05]

$$\begin{bmatrix} dx_1(t) \\ dx_2(t) \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \right) dt + \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1(t-) \\ 0 \end{bmatrix} dM^i(t),$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ 0 \end{bmatrix}, \quad t \geq 0.$$

Reduced Order Model Balanced Truncation (BT)

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + \tilde{E}_i u(t-)\right] dM^i(t),$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t).$$

BT: $\tilde{A} = A_{11}, \quad \tilde{N}_i = N_{i,11}, \quad \tilde{B} = B_1, \quad \tilde{C} = C_1, \quad D = E_i = 0.$

Let (A, B, C, N_i) be balanced. Using a partition of the matrices we obtain

Set $dx_2(t) = 0$ $\qquad N_i = 0 \rightarrow$ **[Liu, Anderson '89]**

$$\begin{bmatrix} dx_1(t) \\ 0 \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \right) dt + \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1(t-) \\ x_2(t-) \end{bmatrix} dM^i(t),$$

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Reduced Order Model Singular Perturbation Approximation (SPA)

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + \tilde{E}_i u(t-)\right] dM^i(t),$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t).$$

SPA: $\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}$, $\tilde{N}_i = N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}$, $\tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2$,
$\tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}$, $\tilde{D} = -C_2A_{22}^{-1}B_2$, $\tilde{E}_i = -N_{i,12}A_{22}^{-1}B_2$.

Let (A, B, C, N_i) be balanced. Using a partition of the matrices we obtain

Set $dx_2(t) = 0$ $\qquad$ $N_i = 0 \rightarrow$ **[Liu, Anderson '89]**

$$\begin{bmatrix} dx_1(t) \\ 0 \end{bmatrix} = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t) \right) dt + \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \begin{bmatrix} x_1(t-) \\ x_2(t-) \end{bmatrix} dM^i(t),$$

$$y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad t \geq 0.$$

Reduced Order Model Simplified Singular Perturbation Approximation (SSPA)

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + \tilde{E}_i u(t-)\right] dM^i(t),$$

$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + \tilde{D}u(t).$$

SSPA: $\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \tilde{N}_i = N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}, \quad \tilde{B} = B_1,$

$\tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \tilde{D} = 0, \quad \tilde{E}_i = 0.$

Why balancing based on P_2 is better than balancing based on P_1 from the theoretical point of view?

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P_1 versus P_2

Why balancing based on P_2 is better than balancing based on P_1 from the theoretical point of view?

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- (S)SPA based on P_1 only allows us to prove an $\mathcal{H}_2$-error bound, an $\mathcal{H}_\infty$-error bound doesn't exist.
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Theorem **[R. '17]**

For SPA and SSPA with reduced order coefficients

$$\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \tilde{N}_i = N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}, \quad \tilde{B} = B_1(-A_{12}A_{22}^{-1}B_2),$$
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we have

$$\sigma(A \otimes I + I \otimes A + \sum_{i,j=1}^{q} N_i \otimes N_j k_{ij}) \subset \mathbb{C}_-$$
$$\Rightarrow \sigma(\tilde{A} \otimes I + I \otimes \tilde{A} + \sum_{i,j=1}^{q} \tilde{N}_i \otimes \tilde{N}_j k_{ij}) \subset \mathbb{C}_-.$$

Theorem [Liu, Anderson '89] for N_i=0

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The prove of the above theorem relies on the following:

Theorem **consequence of [Benner, Damm, Rodriguez Cruz '17]**

For balanced truncation with reduced order coefficients $(A_{11}, B_1, C_1, N_{i,11})$, we have

$$\sigma(A \otimes I + I \otimes A + \sum_{i,j=1}^{q} N_i \otimes N_j k_{ij}) \subset \mathbb{C}_-$$
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Remark

The proof of the Theorem is an open problem when balancing based on P_1, [Benner, R. '17].

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, N_i = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}.$$

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Theorem **[R. '17]**

Let $\tilde{y}$ be the output of SSPA and $P_G = \begin{bmatrix} P_{G,1} \\ P_{G,2} \end{bmatrix}$, then

$$\sup_{t\in[0,T]} \mathbb{E} \left\|y(t) - \tilde{y}(t)\right\|_{\mathbb{R}^p} \leq \left(\mathrm{tr}(\Sigma_2 W)\right)^{\frac{1}{2}} \left\|u\right\|_{L^2_T},$$

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where the scaling matrix is

$$\begin{aligned} W = \operatorname{tr}&\left((B_2 B_2^T - 2(A_{22}P_{G,2} + A_{21}P_{G,1})(A_{22}^{-1}A_{21})^T)\right) \\ &+ \operatorname{tr}\left(2\sum_{i,j=1}^{q}(N_{i,22}P_{G,2} + N_{i,21}P_{G,1})(N_{j,21} - N_{j,22}A_{22}^{-1}A_{21})^T k_{ij}\right) \\ &- \operatorname{tr}\left(\sum_{i,j=1}^{q}(N_{i,21} - N_{i,22}A_{22}^{-1}A_{21})P_r^1(N_{j,21} - N_{j,22}A_{22}^{-1}A_{21})^T k_{ij}\right). \end{aligned}$$

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Remark

The above Theorem is similar in the case of using P_1, see [Benner, R. '17].

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Theorem **[R. '17]**

Let y be the output of the original system and $\Sigma_2 = \operatorname{diag}(\tilde{\sigma}_1 I, \tilde{\sigma}_2 I, \ldots, \tilde{\sigma}_\nu I)$. Then,

$$\|y - \tilde{y}\|_{L^2_T} \leq 2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \ldots + \tilde{\sigma}_\nu) \|u\|_{L^2_T},$$

where $\tilde{y}$ the output from

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Theorem [Liu, Anderson '89] for $N_i = 0$

Let y be the output of the original system and $\Sigma_2 = \operatorname{diag}(\tilde{\sigma}_1 I, \tilde{\sigma}_2 I, \ldots, \tilde{\sigma}_\nu I)$. Then,

$$\|y - \tilde{y}\|_{L^2_T} \leq 2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \ldots + \tilde{\sigma}_\nu) \|u\|_{L^2_T},$$

where $\tilde{y}$ the output from

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + E_i u(t-)\right] dM^i(t),$$
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$$\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \tilde{N}_i = N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}, \quad \tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$
$$\tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \tilde{D} = -C_2A_{22}^{-1}B_2, \quad \tilde{E}_i = -N_{i,12}A_{22}^{-1}B_2.$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, N_i = \begin{bmatrix} N_{i,11} & N_{i,12} \\ N_{i,21} & N_{i,22} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix}.$$

Theorem [Liu, Anderson '89] for $N_i = 0$

Let y be the output of the original system and $\Sigma_2 = \mathrm{diag}(\tilde{\sigma}_1 I, \tilde{\sigma}_2 I, \ldots, \tilde{\sigma}_\nu I)$. Then,

$$\|y - \tilde{y}\|_{L^2_T} \leq 2(\tilde{\sigma}_1 + \tilde{\sigma}_2 + \ldots + \tilde{\sigma}_\nu) \|u\|_{L^2_T},$$

where $\tilde{y}$ the output from

$$d\tilde{x}(t) = \left[\tilde{A}\tilde{x}(t) + \tilde{B}u(t)\right] dt + \left[\tilde{N}_i\tilde{x}(t-) + E_i u(t-)\right] dM^i(t),$$
$$\tilde{y}(t) = \tilde{C}\tilde{x}(t) + Du(t)$$

with the coefficients

$$\tilde{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \tilde{N}_i = N_{i,11} - N_{i,12}A_{22}^{-1}A_{21}, \quad \tilde{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$
$$\tilde{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \tilde{D} = -C_2A_{22}^{-1}B_2, \quad \tilde{E}_i = -N_{i,12}A_{22}^{-1}B_2.$$

Remark

The above theorem is not true when using P_1 instead, see [Damm, Benner, '14].

Difficulties

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- Proofs are conducted in time domain.
- No link to balanced truncation.
- Change in the structure from original to reduced model, no balanced ROM ($\mathcal{H}_\infty$-case).

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Thank You!