Preferential attachment with choice

Jonathan Jordan (joint with John Haslegrave)

School of Mathematics and Statistics, University of Sheffield

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Motivation: Achlioptas processes

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Some interest (papers by Malyshkin & Paquette and Krapivsky & Redner) in similar modifications to preferential attachment process.

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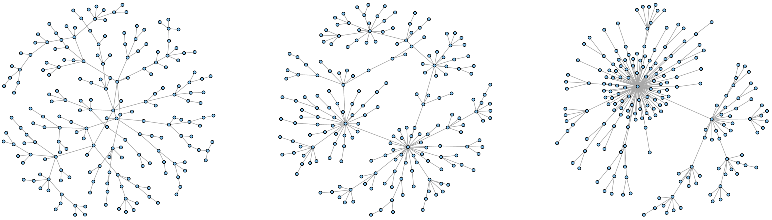
We will generally think of r being at least 2, since the case $r = s = 1$ is standard preferential attachment.

Pictures



200-vertex simulations, produced using igraph in R. Left to right: $r = 2, s = 2$, standard preferential attachment, $r = 2, s = 1$. The maximum degrees in these simulations are are 6, 30 and 90 respectively.

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- A non-degenerate limit distribution with a heavy tail (power law or similar), similar to standard preferential attachment.
- A dominant vertex, with degree of the same order as the size of the graph.
- A non-degenerate limit distribution with a doubly exponential tail.

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They suggested:

- When $s = 1$ ("greedy choice") a dominant vertex occurs if $r \geq 3$, and a degree distribution with tail decay $(n \log n)^{-2}$ if $r = 2$;

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- When $s > 1$ ("meek choice") doubly exponential decay happens whatever the values of r and s. (Even for r large and $s = 2$.)

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Their results match Krapivsky & Redner's in these cases: doubly exponential decay for $r = 2, s = 2$, $(n \log n)^{-2}$ decay for $r = 2, s = 1$, and a dominant vertex for $r > 2, s = 1$.

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- For $s = 2$, $r(s) = 7$. In particular, a non-degenerate limiting degree distribution does not exist if $r \geq 7, s = 2$.

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Then $F_m(k)/2m$ is the probability of selecting a vertex of degree at most k with a single preferential choice.

Evolution of $F_m(k)$

Write

$$f_k(x,p) = (k+1)B_{r,s}(p) - kB_{r,s}(x) - 2x + 1$$

for $x, p \in [0,1]$ and $k \geq 0$.

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By considering the degree of the vertex selected as neighbour to the new vertex, we find $\mathbb{E}\left(\frac{F_{m+1}(k)}{2(m+1)} - \frac{F_m(k)}{2m} | \mathcal{F}_m\right)$ is

$$\frac{1}{2(m+1)} f_k\left(\frac{F_m(k)}{2m}, \frac{F_m(k-1)}{2m}\right).$$

The p_k

This suggests that if $\frac{F_m(k)}{2m} \to p_k$ a.s. as $m \to \infty$, we expect $f_k(p_k, p_{k-1}) = 0$: ideas similar to *stochastic approximation* processes.

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Stochastic approximation intuition now suggests $\frac{F_m(k)}{2m} \to p_k$ a.s. as $m \to \infty$. More precise results in paper.

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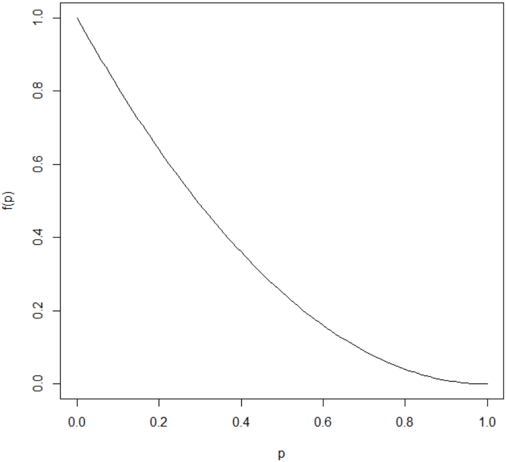
- *Provided* $s \geq 2$, *if* $p_* = 1$ *then* $-\log(1 - p_k) = \Omega(s^k)$.
- *The only other case with* $r > 1$ *where* $p_* = 1$ *is* $r = 2, s = 1$, *and then* $1 - p_k = (2 + o(1))/\log k$.

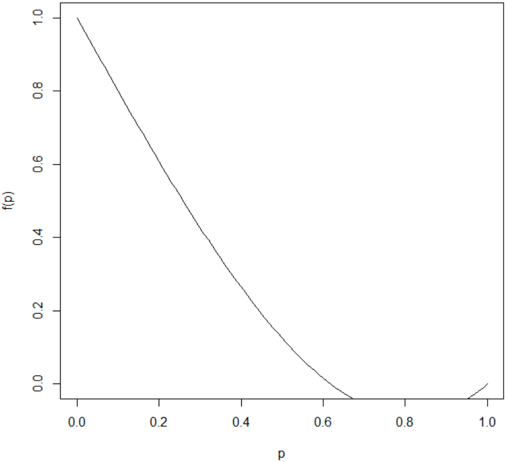
Plots of $f(p)$

$r = 2, s = 1$



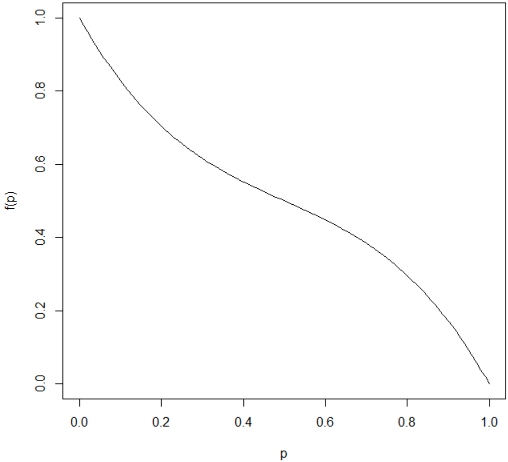
Plots of $f(p)$

$r = 3, s = 1$



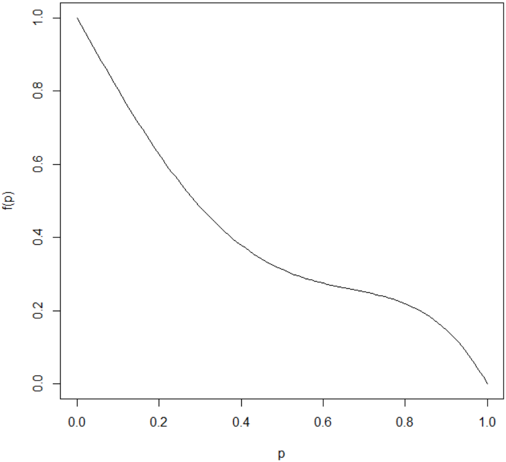
Plots of $f(p)$

$r = 3, s = 2$



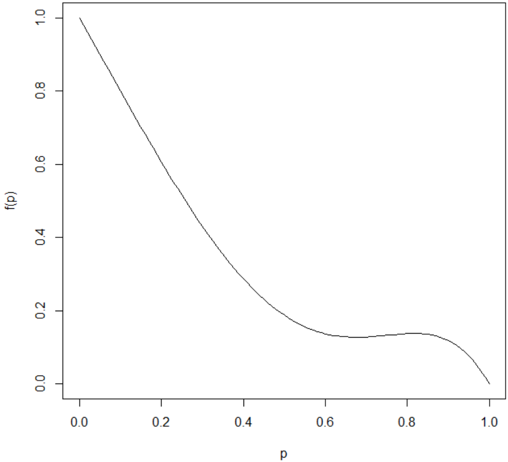
Plots of $f(p)$

$r = 4, s = 2$



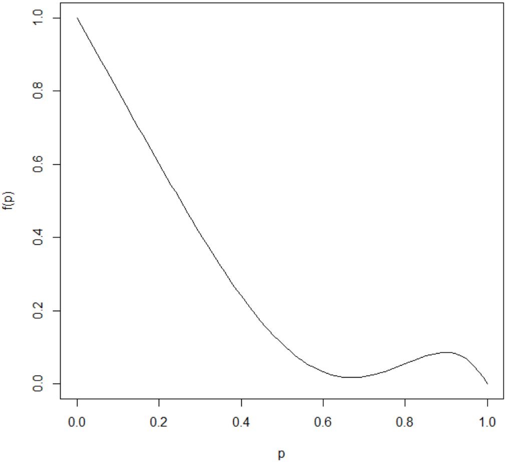
Plots of $f(p)$

$r = 5, s = 2$



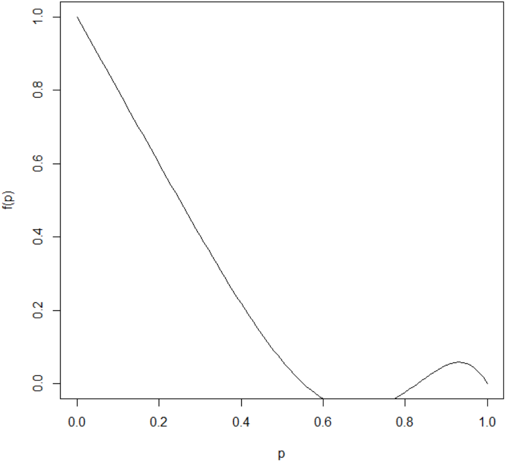
Plots of $f(p)$

$r = 6, s = 2$



Plots of $f(p)$

$r = 7, s = 2$



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In fact we can show $r(s) > 2s$ and $r(s)/s \to 2$ but $s^{-1/2}(r(s) - 2s) \to \infty$, by further analysis of the function f.

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Step 2: Step 1 implies doubly-exponential decay provided we can find some k_0 with

$$q_{k_0} < \left(\frac{2}{\binom{r}{s}(k_0+3)}\right)^{1/(s-1)} . \tag{1}$$

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For $r = 2, s = 2$, (1) satisfied for $k_0 = 4$, for $r = 3, s = 2$ for $k_0 = 18$, for $r = 4, s = 2$ for $k_0 = 98$, for $r = 5, s = 2$ for $k_0 = 2416$. For $r = 6, s = 2$ $k_0 > e^{23}$.

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Simulations don't easily distinguish between doubly exponential decay above very large threshold and a dominant vertex.

References

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